

Note on an Inequality of N. G. de Bruijn

J. Rubi3-Massegu3, J. L. D3az-Barrero and P. Rubi3-D3az

Abstract

In this note an inequality of N. G. de Bruijn is used to obtain inequalities involving finite sums of sequences of real and complex numbers.

Keywords: Inequalities in the complex plane, multivariate inequalities, finite sums.

1 INTRODUCTION

In 1960 N. G. de Bruijn ([1], [2]) established the following refinement of the classical Cauchy-Buniakowsky-Schwarz inequality:

Theorem A *If a_1, a_2, \dots, a_n is a sequence of real numbers and z_1, z_2, \dots, z_n is a sequence of complex numbers, then*

$$\left| \sum_{k=1}^n a_k z_k \right|^2 \leq \frac{1}{2} \sum_{k=1}^n a_k^2 \left[\sum_{k=1}^n |z_k|^2 + \left| \sum_{k=1}^n z_k^2 \right| \right] \quad (1)$$

Equality holds when there is $\alpha \in \mathbb{C}$ such that $\alpha^2 \sum_{k=1}^n z_k^2 \geq 0$ and $a_k = \operatorname{Re}(\alpha z_k)$ for all k .

In this paper we employ Theorem A to obtain some inequalities for sums of complex numbers similar to those presented in ([3], [4], [5] [6], [7]).

2 THE INEQUALITIES

We now establish the main result.

Theorem 1 *Let z_1, z_2, \dots, z_n , ($n \geq 2$) be a sequence of complex numbers. Then,*

$$\sum_{1 \leq i < j \leq n} |z_i + z_j|^2 \leq \left(\frac{3n-4}{2} \right) \sum_{k=1}^n |z_k|^2 + \frac{n}{2} \left| \sum_{k=1}^n z_k^2 \right|.$$

Equality holds when there is $\alpha \in \mathbb{C}$ such that $\alpha^2 \sum_{k=1}^n z_k^2 \geq 0$ and $a_k = \operatorname{Re}(\alpha z_k)$ for all k . In particular, the constants $(3n-4)/2$ and $n/2$ are the best possible.

Proof. We begin with an identity.

Lemma 1 *If $z_1, z_2, \dots, z_n, (n \geq 2)$ is a sequence of complex numbers, then*

$$\sum_{1 \leq i < j \leq n} |z_i + z_j|^2 = (n-2) \sum_{k=1}^n |z_k|^2 + \left| \sum_{k=1}^n z_k \right|^2.$$

Proof. The proof is by induction on n . For $n = 2$ the identity is trivial. Assume

$$\sum_{1 \leq i < j \leq n} |z_i + z_j|^2 = (n-2) \sum_{k=1}^n |z_k|^2 + \left| \sum_{k=1}^n z_k \right|^2.$$

Then,

$$\begin{aligned} \sum_{1 \leq i < j \leq n+1} |z_i + z_j|^2 &= (n-2) \sum_{k=1}^n |z_k|^2 + \left| \sum_{k=1}^n z_k \right|^2 + \sum_{k=1}^n |z_k + z_{n+1}|^2 \\ &= (n-1) \sum_{k=1}^{n+1} |z_k|^2 + |z_{n+1}|^2 + \sum_{k=1}^n (z_k \bar{z}_{n+1} + \bar{z}_k z_{n+1}) \\ &= (n-1) \sum_{k=1}^{n+1} |z_k|^2 + \left| \sum_{k=1}^{n+1} z_k \right|^2. \end{aligned}$$

The proof is complete. □

Setting $a_k = 1, (1 \leq k \leq n)$ into (1), we have

$$\left| \sum_{k=1}^n z_k \right|^2 \leq \frac{n}{2} \left(\sum_{k=1}^n |z_k|^2 + \left| \sum_{k=1}^n z_k^2 \right| \right).$$

On the other hand, from the previous lemma, we have

$$\left| \sum_{k=1}^n z_k \right|^2 = \sum_{1 \leq i < j \leq n} |z_i + z_j|^2 - (n-2) \sum_{k=1}^n |z_k|^2.$$

Substituting the preceding expression into the previous one, and rearranging terms, the statement immediately follows and the proof is complete. □

Lemma 2 Let $z_1, z_2, \dots, z_n, (n \geq 2)$ be complex numbers. Then,

$$\sum_{1 \leq i < j \leq n} |z_i - z_j|^2 = n \sum_{k=1}^n |z_k|^2 - \left| \sum_{k=1}^n z_k \right|^2.$$

Proof. The case $n = 2$ is easily checked. Assume

$$\sum_{1 \leq i < j \leq n} |z_i - z_j|^2 = n \sum_{k=1}^n |z_k|^2 - \left| \sum_{k=1}^n z_k \right|^2.$$

Then,

$$\begin{aligned} \sum_{1 \leq i < j \leq n+1} |z_i - z_j|^2 &= n \sum_{k=1}^n |z_k|^2 - \left| \sum_{k=1}^n z_k \right|^2 + \sum_{k=1}^n |z_k|^2 + n|z_{n+1}|^2 \\ &\quad - \sum_{k=1}^n (\bar{z}_k z_{n+1} + z_k \bar{z}_{n+1}) \\ &= (n+1) \sum_{k=1}^{n+1} |z_k|^2 - \left| \sum_{k=1}^n z_k \right|^2 - |z_{n+1}|^2 \\ &\quad - \bar{z}_{n+1} \left(\sum_{k=1}^n z_k \right) - z_{n+1} \left(\sum_{k=1}^n \bar{z}_k \right) \\ &= (n+1) \sum_{k=1}^{n+1} |z_k|^2 - \left| \sum_{k=1}^{n+1} z_k \right|^2. \end{aligned}$$

The statement is proved. □

Notice that the preceding lemma gives an explicit expression for number $R = n \sum_{k=1}^n |z_k|^2 - \left| \sum_{k=1}^n z_k \right|^2$, which is always positive because of the Cauchy-Buniakowsky-Schwarz inequality, in terms of a finite sum of positive numbers.

Theorem 2 Let $z_1, z_2, \dots, z_n, (n \geq 2)$ be complex numbers. Then,

$$\sum_{1 \leq i < j \leq n} |z_i - z_j|^2 \geq \frac{n}{2} \left(\sum_{k=1}^n |z_k|^2 - \left| \sum_{k=1}^n z_k^2 \right| \right).$$

Equality holds when there is $\alpha \in \mathbb{C}$ such that $\alpha^2 \sum_{k=1}^n z_k^2 \geq 0$ and $a_k = \operatorname{Re}(\alpha z_k)$ for all k .

Proof. Adding up the identities in Lemma 1 and Lemma 2, we get

$$\sum_{1 \leq i < j \leq n} |z_i + z_j|^2 + \sum_{1 \leq i < j \leq n} |z_i - z_j|^2 = 2(n-1) \sum_{k=1}^n |z_k|^2.$$

Now, applying Theorem 1, we have

$$2(n-1) \sum_{k=1}^n |z_k|^2 - \sum_{1 \leq i < j \leq n} |z_i - z_j|^2 \leq \left(\frac{3n-4}{2} \right) \sum_{k=1}^n |z_k|^2 + \frac{n}{2} \left| \sum_{k=1}^n z_k^2 \right|$$

or equivalently

$$\frac{n}{2} \sum_{k=1}^n |z_k|^2 \leq \sum_{1 \leq i < j \leq n} |z_i - z_j|^2 + \frac{n}{2} \left| \sum_{k=1}^n z_k^2 \right|$$

from which the statement immediately follows and the proof is complete. \square

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Applied Mathematics III. Universitat Politècnica de Catalunya. Jordi Girona 1-3, C2, 08034 Barcelona. Spain. jose.luis.diaz@upc.edu