# NOTE ON A CLASS OF COMPLETELY MONOTONIC FUNCTIONS INVOLVING THE POLYGAMMA FUNCTIONS

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ABSTRACT. In this article, some monotonicity of the function $x^{\alpha}\left|\psi^{(i)}(x+\beta)\right|$ and the complete monotonicity of the functions $\frac{\alpha}{x}\left|\psi^{(i)}(x+\beta)\right|-\left|\psi^{(i+1)}(x+\beta)\right|$ and $\alpha\left|\psi^{(i)}(x+\beta)\right|-x\left|\psi^{(i+1)}(x+\beta)\right|$ in $(0,\infty)$ for $i\in\mathbb{N}$, $\alpha>0$ and $\beta\geq 0$ are investigated, where $\psi^{(i)}(x)$ is the well known polygamma functions. Moreover, lower and upper bounds for infinite series whose coefficients involves the Bernoulli numbers are established.

## 1. Introduction

Recall [7, 11, 14] that a function $f$ is called completely monotonic on an interval $I$ if $f$ has derivatives of all orders on $I$ and $0\leq(-1)^k f^{(k)}(x)<\infty$ for all $k\geq 0$ on $I$. The well known Bernstein's Theorem [14, p. 161] states that $f\in\mathcal{C}[(0,\infty)]$ if and only if $f(x)=\int_0^\infty e^{-xs}\,\mathrm{d}\,\mu(s)$, where $\mu$ is a nonnegative measure on $[0,\infty)$ such that the integral converges for all $x>0$. The class of completely monotonic functions on $I$ is denoted by $\mathcal{C}[I]$. For more information on $\mathcal{C}[I]$, please refer to [5, 6, 7, 8, 9, 10, 11, 14] and the references therein.

By using the convolution theorem of Laplace transforms, the increasingly monotonicity of $x^{\alpha}\left|\psi^{(i)}(x+1)\right|$ is presented in [9, 10]: The function $x^{\alpha}\left|\psi^{(i)}(x+1)\right|$ is strictly increasing in $(0,\infty)$ if and only if $\alpha\geq i$, where $\psi(x)$, the logarithmic derivative of the classical Euler's gamma function $\Gamma(x)$, is called psi function and $\psi^{(i)}(x)$ for $i\in\mathbb{N}$ are called polygamma functions. In [3], in order to show the subadditive property of the function $\psi^{(i)}(a+e^x)$, it was proved that the function $x\psi'(x+a)$ is strictly increasing on $[0,\infty)$ for $a\geq 1$. In [2], it was also showed, using the convolution theorem of Laplace transforms, that the function $x^c\left|\psi^{(k)}(x)\right|$ for $k\geq 1$ is strictly decreasing in $(0,\infty)$ if and only if $c\leq k$ and is strictly increasing in $(0,\infty)$ if and only if $c\geq k+1$. In [4], the monotonicity of the more general function $x^{\alpha}\left|\psi^{(i)}(x+\beta)\right|$ was studied without using the convolution theorem of Laplace transforms and, except the above results, the following conclusions are obtained: For $i\in\mathbb{N}$, $\alpha>0$ and $\beta\geq 0$,

(1) the function $x^{\alpha}\left|\psi^{(i)}(x+\beta)\right|$ is strictly increasing in $(0,\infty)$ if $(\alpha,\beta)\in\left\{\alpha\geq i,\frac12\leq\beta<1\right\}\cup\left\{\alpha\geq i,\beta\geq\frac{\alpha-i+1}{2}\right\}\cup\left\{\alpha\geq i+1,\beta\leq\frac{\alpha-i+1}{2}\right\}$ and only if $\alpha\geq i$;
(2) $\frac{\alpha}{x}\left|\psi^{(i)}(x)\right|-\left|\psi^{(i+1)}(x)\right|\in\mathcal{C}[(0,\infty)]$ if and only if $\alpha\geq i+1$;
(3) $\left|\psi^{(i+1)}(x)\right|-\frac{\alpha}{x}\left|\psi^{(i)}(x)\right|\in\mathcal{C}[(0,\infty)]$ if and only if $0<\alpha\leq i$;

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(4) $\frac{\alpha}{x}\left|\psi^{(i)}(x+1)\right|-\left|\psi^{(i+1)}(x+1)\right|\in\mathcal{C}[(0,\infty)]$ if and only if $\alpha\ge i$;
(5) $\frac{\alpha}{x}\left|\psi^{(i)}(x+\beta)\right|-\left|\psi^{(i+1)}(x+\beta)\right|\in\mathcal{C}[(0,\infty)]$ if $(\alpha,\beta)\in\left\{\alpha\ge i+1,\beta\le\frac{\alpha-i+1}{2}\right\}\cup\left\{i\le\alpha\le\frac{(i+1)(i+4\beta-2)}{i+2\beta},\frac12\le\beta<1\right\}\cup\left\{i\le\alpha\le i+1,\beta\ge\frac{\alpha-i+1}{2}\right\}$ and only if $\alpha\ge i$;
(6) $\alpha\left|\psi^{(i)}(x+\beta)\right|-x\left|\psi^{(i+1)}(x+\beta)\right|\in\mathcal{C}[(0,\infty)]$ if $(\alpha,\beta)\in\left\{i\le\alpha\le i+1,\beta\ge\frac{\alpha-i+1}{2}\right\}\cup\left\{\alpha\ge i+1,\beta\le\frac{\alpha-i+1}{2}\right\}$ and only if $\alpha\ge i$.

The main purpose of this paper is to research further the monotonic properties of the function $x^\alpha\left|\psi^{(i)}(x+\beta)\right|$ and to obtain some more better conclusions than those mentioned above.

Our main results are the following four theorems.

**Theorem 1.** *For $i\in\mathbb{N}$, $\alpha\ge0$ and $\beta\ge0$.*

(1) *The function $x^\alpha\left|\psi^{(i)}(x)\right|$ in $(0,\infty)$ is strictly increasing if and only if $\alpha\ge i+1$ and strictly decreasing if and only if $0\le\alpha\le i$.*
(2) *For $\beta\ge\frac12$, the function $x^\alpha\left|\psi^{(i)}(x+\beta)\right|$ is strictly increasing in $[0,\infty)$ if and only if $\alpha\ge i$.*
(3) *Let $\delta:(0,\infty)\to\left(0,\frac12\right)$ be defined by*

$$\delta(t)=\frac{e^t(t-1)+1}{(e^t-1)^2}\tag{1}$$

*for $t\in(0,\infty)$ and $\delta^{-1}:\left(0,\frac12\right)\to(0,\infty)$ stand for the inverse function of $\delta$. If $0<\beta<\frac12$ and*

$$\alpha\ge i+1-\left[\frac{e^{\delta^{-1}(\beta)}}{e^{\delta^{-1}(\beta)}-1}+\beta-1\right]\delta^{-1}(\beta),\tag{2}$$

*then the function $x^\alpha\left|\psi^{(i)}(x+\beta)\right|$ is strictly increasing in $(0,\infty)$.*

*Remark* 1. It is noted that

$$0<\left[\frac{e^{\delta^{-1}(\beta)}}{e^{\delta^{-1}(\beta)}-1}+\beta-1\right]\delta^{-1}(\beta)<1$$

for $\beta\in(0,1)$, since $\lim_{\beta\to0+}[\beta\delta^{-1}(\beta)]=0$.

**Theorem 2.** *Let $i\in\mathbb{N}$, $\alpha\ge0$ and $\beta\ge0$.*

(1) $\alpha\left|\psi^{(i)}(x)\right|-x\left|\psi^{(i+1)}(x)\right|\in\mathcal{C}[(0,\infty)]$ *if and only if $\alpha\ge i+1$.*
(2) $x\left|\psi^{(i+1)}(x)\right|-\alpha\left|\psi^{(i)}(x)\right|\in\mathcal{C}[(0,\infty)]$ *if and only if $0\le\alpha\le i$.*
(3) *If $\beta\ge\frac12$, then $\alpha\left|\psi^{(i)}(x+\beta)\right|-x\left|\psi^{(i+1)}(x+\beta)\right|\in\mathcal{C}[(0,\infty)]$ if and only if $\alpha\ge i$.*
(4) *If $0<\beta<\frac12$ and inequality (2) holds true, then $\alpha\left|\psi^{(i)}(x+\beta)\right|-x\left|\psi^{(i+1)}(x+\beta)\right|\in\mathcal{C}[(0,\infty)]$.*

**Theorem 3.** *Let $i\in\mathbb{N}$, $\alpha\ge0$ and $\beta\ge0$.*

(1) $\frac{\alpha}{x}\left|\psi^{(i)}(x)\right|-\left|\psi^{(i+1)}(x)\right|\in\mathcal{C}[(0,\infty)]$ *if and only if $\alpha\ge i+1$.*
(2) $\left|\psi^{(i+1)}(x)\right|-\frac{\alpha}{x}\left|\psi^{(i)}(x)\right|\in\mathcal{C}[(0,\infty)]$ *if and only if $0\le\alpha\le i$.*
(3) *If $\beta\ge\frac12$, then $\frac{\alpha}{x}\left|\psi^{(i)}(x+\beta)\right|-\left|\psi^{(i+1)}(x+\beta)\right|\in\mathcal{C}[(0,\infty)]$ if and only if $\alpha\ge i$.*
(4) *If $0<\beta<\frac12$ and inequality (2) holds true, then $\frac{\alpha}{x}\left|\psi^{(i)}(x+\beta)\right|-\left|\psi^{(i+1)}(x+\beta)\right|\in\mathcal{C}[(0,\infty)]$.*

**Theorem 4.** *Let $0 < \beta < \frac{1}{2}$ and $\delta^{-1}$ be the inverse function of $\delta$ defined by* (1)*. Then the following inequalities holds for $t \in (0, \infty)$:*

$$\frac{1}{2} > \sum_{k=1}^{\infty} B_{2k} \frac{t^{2k-1}}{(2k-1)!} > 0, \tag{3}$$

$$\frac{t}{2} > \sum_{k=0}^{\infty} B_{2k+2} \frac{t^{2k+2}}{(2k+2)!} > \max\left\{0, \frac{t}{2} - 1\right\}, \tag{4}$$

$$\sum_{k=0}^{\infty} B_{2k+2} \frac{t^{2k+2}}{(2k+2)!} > \left(\frac{1}{2} - \beta\right)t + \left[\frac{e^{\delta^{-1}(\beta)}}{e^{\delta^{-1}(\beta)} - 1} - \beta + 1\right]\delta^{-1}(\beta) - 1, \tag{5}$$

*where $B_k$ stands for the Bernoulli numbers defined by*

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} \frac{B_k x^k}{k!}. \tag{6}$$

## 2. Lemmas

In order to prove our main results, the following lemmas are necessary.

**Lemma 1** ([1, 12, 13])**.** *The polygamma functions $\psi^{(k)}(x)$ are expressed for $x > 0$ and $k \in \mathbb{N}$ as*

$$\psi^{(k)}(x) = (-1)^{k+1} \int_0^{\infty} \frac{t^k e^{-xt}}{1 - e^{-t}} \,\mathrm{d}t. \tag{7}$$

*For $x > 0$ and $r > 0$,*

$$\frac{1}{x^r} = \frac{1}{\Gamma(r)} \int_0^{\infty} t^{r-1} e^{-xt} \,\mathrm{d}t. \tag{8}$$

*For $i \in \mathbb{N}$ and $x > 0$,*

$$\psi^{(i-1)}(x+1) = \psi^{(i-1)}(x) + \frac{(-1)^{i-1}(i-1)!}{x^i}. \tag{9}$$

**Lemma 2** ([5, 6])**.** *Let $f(x)$ be defined in an infinite interval $I$. If $\lim_{x\to\infty} f(x) = 0$ and $f(x) - f(x+\varepsilon) \gtreqless 0$ for any given $\varepsilon > 0$, then $f(x) \gtreqless 0$ in $I$.*

## 3. Proofs of theorems

*Proof of Theorem 1.* Direct calculation and rearrangement yields

$$\begin{aligned}\frac{g'_{i,\alpha,\beta}(x)}{x^{\alpha-1}} &= \alpha\left|\psi^{(i)}(x+\beta)\right| - x\left|\psi^{(i+1)}(x+\beta)\right| \\ &= (-1)^{i+1}\left[\alpha\psi^{(i)}(x+\beta) + x\psi^{(i+1)}(x+\beta)\right]\end{aligned} \tag{10}$$

and

$$\lim_{x\to\infty} \frac{g'_{i,\alpha,\beta}(x)}{x^{\alpha-1}} = 0. \tag{11}$$

Straightforwardly computing in virtue of formulas (9), (8) and (7) gives

$$
\begin{aligned}
\frac{g'_{i,\alpha,\beta}(x)}{x^{\alpha-1}} - \frac{g'_{i,\alpha,\beta}(x+1)}{(x+1)^{\alpha-1}}
&= (-1)^{i+1}\big\{\alpha\big[\psi^{(i)}(x+\beta) - \psi^{(i)}(x+\beta+1)\big] \\
&\quad + x\big[\psi^{(i+1)}(x+\beta) - \psi^{(i+1)}(x+\beta+1)\big] - \psi^{(i+1)}(x+\beta+1)\big\} \\
&= \frac{i!\alpha}{(x+\beta)^{i+1}} - \frac{(i+1)!x}{(x+\beta)^{i+2}} - \frac{(i+1)!}{(x+\beta)^{i+2}} + (-1)^{i+2}\psi^{(i+1)}(x+\beta) \\
&= (-1)^{i+2}\psi^{(i+1)}(x+\beta) + \frac{i!(\alpha-i-1)}{(x+\beta)^{i+1}} + \frac{(i+1)!(\beta-1)}{(x+\beta)^{i+2}} \qquad (12) \\
&= \int_0^\infty \left[\frac{t}{1-e^{-t}} + (\beta-1)t + \alpha - i - 1\right] t^i e^{-(x+\beta)t} \,\mathrm{d}t \\
&\triangleq \int_0^\infty h_{i,\alpha,\beta}(t) t^i e^{-(x+\beta)t} \,\mathrm{d}t.
\end{aligned}
$$

If $\beta = 0$, the function $h'_{i,\alpha,0}(t) = -\frac{1+(t-1)e^t}{(e^t-1)^2} < 0$ and $h_{i,\alpha,0}(t)$ is decreasing in $(0,\infty)$ with $\lim_{t\to0+} h_{i,\alpha,0}(t) = \alpha - i$ and $\lim_{t\to\infty} h_{i,\alpha,0}(t) = \alpha - i - 1$. For $\alpha \ge i+1$, the functions $h_{i,\alpha,0}(t)$ and $\frac{g'_{i,\alpha,0}(x)}{x^{\alpha-1}} - \frac{g'_{i,\alpha,0}(x+1)}{(x+1)^{\alpha-1}}$ are positive in $(0,\infty)$. Combining this with (11) and considering Lemma 2, it is obtained that the functions $\frac{g'_{i,\alpha,0}(x)}{x^{\alpha-1}}$ and $g'_{i,\alpha,0}(x)$ are positive in $(0,\infty)$, which means that the function $g_{i,\alpha,0}(x)$ is strictly increasing in $(0,\infty)$ for $\alpha \ge i+1$. Similarly, for $\alpha \le i$, the function $g_{i,\alpha,0}(x)$ is strictly decreasing in $(0,\infty)$.

If $\beta > 0$, then the function $h'_{i,\alpha,\beta}(t) = \frac{e^t(e^t-t-1)}{(e^t-1)^2} + \beta - 1 \triangleq \lambda(t) + \beta - 1$ with $\lambda'(t) = \frac{e^t[e^t(t-2)+t+2]}{(e^t-1)^3} \triangleq \frac{\lambda_1(t)}{(e^t-1)^3}$ and $\lambda_1'(t) = 1 + (t-1)e^t > 0$ in $(0,\infty)$, and the function $\lambda_1(t)$ is increasing with $\lambda_1(0) = 0$, thus $\lambda_1(t) > 0$ and $\lambda'(t) > 0$. Hence, the functions $\lambda(t)$ and $h'_{i,\alpha,\beta}(t)$ are strictly increasing in $(0,\infty)$ with $\lim_{t\to0+} h'_{i,\alpha,\beta}(t) = \beta - \frac12$ and $\lim_{t\to\infty} h'_{i,\alpha,\beta}(t) = \beta$. Thus, if $\beta \ge \frac12$, the function $h'_{i,\alpha,\beta}(t)$ is positive and the function $h_{i,\alpha,\beta}(t)$ is strictly increasing in $(0,\infty)$ with $\lim_{t\to0+} h_{i,\alpha,\beta}(t) = \alpha - i$ and $\lim_{t\to\infty} h_{i,\alpha,\beta}(t) = \infty$. Accordingly, for $\alpha \ge i$ and $\beta \ge \frac12$, the function $h_{i,\alpha,\beta}(t) > 0$ in $(0,\infty)$. Therefore, for $\alpha \ge i$ and $\beta \ge \frac12$, by the same argument as above, it is deduced that the function $g_{i,\alpha,\beta}(x)$ is strictly increasing in $(0,\infty)$.

If $0 < \beta < \frac12$, since the function $h'_{i,\alpha,\beta}(t)$ is strictly increasing in $(0,\infty)$ with $\lim_{t\to0+} h'_{i,\alpha,\beta}(t) = \beta - \frac12 < 0$ and $\lim_{t\to\infty} h'_{i,\alpha,\beta}(t) = \beta > 0$, then the function $h_{i,\alpha,\beta}(t)$ attains its unique minimum at some point $t_0 \in (0,\infty)$. It is easy to see that the function $\delta(t)$ defined by (1) satisfies $\delta(t_0) = \beta$ for $0 < \beta < \frac12$, equals $-[\lambda(t)+1]$ and is positive and strictly decreasing with $\lim_{t\to0+} \delta(t) = \frac12$ and $\lim_{t\to\infty} \delta(t) = 0$. Therefore, the unique minimum of $h_{i,\alpha,\beta}(t)$ equals

$$
\frac{\delta^{-1}(\beta)e^{\delta^{-1}(\beta)}}{e^{\delta^{-1}(\beta)} - 1} + (\beta-1)\delta^{-1}(\beta) + \alpha - i - 1,
$$

where $\delta^{-1}$ is the inverse function of $\delta$ defined by (1) and is strictly decreasing in $(0,\frac12)$ with $\lim_{s\to0+} \delta^{-1}(s) = \infty$ and $\lim_{s\to\frac12-} \delta^{-1}(s) = 0$. As a result, while inequality (2) holds for $0 < \beta < \frac12$, the function $h_{i,\alpha,\beta}(t)$ is positive in $(0,\infty)$. Consequently, if $0 < \beta < \frac12$ and inequality (2) is valid, then the function $g_{i,\alpha,\beta}(x)$ is strictly increasing in $(0,\infty)$. The sufficiency is proved.

Now we are in a position to prove the necessity. In [8], it was proved that $\psi(x)-\ln x+\frac{\alpha}{x}\in\mathcal{C}[(0,\infty)]$ if and only if $\alpha\ge1$ and $\ln x-\frac{\alpha}{x}-\psi(x)\in\mathcal{C}[(0,\infty)]$ if and only if $\alpha\le\frac12$. From this it is deduced that inequality

$$\frac{(k-1)!}{x^k}+\frac{k!}{2x^{k+1}}<(-1)^{k+1}\psi^{(k)}(x)=\bigl|\psi^{(k)}(x)\bigr|<\frac{(k-1)!}{x^k}+\frac{k!}{x^{k+1}} \tag{13}$$

holds in $(0,\infty)$ for $k\in\mathbb{N}$.

If $g_{i,\alpha,0}(x)$ is strictly decreasing in $(0,\infty)$, then

$$x^{i+1-\alpha}g'_{i,\alpha,0}(x)=\alpha x^i\bigl|\psi^{(i)}(x)\bigr|-x^{i+1}\bigl|\psi^{(i+1)}(x)\bigr|<0. \tag{14}$$

Applying (13) into (14) leads to

$$\begin{aligned}
0&\ge\lim_{x\to\infty}x^{i+1-\alpha}g'_{i,\alpha,0}(x)\\
&\ge\alpha\lim_{x\to\infty}x^i\left[\frac{(i-1)!}{x^i}+\frac{i!}{2x^{i+1}}\right]-\lim_{x\to\infty}x^{i+1}\left[\frac{i!}{x^{i+1}}+\frac{(i+1)!}{x^{i+2}}\right]\\
&=(i-1)!(\alpha-i),
\end{aligned}$$

which means $\alpha\le i$.

If $g_{i,\alpha,0}(x)$ is strictly increasing in $(0,\infty)$, then

$$x^{i+2-\alpha}g'_{i,\alpha,0}(x)=\alpha x^{i+1}\bigl|\psi^{(i)}(x)\bigr|-x^{i+2}\bigl|\psi^{(i+1)}(x)\bigr|>0 \tag{15}$$

and, applying (9) into (15) and using (13),

$$\begin{aligned}
0&\le\lim_{x\to0+}x^{i+2-\alpha}g'_{i,\alpha,0}(x)\\
&=\lim_{x\to0+}\left\{\alpha x^{i+1}\bigl|\psi^{(i)}(x)\bigr|-x^{i+2}\left[\bigl|\psi^{(i+1)}(x+1)\bigr|+\frac{(i+1)!}{x^{i+2}}\right]\right\}\\
&=\alpha\lim_{x\to0+}x^{i+1}\bigl|\psi^{(i)}(x)\bigr|-(i+1)!-\lim_{x\to0+}x^{i+2}\bigl|\psi^{(i+1)}(x+1)\bigr|\\
&\le\alpha\lim_{x\to0+}x^{i+1}\left[\frac{(i-1)!}{x^i}+\frac{i!}{x^{i+1}}\right]-(i+1)!\\
&\quad-\lim_{x\to0+}x^{i+2}\left[\frac{i!}{(x+1)^{i+1}}+\frac{(i+1)!}{2(x+1)^{i+2}}\right]\\
&=i!(\alpha-i-1),
\end{aligned}$$

which means $\alpha\ge i+1$.

If the function $g_{i,\alpha,\beta}(x)$ is strictly increasing in $(0,\infty)$ for $\beta>0$, then

$$x^{i+1-\alpha}g'_{i,\alpha,\beta}(x)=\alpha x^i\bigl|\psi^{(i)}(x+\beta)\bigr|-x^{i+1}\bigl|\psi^{(i+1)}(x+\beta)\bigr|>0. \tag{16}$$

Applying (13) in (16) and taking limit leads to

$$\begin{aligned}
0&\le\lim_{x\to\infty}x^{i+1-\alpha}g'_{i,\alpha,\beta}(x)\\
&\le\alpha\lim_{x\to\infty}x^i\left[\frac{(i-1)!}{(x+\beta)^i}+\frac{i!}{(x+\beta)^{i+1}}\right]\\
&\quad-\lim_{x\to\infty}x^{i+1}\left[\frac{i!}{(x+\beta)^{i+1}}+\frac{(i+1)!}{2(x+\beta)^{i+2}}\right]\\
&=(i-1)!(\alpha-i),
\end{aligned}$$

which means $\alpha\ge i$. The proof of Theorem 1 is complete. □

*Proof of Theorem 2.* If $h_{i,\alpha,\beta}(t) \gtrless 0$ in $(0,\infty)$, then $\pm\int_0^\infty h_{i,\alpha,\beta}(t)t^ie^{-(x+\beta)t}\,\mathrm{d}\,t \in \mathcal{C}[(-\beta,\infty)]$, which is equivalent to $\pm\Bigl[\frac{g'_{i,\alpha,\beta}(x)}{x^{\alpha-1}} - \frac{g'_{i,\alpha,\beta}(x+1)}{(x+1)^{\alpha-1}}\Bigr] \in \mathcal{C}[(0,\infty)]$ by (12), and then, by definition,

$$(-1)^j\left[\frac{g'_{i,\alpha,\beta}(x)}{x^{\alpha-1}} - \frac{g'_{i,\alpha,\beta}(x+1)}{(x+1)^{\alpha-1}}\right]^{(j)} = (-1)^j\left[\frac{g'_{i,\alpha,\beta}(x)}{x^{\alpha-1}}\right]^{(j)} - (-1)^j\left[\frac{g'_{i,\alpha,\beta}(x+1)}{(x+1)^{\alpha-1}}\right]^{(j)} \gtreqless 0$$

in $(0,\infty)$ for $j \ge 0$. Further, formulas (7) and (10) imply

$$\lim_{x\to\infty}\left[\frac{g'_{i,\alpha,\beta}(x)}{x^{\alpha-1}}\right]^{(j)} = \lim_{x\to\infty}(-1)^j\left[\frac{g'_{i,\alpha,\beta}(x)}{x^{\alpha-1}}\right]^{(j)} = 0. \tag{17}$$

By (17) and Lemma 2, it is concluded that $(-1)^j\Bigl[\frac{g'_{i,\alpha,\beta}(x)}{x^{\alpha-1}}\Bigr]^{(j)} \gtreqless 0$ and

$$\pm\frac{g'_{i,\alpha,\beta}(x)}{x^{\alpha-1}} = \pm\bigl[\alpha\bigl|\psi^{(i)}(x+\beta)\bigr| - x\bigl|\psi^{(i+1)}(x+\beta)\bigr|\bigr] \in \mathcal{C}[(0,\infty)]$$

if $h_{i,\alpha,\beta}(t) \gtrless 0$ in $(0,\infty)$. The proof of Theorem 1 tells us that the function $h_{i,\alpha,\beta}(t)$ is positive in $(0,\infty)$ if either $\beta = 0$ and $\alpha \ge i+1$, or $\beta \ge \frac12$ and $\alpha \ge i$, or $0 < \beta < \frac12$ and inequality (2) validating, and that $h_{i,\alpha,\beta}(t)$ is negative in $(0,\infty)$ if $\beta = 0$ and $\alpha \le i$. As a result, the function $\alpha\bigl|\psi^{(i)}(x+\beta)\bigr| - x\bigl|\psi^{(i+1)}(x+\beta)\bigr|$ is completely monotonic in $(0,\infty)$ for either $\beta = 0$ and $\alpha \ge i+1$, or $\beta \ge \frac12$ and $\alpha \ge i$, or $0 < \beta < \frac12$ and inequality (2) being true, and $x\bigl|\psi^{(i+1)}(x+\beta)\bigr| - \alpha\bigl|\psi^{(i)}(x+\beta)\bigr| \in \mathcal{C}[(0,\infty)]$ for $\beta = 0$ and $\alpha \le i$.

The proofs of necessities are the same as those in Theorem 1. The proof of Theorem 2 is complete. □

*Proof of Theorem 3.* This follows from Theorem 2 and the following facts that

$$\pm\left[\frac{\alpha}{x}\bigl|\psi^{(i)}(x+\beta)\bigr| - \bigl|\psi^{(i+1)}(x+\beta)\bigr|\right] = \pm\frac{1}{x}\bigl\{\alpha\bigl|\psi^{(i)}(x+\beta)\bigr| - x\bigl|\psi^{(i+1)}(x+\beta)\bigr|\bigr\},$$

$\frac1x \in \mathcal{C}[(0,\infty)]$, and that the product of two completely monotonic functions is also completely monotonic on the union of their domains. □

*Proof of Theorem 4.* Let $B_k(x)$ be the Bernoulli polynomials defined [1, 12, 13] by

$$\frac{te^{xt}}{e^t-1} = \sum_{k=0}^\infty B_k(x)\frac{t^k}{k!}. \tag{18}$$

It is well known that the Bernoulli numbers $B_k$ and $B_k(x)$ are connected by $B_k(1) = (-1)^kB_k(0) = (-1)^kB_k$ and $B_{2k+1}(0) = B_{2k+1} = 0$ for $k \ge 1$, and that the first few Bernoulli numbers and polynomials are

$$B_0 = 1, \quad B_1 = -\frac12, \quad B_2 = \frac16, \quad B_4 = -\frac1{30},$$

$$B_0(x) = 1, \quad B_1(x) = x - \frac12, \quad B_2(x) = x^2 - x + \frac16, \quad B_3(x) = x^3 - \frac32x^2 + \frac12x.$$

Using these notations, the functions $h_{i,\alpha,\beta}(t)$ and $h'_{i,\alpha,\beta}(t)$ can be rewritten as

$$h_{i,\alpha,\beta}(t) = \frac{te^t}{e^t-1} + (\beta-1)t + \alpha - i - 1$$

$$= \alpha - i + \left(\beta - \frac{1}{2}\right)t + \sum_{k=2}^{\infty} B_k(1)\frac{t^k}{k!}$$
$$= \alpha - i + \left(\beta - \frac{1}{2}\right)t + \sum_{k=2}^{\infty} (-1)^k B_k \frac{t^k}{k!}$$
$$= \alpha - i + \left(\beta - \frac{1}{2}\right)t + \sum_{k=1}^{\infty} (-1)^{k+1} B_{k+1} \frac{t^{k+1}}{(k+1)!}$$
$$= \alpha - i + \left(\beta - \frac{1}{2}\right)t + \sum_{k=0}^{\infty} B_{2k+2} \frac{t^{2k+2}}{(2k+2)!},$$
$$h'_{i,\alpha,\beta}(t) = \beta - \frac{1}{2} + \sum_{k=1}^{\infty} B_{2k} \frac{t^{2k-1}}{(2k-1)!}.$$

The proof of Theorem 1 states that

(1) $h'_{i,\alpha,0}(t) < 0$ in $(0,\infty)$;
(2) if $\alpha \ge i+1$, then $h_{i,\alpha,0}(t) > 0$ in $(0,\infty)$;
(3) if $0 < \alpha \le i$, then $h_{i,\alpha,0}(t) < 0$ in $(0,\infty)$;
(4) if $\beta \ge \frac{1}{2}$, then $h'_{i,\alpha,\beta}(t) > 0$ in $(0,\infty)$;
(5) if $\alpha \ge i$ and $\beta \ge \frac{1}{2}$, then $h_{i,\alpha,\beta}(t) > 0$ in $(0,\infty)$;
(6) if $0 < \beta < \frac{1}{2}$ and inequality (2) holds true, then $h_{i,\alpha,\beta}(t) > 0$ in $(0,\infty)$.

From these and standard argument, Theorem 4 is proved. □

## References


[1] M. Abramowitz and I. A. Stegun (Eds), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Reprint of the 1972 edition. A Wiley-Interscience Publication. Selected Government Publications. John Wiley & Sons, Inc., New York; National Bureau of Standards, Washington, DC, 1984.

[2] H. Alzer, *Mean-value inequalities for the polygamma functions*, Aequationes Math. **61** (2001), no. 1, 151–161.

[3] P. Gao, *A subadditive property of the digamma function*, RGMIA Res. Rep. Coll. **8** (2005), no. 3, Art. 9. Available online at `http://rgmia.vu.edu.au/v8n3.html`.

[4] B.-N. Guo, R.-J. Chen, and F. Qi, *A class of completely monotonic functions involving the polygamma functions*, J. Math. Anal. Approx. Theory **2** (2007), no. 1, in press.

[5] F. Qi, *A completely monotonic function involving divided difference of psi function and an equivalent inequality involving sum*, RGMIA Res. Rep. Coll. **9** (2006), no. 4, Art. 5. Available online at `http://rgmia.vu.edu.au/v9n4.html`.

[6] F. Qi, *A completely monotonic function involving divided differences of psi and polygamma functions and an application*, RGMIA Res. Rep. Coll. **9** (2006), no. 4, Art. 8. Available online at `http://rgmia.vu.edu.au/v9n4.html`.

[7] F. Qi, *Certain logarithmically $N$-alternating monotonic functions involving gamma and $q$-gamma functions*, RGMIA Res. Rep. Coll. **8** (2005), no. 3, Art. 5. Available online at `http://rgmia.vu.edu.au/v8n3.html`.

[8] F. Qi, *Three classes of logarithmically completely monotonic functions involving gamma and psi functions*, Integral Transforms Spec. Funct. **18** (2007), in press. RGMIA Res. Rep. Coll. **9** (2006), Suppl., Art. 6. Available online at `http://rgmia.vu.edu.au/v9(E).html`.

[9] F. Qi and W. Li, *Two logarithmically completely monotonic functions connected with the gamma function*, RGMIA Res. Rep. Coll. **8** (2005), no. 3, Art. 13. Available online at `http://rgmia.vu.edu.au/v8n3.html`.

[10] F. Qi, Q. Yang and W. Li, *Two logarithmically completely monotonic functions connected with gamma function*, Integral Transforms Spec. Funct. **17** (2006), no. 7, 539–542.

[11] H. van Haeringen, *Completely Monotonic and Related Functions*, Report 93-108, Faculty of Technical Mathematics and Informatics, Delft University of Technology, Delft, The Netherlands, 1993.
[12] Zh.-X. Wang and D.-R. Guo, *Special Functions*, Translated from the Chinese by D.-R. Guo and X.-J. Xia, World Scientific Publishing, Singapore, 1989.
[13] Zh.-X. Wang and D.-R. Guo, *Tèshū Hánshù Gàilùn* (*A Panorama of Special Functions*), The Series of Advanced Physics of Peking University, Peking University Press, Beijing, China, 2000. (Chinese)
[14] D. V. Widder, *The Laplace Transform*, Princeton University Press, Princeton, 1941.

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