

# Two-sided inequalities for generalized hypergeometric function

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**Abstract.** We find two-sided inequalities for the generalized hypergeometric function with positive parameters restricted by certain additional conditions. Our lower bounds are asymptotically precise at  $x = 0$ , while the upper bounds are either asymptotically precise or at least agree with  ${}_qF_q((a_{q+1}), (b_q); -x)$  at  $x = \infty$ . Inequalities are derived as corollaries of a theorem asserting the monotony of the quotient of two generalized hypergeometric functions with shifted parameters. The proofs hinge on a generalized Stieltjes representation of the generalized hypergeometric function. This representation also provides yet another method to derive the second Thomae relation for  ${}_3F_2(1)$  and leads to an integral representations of  ${}_4F_3(x)$  in terms of the Appell function  $F_3$ . In the last section of the paper we list some open questions and conjectures.

Keywords: *Generalized hypergeometric function, generalized Stieltjes function, hypergeometric inequality, Thomae relations*

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**1. Introduction.** We will use standard notation for generalized hypergeometric function:

$${}_qF_q((a_{q+1}); (b_q); z) = {}_qF_q \left( \begin{matrix} (a_{q+1}) \\ (b_q) \end{matrix} \middle| z \right) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_{q+1})_n}{(b_1)_n \cdots (b_q)_n n!}, \quad (1)$$

where  $(a_q) = a_1, a_2, \dots, a_q$  is the parameter list and  $(a)_n = a(a+1) \cdots (a+n-1)$  is shifted factorial. The series converges for  $|z| < 1$  and can be analytically continued onto entire complex plane cut along  $[1, \infty]$ . The celebrated continued fraction of Gauss

$$\frac{{}_2F_1(a, b; c; -x)}{{}_2F_1(a-1, b; c-1; -x)} = \frac{c-1}{c-1+} \frac{b(c-a)x}{c} + \frac{a(c-b)x}{c+1} + \cdots + \frac{(b+n)(c-a+n)x}{c+2n} + \frac{(a+n)(c-b+n)x}{c+2n+1} + \cdots \quad (2)$$

implies on setting  $a = 1$  the classical representation

$${}_2F_1(1, b; c; -x) = \frac{c-1}{c-1+} \frac{b(c-1)x}{c} + \frac{(c-b)x}{c+1} + \cdots + \frac{(b+n)(c-1+n)x}{c+2n} + \frac{(n+1)(c-b+n)x}{c+2n+1} + \cdots \quad (3)$$

This continued fraction converges and has positive elements when  $x > 0$  and  $c > b > 0$ . Hence, its even convergents form an increasing sequence and approximate the value of  ${}_2F_1(1, b; c; -x)$  from

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below, while the odd convergents form a decreasing sequence and approximate  ${}_2F_1(1, b; c; -x)$  from above (see details in [LW]). Taking first three terms this implies:

$$\frac{1}{1 + bx/c} < {}_2F_1(1, b; c; -x) < \frac{c(c+1) + (c-b)x}{c(c+1) + c(b+1)x} < 1. \quad (4)$$

Gauss derived his continued fraction (2) from contiguous relations for the hypergeometric function. Another way to explain both continued fraction (3) and inequality (4) is through Euler representation

$${}_2F_1(1, b; c; -x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{1+xt} dt, \quad (5)$$

which shows that  ${}_2F_1(1, b; c; -x)$  is a Stieltjes function, i.e. the function of the form

$$f(x) = \int_0^{1/R} \frac{d\phi(u)}{1+xu}$$

with bounded nondecreasing function  $\phi$  taking infinitely many values (see [BG, Chapter 5]). For such functions both continued fraction representation with positive elements and two-sided estimates through Padé approximants are well known (see [Baker, BG]). Indeed, it is easy to check that

$$\begin{aligned} \frac{1}{1 + bx/c} &= 1 - \frac{b}{c}x + \mathcal{O}(x^2), \quad x \rightarrow 0, \\ \frac{c(c+1) + (c-b)x}{c(c+1) + c(b+1)x} &= 1 - \frac{b}{c}x + \frac{b(b+1)}{c(c+1)}x^2 + \mathcal{O}(x^3), \quad x \rightarrow 0. \end{aligned}$$

Hence, the fraction on the left-hand side of (4) is the Padé approximant of  ${}_2F_1(1, b; c; -x)$  of order  $[0/1]$  while the fraction on the right-hand side is the Padé approximation of order  $[1/1]$  (see [Baker, BG]). The sequences of diagonal and paradiagonal one-point Padé approximants form upper and lower bounds uniformly converging to the Stieltjes function, i.e.

$$[M/M - 1]_f < [M - 1/M]_f < f < [M/M]_f < [M + 1/M - 1]_f \quad (6)$$

for each positive integer  $M$ . Moreover, these bounds are best with respect to given number of power series coefficients [Baker, Theorem 15.2]. Inequality (4) is a particular case when  $M = 1$ . These results have been recently generalized to multi-point Padé approximants by Gilewicz, Pindor, Telega and Tokarzewski (see [GPTT, TokarTelega]). These papers deal, however, with diagonal and superdiagonal approximants. The subdiagonal approximants used here are not considered. Some applications of Padé approximants to inequalities for special functions are discussed in [SitnDokl, SitnPrepr1].

The lower bound in (4) is not only asymptotically precise at  $x = 0$  but also agrees with  ${}_2F_1(1, b; c; -x)$  at  $x = \infty$ . One problem with (4) is that the same, unfortunately, is not true for the upper bound, which reduces to the constant  $(c-b)/[c(b+1)]$  at  $x = \infty$ . If we wish to generalize (4) to  ${}_{q+1}F_q$  we are also faced with the problem that neither an analogue of the Gauss continued fraction for the general  ${}_{q+1}F_q$  is known nor multiple Euler representation generalizing (5) has the form of a Stieltjes function. In this paper we solve both problems and find two-sided estimates for  $f(x) = {}_{q+1}F_q(1, (a_q); (b_q); -x)$ , where the lower bound is asymptotically precise at  $x = 0$ , the upper bound is asymptotically precise at  $x = \infty$  and both agree with the values of  $f(x)$  at the endpoints of  $[0, \infty]$ . Next, we extend our inequalities to  ${}_{q+1}F_q(\sigma, (a_q); (b_q); -x)$  for some values of  $\sigma$ . These results are derived from a somewhat stronger statement about the monotony of a special quotient of hypergeometric functions. Our method is based on a generalized Stieltjes representation for  ${}_{q+1}F_q((a_{q+1}); (b_q); -x)$  but does not utilize the relationship with Padé approximants. In the last section of the paper we list some open problems and conjectures.

Inequalities for general  ${}_{q+1}F_q$  are surprisingly rare in the literature. Let us mention a few papers dealing with inequalities for the Gauss or generalized hypergeometric function. Carlson studied in [Carlson] some inequalities for  $R$ -hypergeometric function, which can be expressed in terms of Laurichella  $F_D$  function and as such is a generalization of  ${}_2F_1$  to multivariate case. For  ${}_2F_1(a, b; c; x)$  his restrictions on parameters are:  $c > b > 0$ ,  $x < 1$ . See section 3 for detailed comparison of our results for  ${}_2F_1(a, b; c; x)$  with those of Carlson. Inequalities for  ${}_{q+1}F_q(1, (a_q); (b_q); x)$ ,  $q > 1$  are not considered in Carlson's paper.

Buschman [Buschman] uses determinant representations to obtain two-sided inequalities for the Gauss hypergeometric function  ${}_2F_1(a + n, b; c; x)$  in terms of  ${}_2F_1(a, b; c; x)$  for positive parameters and  $x \in (0, 1)$ . His results were later improved and extended by Joshi and Arya [JA1, JA2].

Inequalities of different nature for  ${}_pF_q$  and  $\Re({}_pF_q)$  have been obtained by Jahangiri and Silvia in [JahSil] for the special case when  ${}_pF_q$  is subordinate (in the sense of analytic function theory) to the linear fractional transformation.

The papers [ABRVV, PV, Baricz1, Baricz2, SitnPrepr2] (and many more found in the references in the above papers) consider inequalities for the Gauss function  ${}_2F_1(a, b; c; x)$ , the Kummer function  ${}_1F_1(a; c; x)$ , the Bessel function  ${}_0F_1(a; x)$  and their ratios for  $x \in (0, 1)$  (cf. [SitnDokl]). The emphasis in these papers is on fine properties near the singular point  $x = 1$  of  ${}_2F_1(a, b; c; x)$ . Note also that the method for proving the monotony of ratios based on Lemma 2.1 from [PV] cannot be applied to prove our Theorem 1. Since our results are valid also for  $x < 0$ , they complement those from [ABRVV, PV, Baricz1, Baricz2] for  ${}_2F_1$ . General  ${}_pF_q$  is not discussed in the above papers, except results in [SitnPrepr2] are applicable in case that some upper and lower parameters differ by half-integer.

**2. Generalized Stieltjes representation and inequalities for  ${}_{q+1}F_q$ .** We begin with a representation of  ${}_{q+1}F_q(z)$  as a generalized Stieltjes transform.

**Lemma 1** For  $\Re b_k > \Re a_k > 0$ ,  $k = 1, 2, \dots, q$ , and  $|\arg(1 + z)| < \pi$  we have

$${}_{q+1}F_q \left( \begin{matrix} \sigma, (a_q) \\ (b_q) \end{matrix} \middle| -z \right) = A_0 \int_0^1 \frac{s^{a_1-1} g((a_q); (b_q); s) ds}{(1 + sz)^\sigma}, \quad (7)$$

where

$$g((a_q); (b_q); s) = \int_{\Lambda_q(s)} [1 - s/(t_2 \cdots t_q)]^{b_1 - a_1 - 1} \prod_{k=2}^q t_k^{a_k - a_1 - 1} (1 - t_k)^{b_k - a_k - 1} dt_2 \cdots dt_q \quad (8)$$

and

$$\Lambda_q(s) = [0, 1]^{q-1} \cap \{t_2, \dots, t_q : t_2 \cdots t_q > s\}, \quad (9)$$

$$A_0 = \Gamma \left[ \begin{matrix} (b_q) \\ (a_q), (b_q - a_q) \end{matrix} \right] \equiv \frac{\Gamma(b_1) \cdots \Gamma(b_q)}{\Gamma(a_1) \cdots \Gamma(a_q) \Gamma(b_1 - a_1) \cdots \Gamma(b_q - a_q)}. \quad (10)$$

The function  $g((a_q); (b_q); s)$  is invariant under simultaneous shift of all parameters:

$$g((a_q) + \delta; (b_q) + \delta; s) = g((a_q); (b_q); s) \quad (11)$$

for any complex  $\delta$ .

**Proof.** The multiple Euler integral representation for  ${}_{q+1}F_q(z)$  reads [PBM, formula 7.2.3(10)]:

$${}_{q+1}F_q \left( \begin{matrix} \sigma, (a_q) \\ (b_q) \end{matrix} \middle| -z \right) = \Gamma \left[ \begin{matrix} (b_q) \\ (a_q), (b_q - a_q) \end{matrix} \right] \int_{[0,1]^q} \frac{\prod_{k=1}^q t_k^{a_k-1} (1 - t_k)^{b_k - a_k - 1}}{(1 + t_1 t_2 \cdots t_q z)^\sigma} dt_1 \cdots dt_q. \quad (12)$$

Integration in (12) is over  $q$ -dimensional unit cube,  $\Re b_k > \Re a_k > 0$  for all  $k = 1, 2, \dots, q$ , and  $|\arg(1+z)| < \pi$ . Formula (12) is obtained by repeated application of the generalized Euler integral [AAR, formula (2.2.2)]

$${}_{q+1}F_q \left( \begin{matrix} (a_q), a_{q+1} \\ (b_{q-1}), b_q \end{matrix} \middle| z \right) = \frac{\Gamma(b_q)}{\Gamma(a_{q+1})\Gamma(b_q - a_{q+1})} \int_0^1 t^{a_{q+1}-1} (1-t)^{b_q - a_{q+1} - 1} {}_qF_{q-1} \left( \begin{matrix} (a_q) \\ (b_{q-1}) \end{matrix} \middle| zt \right) dt,$$

where the last step with  $q = 1$  is the standard Euler integral for  ${}_2F_1$ .

We make the variable change  $s = t_1 t_2 \cdots t_q$ ,  $t_1 = s/(t_2 \cdots t_q)$ , leaving  $t_2, \dots, t_q$  unchanged. The Jacobian of such transformation is

$$J(s, t_2, \dots, t_q) = \frac{1}{t_2 \cdots t_q}.$$

Hence, we get from (12):

$$\begin{aligned} {}_{q+1}F_q \left( \begin{matrix} \sigma, (a_q) \\ (b_q) \end{matrix} \middle| -z \right) &= A_0 \int_0^1 \frac{ds}{(1+sz)^\sigma} \\ &\times \int_{\Lambda_q(s)} [s/(t_2 \cdots t_q)]^{a_1-1} [1 - s/(t_2 \cdots t_q)]^{b_1 - a_1 - 1} \prod_{k=2}^q t_k^{a_k-1} (1-t_k)^{b_k - a_k - 1} \frac{dt_2 \cdots dt_q}{t_2 \cdots t_q} \\ &= A_0 \int_0^1 \frac{s^{a_1-1} ds}{(1+sz)^\sigma} \int_{\Lambda_q(s)} [1 - s/(t_2 \cdots t_q)]^{b_1 - a_1 - 1} \prod_{k=2}^q t_k^{a_k - a_1 - 1} (1-t_k)^{b_k - a_k - 1} dt_2 \cdots dt_q, \end{aligned}$$

where  $\Lambda_q(s)$  and  $A_0$  are defined in (9) and (10), respectively. Introducing the notation (8) for the inner integral we arrive at formula (7). The shift invariance (11) is obvious from the definition of  $g((a_q); (b_q); s)$ .  $\square$

Representation (7) is a key ingredient in the proof of the following theorem.

**Theorem 1** Suppose  $\delta > 0$ ,  $b_k > a_k > 0$ ,  $k = 1, \dots, q$ . The function

$$f(\sigma, (a_q); (b_q); \delta; x) \equiv \frac{{}_{q+1}F_q(\sigma, (a_q) + \delta; (b_q) + \delta; -x)}{{}_{q+1}F_q(\sigma, (a_q); (b_q); -x)} \quad (13)$$

is monotone decreasing if  $\sigma > 0$  and monotone increasing if  $\sigma < 0$  for all  $x > -1$ .

**Proof.** Put

$$A_\delta = \Gamma \left[ \begin{matrix} (b_q) + \delta \\ (a_q) + \delta, (b_q - a_q) \end{matrix} \right].$$

Then by (7) and (11):

$$\begin{aligned} f(\sigma, (a_q); (b_q); \delta; x) &= \frac{A_\delta \int_0^1 s^{a_1 + \delta - 1} (1+sx)^{-\sigma} g((a_q) + \delta; (b_q) + \delta; s) ds}{A_0 \int_0^1 s^{a_1 - 1} (1+sx)^{-\sigma} g((a_q); (b_q); s) ds} \\ &= \frac{A_\delta \int_0^1 s^{a_1 + \delta - 1} (1+sx)^{-\sigma} g((a_q); (b_q); s) ds}{A_0 \int_0^1 s^{a_1 - 1} (1+sx)^{-\sigma} g((a_q); (b_q); s) ds}. \end{aligned}$$

The statement of the theorem is equivalent to saying that  $f'_x(\sigma, (a_q); (b_q); \delta; x) < 0$  when  $\sigma > 0$  and  $f'_x(\sigma, (a_q); (b_q); \delta; x) > 0$  when  $\sigma < 0$ . Differentiating the definition of  $f$  above we see that both inequalities are equivalent to the single inequality

$$\int_0^1 q(s)p(s)ds \int_0^1 h(s)p(s)ds < \int_0^1 q(s)h(s)p(s)ds \int_0^1 p(s)ds, \quad (14)$$

where

$$p(s) = s^{a_1 + \delta - 1} (1 + sx)^{-\sigma} g((a_q); (b_q); s),$$

$$q(s) = s^\delta, \quad h(s) = \frac{s}{1 + xs}.$$

The function  $p(s)$  is positive, while the functions  $q(s)$  and  $h(s)$  are monotone increasing for fixed  $x > -1$  and  $0 < s < 1$ . Hence, the above inequality is an instance of the Chebyshev inequality [MPF, Chapter IX, formula (1.1)].  $\square$

The value of  $f(\sigma, (a_q); (b_q); \delta; x)$  at  $x = \infty$  can be found using the representation of  ${}_{q+1}F_q(z)$  in the neighbourhood of the singular point  $z = \infty$ . Assume for the moment that no numerator parameters differ by an integer. Then, according to [PBM, formula 7.2.3.77],

$${}_{q+1}F_q \left( \begin{matrix} (a_{q+1}) \\ (b_q) \end{matrix} \middle| -z \right) = \Gamma \left[ \begin{matrix} (b_q) \\ (a_{q+1}) \end{matrix} \right] \sum_{k=1}^{q+1} \Gamma \left[ \begin{matrix} a_k, (a_{q+1})' - a_k \\ (b_q) - a_k \end{matrix} \right] {}_{q+1}F_q \left( \begin{matrix} 1 + a_k - (b_q), a_k \\ 1 + a_k - (a_{q+1})' \end{matrix} \middle| -1/z \right) z^{-a_k}, \quad (15)$$

where the prime at  $(a_{q+1})$  means that the term  $a_k$  is excluded from the list. It follows that for  $\sigma < \min(a_1, a_2, \dots, a_q)$  we will have

$${}_{q+1}F_q \left( \begin{matrix} \sigma, (a_q) \\ (b_q) \end{matrix} \middle| -z \right) = \Gamma \left[ \begin{matrix} (a_q) - \sigma, (b_q) \\ (a_q), (b_q) - \sigma \end{matrix} \right] z^{-\sigma} + o(z^{-\sigma}), \quad z \rightarrow \infty. \quad (16)$$

Hence,

$$f(\sigma, (a_q); (b_q); \delta; \infty) = \Gamma \left[ \begin{matrix} (a_q), (a_q) + \delta - \sigma, (b_q) - \sigma, (b_q) + \delta \\ (a_q) - \sigma, (a_q) + \delta, (b_q), (b_q) + \delta - \sigma \end{matrix} \right]. \quad (17)$$

When some of the numerator parameters on the left-hand side of (15) differ by an integer, formula (15) breaks and one has to resort to much more involved [PBM, formula 7.2.3.78]. Formula (17), however, remains valid by continuity due to  $({}_pF_q/\Gamma((b_q)))$  is entire function of parameters - see [PBM, 7.3.2.8]).

Formula (17) and Theorem 1 imply the following inequalities valid for  $x > 0$ :

$$f(\sigma, (a_q); (b_q); \delta; \infty) < f(\sigma, (a_q); (b_q); \delta; x) < 1 = f(\sigma, (a_q); (b_q); \delta; 0), \quad (18)$$

for positive  $\sigma$  and

$$1 < f(\sigma, (a_q); (b_q); \delta; x) < f(\sigma, (a_q); (b_q); \delta; \infty), \quad (19)$$

for negative  $\sigma$ .

Taking  $\delta = 1$  in (17) we will have:

$$f(\sigma, (a_q); (b_q); 1; \infty) = \prod_{i=1}^q \frac{b_i(a_i - \sigma)}{a_i(b_i - \sigma)}. \quad (20)$$

A simple calculation shows that

$$1 - {}_{q+1}F_q(1, (a_q); (b_q); -x) = x {}_{q+1}F_q(1, (a_q) + 1; (b_q) + 1; -x) \prod_{i=1}^q (a_i/b_i).$$

It follows that

$$\begin{aligned} \frac{1 - {}_{q+1}F_q(1, (a_q); (b_q); -x)}{x {}_{q+1}F_q(1, (a_q); (b_q); -x)} &= \frac{{}_{q+1}F_q(1, (a_q) + 1; (b_q) + 1; -x)}{{}_{q+1}F_q(1, (a_q); (b_q); -x)} \prod_{i=1}^q (a_i/b_i) \\ &= f(1, (a_q); (b_q); 1; x) \prod_{i=1}^q (a_i/b_i). \quad (21) \end{aligned}$$

Combined with (18) and (20) this gives:

$$\prod_{i=1}^q \frac{(a_i - 1)}{(b_i - 1)} < \frac{1 - {}_{q+1}F_q(1, (a_q); (b_q); -x)}{{}_x{}_{q+1}F_q(1, (a_q); (b_q); -x)} < \prod_{i=1}^q (a_i/b_i).$$

A simple rearrangement of this formula leads to

**Theorem 2** For  $b_k > a_k > 1$ ,  $k = 1, \dots, q$ , and  $x > 0$  the inequality

$$\frac{1}{1 + x \prod_{i=1}^q (a_i/b_i)} < {}_{q+1}F_q(1, (a_q); (b_q); -x) < \frac{1}{1 + x \prod_{i=1}^q [(a_i - 1)/(b_i - 1)]} \quad (22)$$

holds true.

Some comments are here in order. Since, clearly,

$$\frac{1}{1 + x \prod_{i=1}^q (a_i/b_i)} = 1 - x \prod_{i=1}^q \frac{a_i}{b_i} + \mathcal{O}(x^2), \quad x \rightarrow 0,$$

we conclude that the lower bound in (22) is asymptotically precise at  $x = 0$  and coincides with the Padé approximant to  ${}_{q+1}F_q(1, (a_q); (b_q); -x)$  at zero of order  $[0/1]$ . Hence, inequality from below in (22) can be proved by noticing that  ${}_{q+1}F_q(1, (a_q); (b_q); -x)$  is the Stieltjes function by Lemma 1. This allows to relax the restrictions on parameters to  $b_k > a_k > 0$ . Moreover, a result of Gilewicz and Magnus [GilMagnus] (see also [MPF, Chapter 25, Theorem 2]) implies that **the lower bound in (22) is valid for all  $x > -1$** . We further generalize the lower bound in the following theorem.

**Theorem 3** For  $b_k > a_k > 0$ ,  $k = 1, \dots, q$ ,  $x > -1$  and  $\sigma \geq 1$  the inequality

$$\frac{1}{(1 + x \prod_{i=1}^q (a_i/b_i))^\sigma} < {}_{q+1}F_q(\sigma, (a_q); (b_q); -x) \quad (23)$$

holds true.

**Proof.** The proof is an application of Jensen's inequality

$$\varphi \left( \frac{\int_a^b f(s) d\mu(s)}{\int_a^b d\mu} \right) \leq \frac{\int_a^b \varphi(f(s)) d\mu(s)}{\int_a^b d\mu} \quad (24)$$

valid for convex  $\varphi$ , integrable  $f$  and non-negative measure  $\mu$  [MPF, formula (7.15)]. Take  $\varphi(u) = u^\sigma$ ,  $\sigma \geq 1$ ,  $f(s) = 1/(1 + sx)$ ,  $d\mu(s) = s^{a_1-1} g((a_q); (b_q); s) ds$ . By (7)

$$\int_0^1 f(s) d\mu(s) = \frac{1}{A_0} {}_{q+1}F_q(1, (a_q); (b_q); -x),$$

$$\int_0^1 d\mu = \int_0^1 s^{a_1-1} g((a_q); (b_q); s) ds = \frac{{}_{q+1}F_q(1, (a_q); (b_q); 0)}{A_0} = \frac{1}{A_0}$$

and

$$\int_0^1 \varphi(f(s)) d\mu(s) = \frac{1}{A_0} {}_{q+1}F_q(\sigma, (a_q); (b_q); -x).$$

Hence, (24) reads

$$({}_{q+1}F_q(1, (a_q); (b_q); -x))^\sigma \leq {}_{q+1}F_q(\sigma, (a_q); (b_q); -x).$$

Combined with the lower bound from (22) this yields (23). The restrictions on parameters are explained in the remark preceding this theorem.  $\square$

In a similar fashion but under stronger assumptions we can prove a generalization of the upper bound from (22):

**Theorem 4** For  $b_k > a_k > 1$ ,  $k = 1, \dots, q$ ,  $x > 0$  and  $0 < \sigma \leq 1$  the inequality

$${}_{q+1}F_q(\sigma, (a_q); (b_q); -x) < \frac{1}{(1+x \prod_{i=1}^q [(a_i-1)/(b_i-1)])^\sigma} \quad (25)$$

holds true.

**Proof.** Again apply (24) but this time with  $\varphi(u) = u^{1/\sigma}$ ,  $0 < \sigma \leq 1$ ,  $f(s) = 1/(1+sx)^\sigma$  and  $d\mu(s) = s^{a_1-1}g((a_q); (b_q); s)ds$ . This yields

$$({}_{q+1}F_q(\sigma, (a_q); (b_q); -x))^{1/\sigma} \leq {}_{q+1}F_q(1, (a_q); (b_q); -x).$$

The combination of this inequality with the upper bound from (22) results in (25).  $\square$

Formula (16) and the relation

$$\frac{1}{1+x \prod_{i=1}^q [(a_i-1)/(b_i-1)]} = \frac{1}{x} \prod_{i=1}^q \frac{b_i-1}{a_i-1} + \mathcal{O}(1/x^2), \quad x \rightarrow \infty,$$

show that the upper bound in (22) is asymptotically precise at  $x = \infty$ . This, unfortunately, is not true of the upper bound (25). We conjecture an asymptotically precise upper bound for  $\sigma \geq 1$  in the last section of the paper.

**Remark.** It is interesting to observe that the proofs of Theorems 3 and 4 work for any generalized Stieltjes function, i.e. function of the form

$$f_\sigma(x) = \int_0^{1/R} \frac{d\phi}{(1+xt)^\sigma}$$

with bounded nondecreasing function  $\phi$  taking infinitely many values and  $\sigma > 0$ . We have then

$$[f_1(x)]^\sigma \leq [f_1(0)]^{\sigma-1} f_\sigma(x), \quad \sigma \geq 1,$$

and

$$f_\sigma(x) \leq [f_1(0)]^{1-\sigma} [f_1(x)]^\sigma, \quad 0 < \sigma \leq 1.$$

Hence, by combining these inequalities with inequality (6) we can "exponentiate" the known inequalities between Stieltjes function and its Padé approximants.

**3. The case  $q = 1$ .** In this case we are able to extend the upper bound from (25) to negative  $x$  as follows:

**Theorem 5** For  $b > a + 1 > 1$ ,  $0 < \sigma \leq 1$  and  $-1 < x < 0$  the inequality

$${}_2F_1(\sigma, a; b; -x) < \frac{1}{\left(1 + \frac{a}{b-1}x\right)^\sigma} \quad (26)$$

holds true.

**Proof.** The function  $f$  defined by (13) for  $q = 1$ ,  $\delta = 1$  and  $\sigma = 1$  takes the form

$$f(1, a; b; 1; x) = \frac{{}_2F_1(1, a+1; b+1; -x)}{{}_2F_1(1, a; b; -x)}.$$

The value  $f(1, a; b; 1; -1)$  is finite under assumptions of the theorem and is found by the Gauss formula for  ${}_2F_1(a_1, a_2; b; 1)$ :

$$f(1, a; b; 1; -1) = \frac{{}_2F_1(1, a+1; b+1; 1)}{{}_2F_1(1, a; b; 1)} = \frac{b}{b-1}.$$

By Theorem 1  $f(1, a; b; 1; x)$  is monotone decreasing for  $x \in (-1, 0)$  and so

$$f(1, a; b; 1; 0) = 1 < f(1, a; b; 1; x) < \frac{b}{b-1} = f(1, a; b; 1; -1).$$

According to (21)

$$\frac{a}{b} < \frac{1 - {}_2F_1(1, a; b; -x)}{{}_2F_1(1, a; b; -x)} < \frac{b}{b-1} \frac{a}{b}$$

or

$$\frac{1}{1 + ax/b} < {}_2F_1(1, a; b; -x) < \frac{1}{1 + ax/(b-1)}.$$

Inequality (26) is obtained from this estimate by an application of Jensen's inequality as in the proof of Theorem 4.  $\square$

It is interesting to compare these results with those from [Carlson]. Carlson's inequality for the case  $b > a \geq \sigma > 0$  considered here reads

$$\max \left\{ \frac{(1+x)^{b-a-\sigma}}{(1+x(1-a/b))^{b-\sigma}}, (1+ax/b)^{-\sigma} \right\} < {}_2F_1(\sigma, a; b; -x) < (1+x)^{-\sigma a/b}. \quad (27)$$

Note first that our condition  $b > a \geq \sigma > 0$  is not more restrictive than  $b > \max(a, \sigma) > 0$ , since one can exchange the roles of  $a$  and  $\sigma$ . We see that the lower bound in (27) is an extension (and possibly a refinement due to the competitive term under  $\max$ ) of our inequality (23) to the values of  $\sigma \in (0, 1)$  for the particular case  $q = 1$ .

To compare the upper bounds we note that for  $b > a > \sigma > 0$  according to (16)

$${}_2F_1(\sigma, a; b; -x) = \frac{1}{x^\sigma} \left[ \frac{\Gamma(b)\Gamma(a-\sigma)}{\Gamma(a)\Gamma(b-\sigma)} \right]^\sigma + o(x^{-\sigma}), \quad x \rightarrow \infty,$$

and clearly

$$(1+x)^{-\sigma a/b} = \frac{1}{x^{\sigma a/b}} (1 + \mathcal{O}(1/x)), \quad x \rightarrow \infty.$$

Hence, the upper bound in (27) is never asymptotically precise, albeit agrees with the value  $0 = {}_2F_1(a, b; c; -x)$  at  $x = \infty$ . Our bound in (22) and the conjectured bound in (38) are asymptotically precise and so both are better than Carlson's bound at least for large  $x$ . The upper bound (25) agrees with the main asymptotic term in order of  $x$  but not in the constant, while the upper bound in (27) has the wrong order and so again our bound (25) is better than Carlson's bound at least for large  $x$ .

Finally, for negative  $x$  the upper bound in (27) goes to  $\infty$  as  $x \rightarrow -1$ , while under our restrictions on parameters  ${}_2F_1(\sigma, a; b; -x)$  remains bounded and so does our bound in (25). Hence our bound is guaranteed to be better around  $x = -1$ , while Carlson's bound is more precise when  $x$  is close to 0.

**3. The case  $q = 2$ .** In this case we are able to relax the assumptions on parameters imposed in Lemma 1 and Theorems 1, 2, 3 and 4. To this end we give an alternative proof of Lemma 1 which also shows that the function  $g$  defined by (8) is expressed for  $q = 2$  in terms of  ${}_2F_1$ . Representation (28) below is not new, it is a slightly different form of [PBM, formula 2.21.1.26]. However, we include a short proof which clarifies the source of restrictions on parameters.

**Lemma 2** *Let  $\Re(d + e - b - c) > 0$ ,  $\Re c > 0$ ,  $\Re b > 0$  and  $|\arg(1-x)| < \pi$ , then*

$$\begin{aligned} & {}_3F_2(a, b, c; d, e; x) \\ &= \frac{\Gamma(d)\Gamma(e)}{\Gamma(b)\Gamma(c)\Gamma(d+e-b-c)} \int_0^1 \frac{t^{b-1}(1-t)^{d+e-b-c-1}}{(1-xt)^a} {}_2F_1(e-c, d-c; d+e-b-c; 1-t) dt. \end{aligned} \quad (28)$$



**Proof.** Expand  $(1 - xt)^a$  on the right-hand side of (28) into binomial series and integrate term by term applying the generalized Euler integral [AAR, Theorem 2.2.4]

$$\int_0^1 u^{\gamma-1}(1-u)^{\nu-\gamma-1} {}_2F_1(\alpha, \beta; \gamma; ux) du = \frac{\Gamma(\gamma)\Gamma(\nu-\gamma)}{\Gamma(\nu)} {}_2F_1(\alpha, \beta; \nu; x),$$

where  $\Re\nu > \Re\gamma > 0$ ,  $|\arg(1-x)| < \pi$  (for this formula to be valid at  $x = 1$  the additional restriction  $\Re(\nu - \alpha - \beta) > 0$  must be imposed), and the Gauss formula

$${}_2F_1(\alpha, \beta; \nu; 1) = \frac{\Gamma(\nu)\Gamma(\nu - \alpha - \beta)}{\Gamma(\nu - \alpha)\Gamma(\nu - \beta)},$$

valid under the same restriction. This yields:

$$\begin{aligned} & \int_0^1 t^{b+k-1}(1-t)^{d+e-b-c-1} {}_2F_1(e-c, d-c; d+e-b-c; 1-t) dt \\ &= \int_0^1 u^{d+e-b-c-1}(1-u)^{b+k-1} {}_2F_1(e-c, d-c; d+e-b-c; u) du \\ &= \frac{\Gamma(d+e-b-c)\Gamma(b+k)}{\Gamma(d+e-c+k)} {}_2F_1(e-c, d-c; d+e-c+k; 1) \\ &= \frac{\Gamma(d+e-b-c)\Gamma(b+k)\Gamma(c+k)}{\Gamma(e+k)\Gamma(d+k)}, \quad (29) \end{aligned}$$

which implies (28) on substitution into the series.  $\square$

**Remark.** Representation (28) provides another way to derive Thomae's second fundamental relation for  ${}_3F_2(a, b, c; d, e; 1)$  [Bailey, formula 3.2(2)]. Indeed, apply the connection formula [AAR, formula (2.3.13)]

$$\begin{aligned} {}_2F_1(e-c, d-c; d+e-b-c; 1-t) &= \frac{\Gamma(d+e-b-c)\Gamma(c-b)}{\Gamma(d-b)\Gamma(e-b)} {}_2F_1(d-c, e-c; b-c+1; t) \\ &+ \frac{\Gamma(d+e-b-c)\Gamma(b-c)}{\Gamma(d-c)\Gamma(e-c)} t^{c-b} {}_2F_1(d-b, e-b; c-b+1; t) \end{aligned}$$

on the right-hand side of (28), change  $t \rightarrow 1-t$  and apply (28) again to get

$$\begin{aligned} {}_3F_2 \left[ \begin{matrix} a, b, c \\ d, e \end{matrix} \right] &= \frac{\pi\Gamma(d)\Gamma(e)\Gamma(1-a)\Gamma(d+e-b-c)}{\sin(\pi(c-b))\Gamma(b)\Gamma(c)} \\ &\times \left\{ \frac{{}_3F_2[1-a, 1-c, d+e-a-b-c; d-a-c+1, e-a-c+1]}{\Gamma(d-a-c+1)\Gamma(e-a-c+1)\Gamma(d-b)\Gamma(e-b)} - \text{idem}(b; c) \right\}, \end{aligned}$$

where the unit argument is omitted for conciseness and  $\text{idem}(b; c)$  after an expression means that the preceding expression is repeated with  $b$  and  $c$  interchanged. To obtain the form given in [Bailey, formula 3.2(2)], one needs to apply Thomae's first fundamental relation

$$\begin{aligned} {}_3F_2 \left[ \begin{matrix} d+e-a-b-c, 1-a, 1-c \\ d-a-c+1, e-a-c+1 \end{matrix} \right] \\ = \frac{\Gamma(d-a-c+1)\Gamma(e-a-c+1)\Gamma(b)}{\Gamma(d+e-a-b-c)\Gamma(1+b-a)\Gamma(1+b-c)} {}_3F_2 \left[ \begin{matrix} b, 1+b-e, 1+b-d \\ 1+b-a, 1+b-c \end{matrix} \right]. \quad (30) \end{aligned}$$

Using Lemma 2 we can give a version of Theorem 1 for the ratio of  ${}_3F_2$ s under slightly weaker restrictions on parameters.

**Theorem 6** Let  $d + e - b - c > 0$ ,  $c > 0$ ,  $b > 0$ ,  $\delta > 0$  and  $\min(b, c) < \min(d, e)$ . Then the function

$$f(a, b, c, d, e, \delta; x) \equiv \frac{{}_3F_2(a, b + \delta, c + \delta; d + \delta, e + \delta; -x)}{{}_3F_2(a, b, c; d, e; -x)}$$

is monotone decreasing if  $a > 0$  and monotone increasing if  $a < 0$  for all  $x > -1$ .

**Proof.** Assume without loss of generality  $c = \min(b, c)$  (otherwise exchange the roles of  $b$  and  $c$ ). This and the hypotheses of the theorem imply that  $d - c > 0$  and  $e - c > 0$ . Now follow the proof of Theorem 1 to get inequality (14) with

$$p(s) = \frac{s^{b-1}(1-s)^{d+e-b-c-1}}{(1+xs)^a} {}_2F_1(e-c, d-c; d+e-b-c; 1-s),$$

$$q(s) = s^\delta, \quad h(s) = \frac{s}{1+xs}.$$

The function  $p(s)$  is positive since  $e > c$ ,  $d > c$  and  $d + e - b - c > 0$ , while the functions  $q(s)$  and  $h(s)$  are monotone increasing for fixed  $x > -1$ ,  $0 < s < 1$ . The result follows by the Chebyshev inequality as before.  $\square$

Inequality (22) takes the form

$$\frac{1}{1+xbc/de} < {}_3F_2(1, b, c; d, e; -x) < \frac{1}{1+x(b-1)(c-1)/(d-1)(e-1)} \quad (31)$$

and here is valid under the assumptions of Theorem 6. A particular case of this inequality has been used in [KarpSitnik] to obtain an error bound in the asymptotic expansion of the first incomplete elliptic integral which leads to very precise two-sided inequalities for this integral. Theorems 3 and 4 are also valid here under the assumptions of Theorem 6.

**4. The case  $q = 3$ .** In this section we will show that Lemma 1 leads to new representations of  ${}_4F_3$  as a double integral of  ${}_2F_1$  or as a single integral of the Appell function  $F_3$ . To this end we need to demonstrate that the function  $g((a_3); (b_3); s)$  can be expressed in terms of the Appell function  $F_3$  or as an integral of  ${}_2F_1$ . Indeed, we have by (7)

$$\begin{aligned} & {}_4F_3(\sigma, (a_3); (b_3); -z) \\ &= A_0 \int_0^1 \frac{s^{a_1-1} ds}{(1+sz)^\sigma} \int_{\Lambda_3(s)} [1 - s/(t_2 t_3)]^{b_1-a_1-1} t_2^{a_2-a_1-1} t_3^{a_3-a_1-1} (1-t_2)^{b_2-a_2-1} (1-t_3)^{b_3-a_3-1} dt_2 dt_3, \end{aligned}$$

where

$$\begin{aligned} A_0 &= \frac{\Gamma(b_1)\Gamma(b_2)\Gamma(b_3)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(b_1-a_1)\Gamma(b_2-a_2)\Gamma(b_3-a_3)}, \\ \Lambda_3(s) &= \{t_2, t_3 : t_2 t_3 > s, 0 < t_2 < 1, 0 < t_3 < 1\}. \end{aligned}$$

The double integral (8) here can be written as follows:

$$g((a_3); (b_3); s) = \int_s^1 t_2^{a_2-b_1} (1-t_2)^{b_2-a_2-1} dt_2 \int_{s/t_2}^1 [t_2 t_3 - s]^{b_1-a_1-1} t_3^{a_3-b_1} (1-t_3)^{b_3-a_3-1} dt_3.$$

Make the change of variables

$$x = \frac{t_3 - s/t_2}{1 - s/t_2}, \quad t_3 = x(1 - s/t_2) + s/t_2, \quad 1 - t_3 = (1 - s/t_2)(1 - x), \quad dt_3 = (1 - s/t_2)dx,$$

in the inner integral to get

$$\begin{aligned}
g((a_3); (b_3); s) &= \int_s^1 t_2^{a_2-b_1} (1-t_2)^{b_2-a_2-1} dt_2 \\
&\times \int_0^1 [t_2 x (1-s/t_2)]^{b_1-a_1-1} (x(1-s/t_2) + s/t_2)^{a_3-b_1} ((1-s/t_2)(1-x))^{b_3-a_3-1} (1-s/t_2) dx \\
&= s^{a_3-b_1} \int_s^1 t_2^{b_1+a_2-a_1-a_3-1} (1-t_2)^{b_2-a_2-1} (1-s/t_2)^{b_1+b_3-a_1-a_3-1} dt_2 \int_0^1 \frac{x^{b_1-a_1-1} (1-x)^{b_3-a_3-1}}{(1+x(t_2/s-1))^{b_1-a_3}} dx \\
&= \frac{\Gamma(b_1-a_1)\Gamma(b_3-a_3)}{\Gamma(b_1+b_3-a_1-a_3)} s^{a_3-b_1} \\
&\times \int_s^1 t_2^{b_1+a_2-a_1-a_3-1} (1-t_2)^{b_2-a_2-1} (1-s/t_2)^{b_1+b_3-a_1-a_3-1} {}_2F_1 \left( \begin{matrix} b_1-a_3, b_1-a_1 \\ b_1+b_3-a_1-a_3 \end{matrix} \middle| 1-\frac{t_2}{s} \right) dt_2 \\
&= \frac{\Gamma(b_1-a_1)\Gamma(b_3-a_3)}{\Gamma(b_1+b_3-a_1-a_3)} s^{a_3-a_1} \\
&\times \int_s^1 t_2^{a_2-a_3-1} (1-t_2)^{b_2-a_2-1} (1-s/t_2)^{b_1+b_3-a_1-a_3-1} {}_2F_1 \left( \begin{matrix} b_3-a_1, b_1-a_1 \\ b_1+b_3-a_1-a_3 \end{matrix} \middle| 1-\frac{s}{t_2} \right) dt_2, \quad (32)
\end{aligned}$$

where we used Euler's integral

$$\int_0^1 \frac{x^{b_1-a_1-1} (1-x)^{b_3-a_3-1}}{(1+x(t_2/s-1))^{b_1-a_3}} dx = \frac{\Gamma(b_1-a_1)\Gamma(b_3-a_3)}{\Gamma(b_1+b_3-a_1-a_3)} {}_2F_1 \left( \begin{matrix} b_1-a_3, b_1-a_1 \\ b_1+b_3-a_1-a_3 \end{matrix} \middle| 1-\frac{t_2}{s} \right)$$

and Pfaff's transformation

$${}_2F_1 \left( \begin{matrix} b_1-a_3, b_1-a_1 \\ b_1+b_3-a_1-a_3 \end{matrix} \middle| 1-\frac{t_2}{s} \right) = (t_2/s)^{a_1-b_1} {}_2F_1 \left( \begin{matrix} b_3-a_1, b_1-a_1 \\ b_1+b_3-a_1-a_3 \end{matrix} \middle| 1-\frac{s}{t_2} \right).$$

Finally, make the change of variables

$$y = \frac{1-s/t_2}{1-s}, \quad t_2 = \frac{s}{1-y(1-s)}, \quad 1-t_2 = \frac{(1-s)(1-y)}{1-y(1-s)}, \quad dt_2 = \frac{s(1-s)dy}{(1-y(1-s))^2}$$

in (32) to get:

$$\begin{aligned}
g((a_3); (b_3); s) &= \frac{\Gamma(b_1-a_1)\Gamma(b_3-a_3)}{\Gamma(b_1+b_3-a_1-a_3)} s^{a_2-a_1} (1-s)^{b_1+b_2+b_3-a_1-a_2-a_3-1} \\
&\times \int_0^1 \frac{y^{b_1+b_3-a_1-a_3-1} (1-y)^{b_2-a_2-1}}{(1-y(1-s))^{b_2-a_3}} {}_2F_1 \left( \begin{matrix} b_3-a_1, b_1-a_1 \\ b_1+b_3-a_1-a_3 \end{matrix} \middle| y(1-s) \right) dy. \quad (33)
\end{aligned}$$

This can be further expressed in terms of Appell's function  $F_3$ , defined by [HTF1, formula 5.7(8)]

$$F_3(\alpha_1, \alpha_2; \beta_1, \beta_2; \gamma; z_1, z_2) = \sum_{k,l=0}^{\infty} \frac{(\alpha_1)_k (\alpha_2)_l (\beta_1)_k (\beta_2)_l}{(\gamma)_{k+l}} \frac{z_1^k z_2^l}{k! l!}.$$

Using the double integral representation [HTF1, formula 5.8(3)]

$$F_3(\alpha_1, \alpha_2; \beta_1, \beta_2; \gamma; z_1, z_2) = \frac{\Gamma(\gamma)}{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\gamma - \beta_1 - \beta_2)} \times \int_0^1 \frac{u^{\beta_1-1}}{(1-uz_1)^{\alpha_1}} \int_0^{1-u} \frac{v^{\beta_2-1}(1-u-v)^{\gamma-\beta_1-\beta_2-1}}{(1-vz_2)^{\alpha_2}} dv \quad (34)$$

we can obtain by the substitution  $t = v/(1-u)$  in the inner integral, an application of the Euler integral for  ${}_2F_1$  and changing  $u \rightarrow y = 1-u$ :

$$\int_0^1 \frac{y^{\gamma-\beta_1-1}(1-y)^{\beta_1-1}}{(1-yz_1)^{\alpha_1}} {}_2F_1 \left( \begin{matrix} \alpha_2, \beta_2 \\ \gamma - \beta_1 \end{matrix} \middle| yz_2 \right) dy = \frac{\Gamma(\gamma)(1-z_1)^{-\alpha_1}}{\Gamma(\beta_1)\Gamma(\gamma - \beta_1)} F_3(\alpha_1, \alpha_2; \beta_1, \beta_2; \gamma; z_1/(z_1-1), z_2).$$

This formula is a slightly different guise of [PBM, formula 2.21.1.20]. Combined with (33) it yields

$$g((a_3); (b_3); s) = \frac{\Gamma(b_1 - a_1)\Gamma(b_3 - a_3)\Gamma(b_1 + b_2 + b_3 - a_1 - a_2 - a_3)}{\Gamma(b_2 - a_2)[\Gamma(b_1 + b_3 - a_1 - a_3)]^2} s^{a_2+a_3-a_1-b_2} \times (1-s)^{b_1+b_2+b_3-a_1-a_2-a_3-1} F_3(b_2 - a_3, b_3 - a_1; b_2 - a_2, b_1 - a_1; b_1 + b_2 + b_3 - a_1 - a_2 - a_3; 1 - 1/s, 1 - s). \quad (35)$$

Finally, we obtain the following representations for  ${}_4F_3$ :

$${}_4F_3(\sigma, (a_3); (b_3); -z) = \frac{\Gamma(b_1)\Gamma(b_2)\Gamma(b_3)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(b_2 - a_2)\Gamma(b_1 + b_3 - a_1 - a_3)} \times \int_0^1 \int_0^1 \frac{s^{a_2-1} y^{b_1+b_3-a_1-a_3-1} (1-s)^{b_1+b_2+b_3-a_1-a_2-a_3-1} (1-y)^{b_2-a_2-1}}{(1+sz)^\sigma (1-y(1-s))^{b_2-a_3}} {}_2F_1 \left[ \begin{matrix} b_3 - a_1, b_1 - a_1 \\ b_1 + b_3 - a_1 - a_3 \end{matrix} \middle| y(1-s) \right] ds dy, \quad (36)$$

$${}_4F_3(\sigma, (a_3); (b_3); -z) = B_1 \int_0^1 \frac{s^{a_2+a_3-b_2-1} (1-s)^{b_1+b_2+b_3-a_1-a_2-a_3-1}}{(1+sz)^\sigma} \times F_3(b_2 - a_3, b_3 - a_1; b_2 - a_2, b_1 - a_1; b_1 + b_2 + b_3 - a_1 - a_2 - a_3; 1 - 1/s, 1 - s) ds, \quad (37)$$

where

$$B_1 = \frac{\Gamma(b_1)\Gamma(b_2)\Gamma(b_3)\Gamma(b_1 + b_2 + b_3 - a_1 - a_2 - a_3)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)[\Gamma(b_2 - a_2)]^2[\Gamma(b_1 + b_3 - a_1 - a_3)]^2}.$$

Both representations (36) and (37) are believed to be new. They have been verified by termwise integration of the series expansions of hypergeometric functions occurring in the integrands and comparing coefficients at powers of  $(-z)$ . According to [PBM, formula 7.2.4.74] the function  $F_3(1 - 1/s, 1 - s)$  encountered in (37) can be expressed as the sum of three  ${}_3F_2(s)$ .

**5. Open questions and conjectures.** The cases  $q = 1, 2, 3$  leave little doubt that the function  $g((a_q); (b_q); s)$  defined by (8) can be expressed in terms of multiple hypergeometric functions for all  $q$ . The restrictions on parameters in Lemma 1 are rooted in the definition  $g$  as the integral (8). This observations motivate

**Open problem 1:** How to express the function  $g((a_q); (b_q); s)$  defined by (8) in terms of multivariate hypergeometric functions for  $q > 3$ ? How to extend the validity of Lemma 1 to wider range of parameter values using the analytic continuation of  $g((a_q); (b_q); s)$ ?

Numerical experiments show that the condition  $b_k > a_k > 0$  is sufficient but in no way necessary to the validity of Theorem 1. Hence, our

**Open problem 2:** How to relax the restrictions on parameters in Theorem 1?

Numeric tests also indicate clearly the following

**Conjecture 1.** Theorem 3 is true for all  $\sigma > 0$  and  $\sum b_k > \sum a_k$ .

Asymptotic formula (16) and numerical experiments suggest the following

**Conjecture 2.** For  $0 < \sigma \leq \min(a_1, a_2, \dots, a_q)$ ,  $\sum_{i=1}^q (b_i - a_i) > 0$  and  $x > 0$ :

$${}_{q+1}F_q(\sigma, (a_q); (b_q); -x) < \frac{1}{\left(1 + x \prod_{i=1}^q \frac{\Gamma(a_i)\Gamma(b_i-\sigma)}{\Gamma(b_i)\Gamma(a_i-\sigma)}\right)^\sigma}. \quad (38)$$

Combined with (28) Thomae's first relation (30) leads to the following curious identity

$$\begin{aligned} & \int_0^1 t^{b-1}(1-t)^{d+e-a-b-c-1} {}_2F_1(e-c, d-c; d+e-b-c; 1-t) dt \\ &= \frac{\Gamma(b)\Gamma(c)\Gamma(d+e-a-b-c)}{\Gamma(a)\Gamma(d-a)\Gamma(e-a)} \int_0^1 t^{e-a-1}(1-t)^{a-1} {}_2F_1(e-c, e-b; d+e-b-c; 1-t) dt. \end{aligned}$$

**Open problem 3:** How can one derive the above identity directly from the properties of  ${}_2F_1$ ? Such derivation would immediately give another proof for the first fundamental relation of Thomae.

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