A NOTE ON CERTAIN JORDAN TYPE INEQUALITIES

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ABSTRACT. We offer a survey on Jordan type inequalities. By using the convexity method, some new results will be pointed out.

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1 Introduction

The inequality

$$\sin x \geq \frac{2}{\pi}x, \quad x \in \left[0, \frac{\pi}{2}\right] \tag{1}$$

is called in the literature as "Jordan's inequality" (see e.g. [5], [7]). This simple but useful inequality has evoked the interest of many authors, and many applications, sharpening or extensions have been obtained. On the other hand, such inequalities are rediscovered from time to time, and the same things are published in various journals. Thus one aim of this paper is a survey of such results, by pointing out also the related priority questions.

Jordan's inequality is a consequence of the concavity of the curve $y = \sin x$ on $\left[0, \frac{\pi}{2}\right]$. Since the segment of point $(0,0)$ and $\left(\frac{\pi}{2}, 1\right)$ lies below the curve, we get inequality (1). This shows also that there is strict inequality for $x \in (0, \pi/2)$. For another nice geometric interpretation of Jordan's inequality (via circle arcs), see our book [7] (reproduced also in the recent edition of [15]).

Jordan's inequality is in fact equivalent to the so-called "Kober's inequality", in effect that (see [9], [11])

$$\cos x \geq 1 - \frac{2}{\pi}x, \quad x \in \left[0, \frac{\pi}{2}\right] \tag{2}$$

Indeed, let us apply (1) to $x \to \frac{\pi}{2} - x$. Then (2) follows. This has been shown in our paper [9], and recently in [11]. More generally, by this method, any inequality for $\sin x$ may be transformed into one for $\cos x$. For example, the simple inequality $\sin x \leq x$ becomes

$$\cos x \leq \frac{\pi}{2} - x, \quad x \in \left[0, \frac{\pi}{2}\right], \tag{3}$$

which is complementary to (2).

We want to show an immediate corollary to (1), combined with (3). This is "Stečkin's inequality" (see [5])

$$\text{tg } x > \frac{4}{\pi} \cdot \frac{x}{\pi - 2x}, \quad x \in \left(0, \frac{\pi}{2}\right) \tag{4}$$

Indeed,

$$\text{tg } x = \frac{\sin x}{\cos x} > \left(\frac{2}{\pi}x\right) / \left(\frac{\pi}{2} - x\right) = \frac{4}{\pi} \cdot \frac{x}{\pi - 2x},$$

and (4) follows.

For smaller intervals, Jordan's inequality can be sharpened by the above convexity method (see e.g. [9], [14]). One has

$$\sin x \geq \frac{2}{\pi}\sqrt{2}x, \quad x \in \left[0, \frac{\pi}{4}\right] \tag{5}$$

This follows also from the monotonicity property of

$$g(x) = \frac{\sin x}{x}, \quad x \in \left(0, \frac{\pi}{2}\right).$$

More precisely, this function is nondecreasing, first applied in our paper [8] (see also [10]). This has been rediscovered recently by L. Zhu [19]. Thus, one has

$$\frac{\sin x}{x} \geq \frac{\sin r}{r} \text{ for } 0 < x \leq r < \frac{\pi}{2} \tag{6}$$

For $r = \frac{\pi}{4}$ in (6) one reobtains (5).

In 1999 S.-L. Xu et al. [21] have proved the following double inequality:

$$\frac{2}{\pi} + \frac{1}{\pi^3}(\pi^2 - 4x^2) \leq \frac{\sin x}{x} \leq \frac{2}{\pi} + \frac{\pi - 2}{\pi^3}(\pi^2 - 4x^2), \quad x \in \left(0, \frac{\pi}{2}\right] \tag{7}$$

This has been rediscovered recently by X. Zhang, G. Wang and Y. Chu [17] (see Theorem 1.1, relation (1.4)).

L. Debnath and C.J. Zhao [2] proved the left side of (7) using other method.

L. Zhu [19] generalized (7), by using the "monotone form of l'Hôpital's rule" (see [1]) as follows:

$$\frac{\sin r}{r} + \frac{\sin r - r\cos r}{2r^3}(r^2 - x^2) \leq \frac{\sin x}{x} \leq \frac{\sin r}{r} + \frac{r - \sin r}{r^3}(r^2 - x^2) \tag{8}$$

for $0 < x \leq r \leq \pi/2$.

We note that, in [17], the same method is used for the proof of (7).

J. Sándor [12] (see also [18]), by showing that the function $g(x) = \frac{\sin x}{x}$, $g(0) = 1$ is strictly concave on $\left[0, \frac{\pi}{2}\right]$ has proved that

$$\frac{\sin x}{x} \geq \frac{2}{\pi} + \frac{\pi - 2}{\pi^2}(\pi - 2x) = 1 + \frac{2(2-\pi)}{\pi^2}x, \quad x \in \left[0, \frac{\pi}{2}\right] \tag{9}$$

(where $\frac{\sin 0}{0} := 1$). This has been rediscovered recently in [17] (see Theorem 1.1, left side of (1.3)).

In fact, inequality (9) follows by the above concavity property, by remarking that $y(x) = 1 + \frac{2(2-\pi)}{\pi^2}x$ is the line passing on the points $A(0,1)$ and $B\left(\frac{\pi}{2}, \frac{2}{\pi}\right)$, and that $y(x) \leq g(x)$. By writing the tangent line to $g(x)$ at point B, we get $\frac{\sin x}{x} \leq \frac{4(\pi - x)}{\pi^2}$, i.e.

$$\frac{\sin x}{x} \leq \frac{2}{\pi} + \frac{2}{\pi^2}(\pi - 2x) - \frac{4(\pi - x)}{\pi^2}, \quad x \in \left[0, \frac{\pi}{2}\right], \tag{10}$$

rediscovered again in [17] (see Theorem 1.1, right side of (1.3)).

Finally, note that by letting $x \to \frac{\pi}{2} - x$ in (9) and (10), one obtains

$$1 - \frac{2}{\pi}x + \frac{\pi - 2}{\pi^2}x(\pi - 2x) \le \cos x \le 1 - \frac{2}{\pi}x + \frac{2}{\pi^2}x(\pi - 2x), \tag{11}$$

rediscovered in [17] (Theorem 1.1, (1.5)).

We note that many related inequalities for $\frac{\sin x}{x}$ are known in the literature.

From the series expansion of $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$ we easily get

$$1 - \frac{x^2}{6} < \frac{\sin x}{x} < 1 - \frac{x^2}{6} + \frac{x^4}{120}, \quad 0 < x < \frac{\pi}{2}; \tag{12}$$

see also [4].

R. Euler and J. Sadek [3] have shown that

$$\frac{\sin x}{x} > \frac{1}{1 + \operatorname{tg} x}, \quad x \in \left(0, \frac{\pi}{2}\right). \tag{13}$$

R. Redheffer ([5], [7]) has proved that

$$\frac{\sin x}{x} > \frac{\pi^2 - x^2}{\pi^2 + x^2}, \quad x \in (0, \pi). \tag{14}$$

For a generalization see our recent book [14].

We note that $\frac{\pi^2 - x^2}{\pi^2 + x^2} > \frac{\pi - x}{\pi}$, so we get a weaker result, but many times with importance in applications (e.g. Fourier series):

$$\frac{\sin x}{x} > \frac{\pi - x}{x}, \quad x \in (0, \pi) \tag{15}$$

J.B. Wilker [20] has shown that

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\operatorname{tg} x}{x} - 2 > 0, \quad x \in \left(0, \frac{\pi}{2}\right); \tag{16}$$

for related results, see also [16].

By the convexity of $y = \text{tg } x$, $x \in \left[0, \frac{\pi}{4}\right]$ we easily get (see [9]):

$$\text{tg } x \le \frac{4}{\pi} x \text{ for } x \in \left[0, \frac{\pi}{4}\right] \tag{17}$$

By putting $x \to \frac{x}{2}$, we get

$$\text{tg } \frac{x}{2} \le \frac{2}{\pi} x (\le \sin x) \text{ for } x \in \left[0, \frac{\pi}{2}\right] \tag{18}$$

Applying (17) to $x \to \frac{\pi}{2} - x$, we get

$$\text{ctg } x \le 2 - \frac{4}{\pi} x \text{ for } x \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right] \tag{19}$$

For the inverse trigonometric functions one has e.g. (see [9]):

$$\sqrt{2}x + \frac{\pi}{4} - 1 \le \arcsin x \le \frac{\pi}{2} x, \quad x \in [0, 1]; \tag{20}$$

$$\frac{\pi}{4} x \le \text{arctg } x \le \frac{x-1}{2} + \frac{\pi}{4}, \quad x \in [0, 1] \tag{21}$$

Another double inequality is (see [9], relation (34)):

$$\frac{1 + \cos x}{2} \le \frac{\sin x}{x} \le \cos \frac{x}{2}, \quad x \in \left[0, \frac{\pi}{2}\right] \tag{22}$$

We note that the left side of (22) constitutes another refinement of (15) for $x \in (0, \pi/2)$. This follows by Kober's inequality (2).

The famous "Huygens' inequality" states that

$$2 \sin x + \text{tg } x < 3x \text{ for } x \in \left(0, \frac{\pi}{2}\right) \tag{23}$$

For related results, refinements etc., see [7], [13].

By letting $x \to \frac{\pi}{2} - x$ in (23) we easily get

$$2\cos x + \mathrm{ctg}\, x < \frac{3\pi}{2} - 3x, \quad x \in \left(0, \frac{\pi}{2}\right) \tag{24}$$

Finally, we note that Jordan's inequality (1) has many applications in Mathematics. For a surprising recent application in Number theory, see e.g. W.G. Novak [6].

2 Main results

As we have seen in the Introduction, relations (9) and (10), rediscovered recently in [17], are due in fact to the present author. Our method from [12] was based on concavity of the function $g(x) = \frac{\sin x}{x}$, $x \in \left[0, \frac{\pi}{2}\right]$. This method enables us to offer generalizations of type (8) (but different from (8)). Namely, the following holds true:

Theorem 1. *One has*

$$1 + \frac{\sin r - r}{r^2} x \geq \frac{\sin x}{x} \geq \frac{\sin r}{r} + (x - r)\left(\frac{r\cos r - \sin r}{r^2}\right) \tag{25}$$

for $0 < x \leq r \leq \pi/2$.

Proof. The left side of (25) holds true in fact for all $x, r \in \left(0, \frac{\pi}{2}\right]$. Indeed, will write the tangent line $y = y(x)$ to the concave curve $g(x)$ at the point $\left(r, \frac{\sin r}{r}\right)$. Since the equation of the tangent line is $y(x) = 1 + \frac{\sin r - r}{r^2} x$, we get the left side of (25).

Let now $y = y_1(x)$ be the line joining the points $A(0, 1)$ and

$B\left(r, \dfrac{\sin r}{r}\right)$. Since

$$y_1(x) = \frac{\sin r}{r} + (x - r)\left(\frac{r\cos r - \sin r}{r^2}\right)$$

on $[x \in (0, r]$ we get the right side inequality of (25).

Remark. For $r = \dfrac{\pi}{2}$ in the right side of (25), we reobtain relation (9). For the same value of r in the left side of (25), we get the inequality (10).

Theorem 2. *One has:*

a) $\left(\dfrac{\sin x}{x}\right)^2 > \dfrac{x}{\operatorname{tg} x}$ *for* $x \in (0, \pi/2)$*;*

b) $\dfrac{\sin x}{x} > \dfrac{x}{\arcsin x}$ *for* $x \in (0, 1)$*;*

c) $\dfrac{x}{\arcsin x} > \dfrac{\sin \frac{\pi}{2}x}{\frac{\pi}{2}x}$ *for* $x \in (0, 1)$.

Proof. First remark, that b) and c) are corollaries to relation (6). Indeed, since $x < \arcsin x$ for $x \in (0, 1)$, we can write

$$\frac{\sin x}{x} > \frac{\sin(\arcsin x)}{\arcsin x} = \frac{x}{\arcsin x}$$

and b) follows.

For the proof of c) apply the same argument, by using $\dfrac{\pi}{2}x > \arcsin x$ (see right side of relation (20)).

For the proof of a) let us define the application

$$h(x) = \sin^2 x \cdot \operatorname{tg} x - x^3, \quad x \in (0, \pi/2).$$

Since

$$h'''(x) = (6 + 8\cos^2 x)\operatorname{tg}^4 x > 0$$

we get successively $h''(x) > 0$, $h'(x) > 0$ and finally $h(x) > h(0) = 0$. We omit the details.

Corollaries.

$$\frac{\sin x}{x} > \frac{x}{\arcsin x} > \frac{2}{\pi} \textit{ for } x \in (0,1); \tag{26}$$

$$\frac{x}{\arcsin x} > \frac{\sin \frac{\pi}{2}x}{\frac{\pi}{2}x} > \frac{2}{\pi} \textit{ for } x \in (0,1); \tag{27}$$

$$\frac{\sin x}{x} > \sqrt{\frac{x}{\operatorname{tg}\, x}} > \frac{\sqrt{\pi}}{2} > \frac{2}{\pi} \textit{ for } x \in \left(0, \frac{\pi}{4}\right). \tag{28}$$

Proof. (26) is a consequence of b) and (20), while (27) follows by c) and (1). For the proof of (28) apply a) combined with (17). The last inequality is also true, since $\pi^3 > 16$.

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