

COMPLETE MONOTONICITY OF LOGARITHMIC MEAN

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ABSTRACT. In the article, the logarithmic mean is proved to be completely monotonic and an open problem about the logarithmically complete monotonicity of the extended mean values is posed.

1. INTRODUCTION

Recall [11, 28] that a function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I and $(-1)^n f^{(n)}(x) \geq 0$ for $x \in I$ and $n \geq 0$. Recall [2] that if $f^{(k)}(x)$ for some nonnegative integer k is completely monotonic on an interval I , but $f^{(k-1)}(x)$ is not completely monotonic on I , then $f(x)$ is called a completely monotonic function of k -th order on an interval I . Recall also [17, 18, 20] that a function f is said to be logarithmically completely monotonic on an interval I if its logarithm $\ln f$ satisfies $(-1)^k [\ln f(x)]^{(k)} \geq 0$ for $k \in \mathbb{N}$ on I . It has been proved in [3, 10, 17, 18] and other references that a logarithmically completely monotonic function on an interval I is also completely monotonic on I . The logarithmically completely monotonic functions have close relationships with both the completely monotonic functions and Stieltjes transforms. For detailed information, please refer to [3, 10, 11, 21, 28] and the references therein.

For two positive numbers a and b , the logarithmic mean $L(a, b)$ is defined by

$$L(a, b) = \begin{cases} \frac{b-a}{\ln b - \ln a}, & a \neq b; \\ a, & a = b. \end{cases} \quad (1)$$

This is one of the most important means of two positive variables. See [4, 6, 12, 16] and the list of references therein. It is cited on 13 pages at least in [4], see [4, p. 532]. However, any complete monotonicity on mean values is not founded in the authoritative book [4].

The main aim of this paper is to prove the complete monotonicity of the logarithmic mean L .

Our main result is as follows.

Theorem 1. *The logarithmic mean $L_{s,t}(x) = L(x+s, x+t)$ is a completely monotonic function of first order in $x > -\min\{s, t\}$ for $s, t \in \mathbb{R}$ with $s \neq t$.*

As by-product of the proof of Theorem 1, the following logarithmically completely monotonic property of the function $(x+s)^{1-u}(x+t)^u$ for $s, t \in \mathbb{R}$ with $s \neq t$ and $u \in (0, 1)$ is deduced.

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Corollary 1. *Let $s, t \in \mathbb{R}$ with $s \neq t$ and $u \in (0, 1)$. Then $(x + s)^{1-u}(x + t)^u$ is a completely monotonic function of first order in $x > -\min\{s, t\}$. More strongly, the function $\frac{\partial[(x+s)^{1-u}(x+t)^u]}{\partial x} = \left(\frac{x+t}{x+s}\right)^u \left[1 + \frac{u(s-t)}{x+t}\right]$ is logarithmically completely monotonic in $x > -\min\{s, t\}$.*

The extended mean values $E(r, s; x, y)$ can be defined by

$$\begin{aligned} E(r, s; x, y) &= \left[\frac{r}{s} \cdot \frac{y^s - x^s}{y^r - x^r} \right]^{1/(s-r)}, & rs(r-s)(x-y) \neq 0; \\ E(r, 0; x, y) &= \left[\frac{1}{r} \cdot \frac{y^r - x^r}{\ln y - \ln x} \right]^{1/r}, & r(x-y) \neq 0; \\ E(r, r; x, y) &= \frac{1}{e^{1/r}} \left[\frac{x^{x^r}}{y^{y^r}} \right]^{1/(x^r - y^r)}, & r(x-y) \neq 0; \\ E(0, 0; x, y) &= \sqrt{xy}, & x \neq y; \\ E(r, s; x, x) &= x, & x = y; \end{aligned}$$

where x and y are positive numbers and $r, s \in \mathbb{R}$. Its monotonicity, Schur-convexity, logarithmic convexity, comparison, generalizations, applications and history have been investigated in many articles such as [4, 5, 6, 7, 8, 9, 12, 13, 14, 15, 16, 19, 22, 23, 24, 25, 26, 27, 29, 30, 31] and the references therein, especially the book [4] and the expository paper [16].

For $x, y > 0$ and $r, s \in \mathbb{R}$, let $E_{r,s;x,y}^{[1]}(w) = E(r+w, s+w; x, y)$ with $w \in \mathbb{R}$, $E_{r,s;x,y}^{[2]}(w) = E(r, s; x+w, y+w)$ and $E_{r,s;x,y}^{[3]}(w) = E(r+w, s+w; x+w, y+w)$ with $w > -\min\{x, y\}$. Motivated by Theorem 1, it is natural to pose an open problem: What about the (logarithmically) complete monotonicity of the functions $E_{r,s;x,y}^{[i]}(w)$ in w for $1 \leq i \leq 3$?

2. PROOFS OF THEOREM 1 AND COROLLARY 1

Proof of Theorem 1. In [4, p. 386], an integral representation of the logarithmic mean $L(a, b)$ for positive numbers a and b is given:

$$L(a, b) = \int_0^1 a^{1-u} b^u \, du. \quad (2)$$

From this, it follows easily that

$$L_{s,t}(x) = \int_0^1 (x+s)^{1-u} (x+t)^u \, du \quad (3)$$

and

$$\frac{dL_{s,t}(x)}{dx} = \int_0^1 \left(\frac{x+t}{x+s} \right)^u \frac{x + (1-u)t + us}{x+t} \, du > 0. \quad (4)$$

This means that the function $L_{s,t}(x)$ is increasing, and then it is not completely monotonic in $x > -\min\{s, t\}$.

In [1, p. 230, 5.1.32], it is listed that

$$\ln \frac{b}{a} = \int_0^\infty \frac{e^{-au} - e^{-bu}}{u} \, du. \quad (5)$$

Taking logarithm on both sides of equation (4) and utilizing (5) yields

$$\begin{aligned} & \ln \frac{\partial[(x+s)^{1-u}(x+t)^u]}{\partial x} = u \ln \frac{x+t}{x+s} + \ln \frac{x+(1-u)t+us}{x+t} \\ & = u \int_0^\infty \frac{e^{-(x+s)v} - e^{-(x+t)v}}{v} dv + \int_0^\infty \frac{e^{-(x+t)v} - e^{-[x+(1-u)t+us]v}}{v} dv \\ & = \int_0^\infty \frac{ue^{-(x+s)v} + (1-u)e^{-(x+t)v} - e^{-[x+(1-u)t+us]v}}{v} dv. \end{aligned}$$

Employing the well known Jensen's inequality [4, p. 31, Theorem 12] for convex functions and considering that the function e^{-x} is convex gives

$$q_{s,t;u;v}(x) \triangleq ue^{-(x+s)v} + (1-u)e^{-(x+t)v} - e^{-[x+(1-u)t+us]v} > 0. \quad (6)$$

Hence, for positive integer $m \in \mathbb{N}$,

$$(-1)^m \frac{\partial^m}{\partial x^m} \left\{ \ln \frac{\partial[(x+s)^{1-u}(x+t)^u]}{\partial x} \right\} = \int_0^\infty v^{m-1} q_{s,t;u;v}(x) dv > 0. \quad (7)$$

This implies that the function $\frac{\partial[(x+s)^{1-u}(x+t)^u]}{\partial x}$ is logarithmically completely monotonic in $x > -\min\{s, t\}$. Further, since a logarithmically completely monotonic function is also completely monotonic (see [3, 10, 11, 17, 18, 20, 21] and the references therein), the function $\frac{\partial[(x+s)^{1-u}(x+t)^u]}{\partial x}$ is completely monotonic in $x > -\min\{s, t\}$. Therefore, the function

$$\frac{dL_{s,t}(x)}{dx} = \int_0^1 \frac{\partial[(x+s)^{1-u}(x+t)^u]}{\partial x} du \quad (8)$$

is completely monotonic in $x > -\min\{s, t\}$. Theorem 1 is proved. \square

Proof of Corollary 1. This follows from the proof of Theorem 1 directly. \square

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