

APPROXIMATING THE RIEMANN-STIELTJES INTEGRAL VIA SOME MOMENTS OF THE INTEGRAND

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ABSTRACT. Error bounds in approximating the Riemann-Stieltjes integral in terms of some moments of the integrand are given. Applications for p -convex functions and in approximating the Finite Fourier Transform are pointed out as well.

1. INTRODUCTION

In order to approximate the Riemann-Stieltjes integral $\int_a^b f(t) du(t)$ with the arguably simpler expression

$$(1.1) \quad \frac{u(b) - u(a)}{b - a} \cdot \int_a^b f(t) dt,$$

where $\int_a^b f(t) dt$ is the Riemann integral, Dragomir and Fedotov [8] considered in 1998 the following *Grüss type error functional*:

$$(1.2) \quad D(f, u; a, b) := \int_a^b f(t) du(t) - \frac{1}{b - a} [u(b) - u(a)] \int_a^b f(t) dt.$$

If the *integrand* f is Riemann integrable and $-\infty < m \leq f(t) \leq M < \infty$ for any $t \in [a, b]$ while the *integrator* u is L -Lipschitzian, namely,

$$(1.3) \quad |u(t) - u(s)| \leq L|t - s| \quad \text{for each } t, s \in [a, b],$$

then the Riemann-Stieltjes integral $\int_a^b f(t) du(t)$ exists and the following bound holds:

$$(1.4) \quad |D(f, u; a, b)| \leq \frac{1}{2}L(M - m)(b - a).$$

In (1.4) the constant $\frac{1}{2}$ is best possible in the sense that it cannot be replaced by a smaller quantity.

A different bound for the Grüss error functional $D(f, u; a, b)$ in the case that f is K -Lipschitzian and u is of bounded variation has been obtained by the same authors in 2001, see [9], where they showed that

$$(1.5) \quad |D(f, u; a, b)| \leq \frac{1}{2}K(b - a) \bigvee_a^b(u).$$

Here $\bigvee_a^b(u)$ denotes the total variation of u on $[a, b]$. The constant $\frac{1}{2}$ is also best possible.

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For other results concerning different bounds for the functional $D(f, u; a, b)$ under various assumptions on f and u , see the recent papers [3], [4], [6] – [7], [13] and the references therein.

The main aim of the present paper is to provide error bounds in approximating the Riemann-Stieltjes integral $\int_a^b f(t) du(t)$ with the following expression containing moments of the function f , namely the expression

$$\frac{p}{(b-a)^p} \left[u(b) \cdot \int_a^b (t-a)^{p-1} f(t) dt - u(a) \cdot \int_a^b (b-t)^{p-1} f(t) dt \right],$$

where $p > 0$ and the involved integrals exist.

Some inequalities for monotonic integrands and p -convex integrators as well as where u is an integral of a given weight are provided. An application for approximating the Finite Fourier Transform is also given.

The case $p = 1$ reduces to the Grüss error functional and in this way some earlier results are recaptured as well.

2. GENERAL RESULTS

The following identity holds.

Lemma 1. *Let $f, u : [a, b] \rightarrow \mathbb{R}$ such that the Riemann-Stieltjes integral $\int_a^b f(t) du(t)$ and the Riemann integrals $\int_a^b (t-a)^{p-1} f(t) dt$, $\int_a^b (b-t)^{p-1} f(t) dt$ for $p > 0$ exist. Then*

$$\begin{aligned} (2.1) \quad & \int_a^b f(t) du(t) \\ &= \frac{p}{(b-a)^p} \left[u(b) \cdot \int_a^b (t-a)^{p-1} f(t) dt - u(a) \cdot \int_a^b (b-t)^{p-1} f(t) dt \right] \\ & \quad + \int_a^b \left[\frac{(t-a)^p u(b) + (b-t)^p u(a)}{(b-a)^p} - u(t) \right] df(t). \end{aligned}$$

Proof. Integrating by parts of the Riemann-Stieltjes integral, we have

$$\begin{aligned} & \int_a^b \left[\frac{(t-a)^p u(b) + (b-t)^p u(a)}{(b-a)^p} - u(t) \right] df(t) \\ &= \left[\frac{(t-a)^p u(b) + (b-t)^p u(a)}{(b-a)^p} - u(t) \right] f(t) \Big|_a^b \\ & \quad - \int_a^b f(t) d \left[\frac{(t-a)^p u(b) + (b-t)^p u(a)}{(b-a)^p} - u(t) \right] \\ &= [u(b) - u(a)] f(b) - [u(a) - u(a)] f(a) \\ & \quad - \left[\frac{pu(b)}{(b-a)^p} \int_a^b (t-a)^{p-1} f(t) dt \right. \\ & \quad \left. - \frac{pu(a)}{(b-a)^p} \int_a^b (b-t)^{p-1} f(t) dt - \int_a^b f(t) du(t) \right] \end{aligned}$$

$$\begin{aligned}
 &= -\frac{pu(b)}{(b-a)^p} \int_a^b (t-a)^{p-1} f(t) dt + \frac{pu(a)}{(b-a)^p} \int_a^b (b-t)^{p-1} f(t) dt \\
 &\quad + \int_a^b f(t) du(t),
 \end{aligned}$$

which is equivalent with the desired identity (2.1). \square

Remark 1. For $p = 1$ we get the identity:

$$\begin{aligned}
 (2.2) \quad \int_a^b f(t) du(t) &= \frac{u(b) - u(a)}{b-a} \cdot \int_a^b f(t) dt \\
 &\quad + \int_a^b \left[\frac{(t-a)u(b) + (b-t)u(a)}{b-a} - u(t) \right] df(t)
 \end{aligned}$$

that has been obtained in [5], see also [6].

In order to approximate the Riemann-Stieltjes integral $\int_a^b f(t) du(t)$ by the quadrature

$$(2.3) \quad \frac{p}{(b-a)^p} \left[u(b) \cdot \int_a^b (t-a)^{p-1} f(t) dt - u(a) \cdot \int_a^b (b-t)^{p-1} f(t) dt \right],$$

we consider the error functional:

$$\begin{aligned}
 (2.4) \quad F(f, u, p; a, b) &:= \int_a^b f(t) du(t) \\
 &\quad - \frac{p}{(b-a)^p} \left[u(b) \cdot \int_a^b (t-a)^{p-1} f(t) dt - u(a) \cdot \int_a^b (b-t)^{p-1} f(t) dt \right].
 \end{aligned}$$

The following result may be stated.

Theorem 1. Let $f, u : [a, b] \rightarrow \mathbb{R}$ be as in Lemma 1. For $p > 0$, define

$$(2.5) \quad \Delta_p(u; t, a, b) := \frac{(t-a)^p u(b) + (b-t)^p u(a)}{(b-a)^p} - u(t),$$

where $t \in [a, b]$.

If $F(f, u, p; a, b)$ is the error functional defined by (2.4), then:

$$(2.6) \quad |F(f, u, p; a, b)| \leq \begin{cases} \sup_{t \in [a, b]} |\Delta_p(u; t, a, b)| V_a^b(f) \\ \quad \text{if } f \text{ is of bounded variation;} \\ L \int_a^b |\Delta_p(u; t, a, b)| dt \\ \quad \text{if } f \text{ is } L\text{-Lipschitzian;} \\ \int_a^b |\Delta_p(u; t, a, b)| df(t) \\ \quad \text{if } f \text{ is monotonic nondecreasing.} \end{cases}$$

Proof. It is well known that for the Riemann-Stieltjes integral $\int_a^b w(t) dv(t)$ we have the bounds

$$(2.7) \quad \left| \int_a^b w(t) dv(t) \right| \leq \begin{cases} \sup_{t \in [a,b]} |w(t)| \bigvee_a^b(v) & \text{if } v \text{ is of bounded variation;} \\ L \int_a^b |w(t)| dt & \text{if } v \text{ is } L\text{-Lipschitzian;} \\ \int_a^b |w(t)| dv(t) & \text{if } v \text{ is monotonic nondecreasing.} \end{cases}$$

Now, on utilising the representation (2.1) and applying (2.7) for $w(t) := \Delta_p(u; t, a, b)$, $t \in [a, b]$ and $v = f$, we deduce the desired result. \square

Remark 2. For $p = 1$, by denoting

$$\Delta(u; t, a, b) = \Delta_1(u; t, a, b) = \frac{(t-a)u(b) + (b-t)u(a)}{b-a} - u(t)$$

and

$$F(f, u; a, b) = \int_a^b f(t) du(t) - \frac{u(b) - u(a)}{b-a} \cdot \int_a^b f(t) dt,$$

we get from (2.6)

$$(2.8) \quad |F(f, u; a, b)| \leq \begin{cases} \sup_{t \in [a,b]} |\Delta(u; t, a, b)| \bigvee_a^b(f) & \text{if } f \text{ is of bounded variation;} \\ L \int_a^b |\Delta(u; t, a, b)| dt & \text{if } f \text{ is } L\text{-Lipschitzian;} \\ \int_a^b |\Delta(u; t, a, b)| df(t) & \text{if } f \text{ is monotonic nondecreasing.} \end{cases}$$

The inequality (2.8) has been obtained in [5].

Remark 3. If $u(t) = \int_a^t w(s) ds$, $t \in [a, b]$, then from (2.1) we get the representation

$$(2.9) \quad \int_a^b f(t) w(t) dt = \frac{p}{(b-a)^p} \int_a^b w(s) ds \int_a^b (t-a)^{p-1} f(t) dt \\ + \frac{1}{(b-a)^p} \cdot \int_a^b \left[(t-a)^p \int_a^b w(s) ds - (b-a)^p \int_a^t w(s) ds \right] df(t)$$

for any $p > 0$, provided that the involved integrals exist.

For $p = 1$, we obtain the identity due to Cerone in [2].

3. FURTHER BOUNDS FOR MONOTONIC INTEGRANDS

In this section some bounds for the error functional $F(f, u, p; a, b)$ where the integrator f is monotonic nondecreasing are given.

Theorem 2. Let $f, u : [a, b] \rightarrow \mathbb{R}$ be such that f is monotonic nondecreasing, u satisfies the bounds:

$$(3.1) \quad -\infty < n \leq u(t) \leq N < \infty \quad \text{for any } t \in [a, b]$$

and the Riemann-Stieltjes integral $\int_a^b f(t) du(t)$ exists. Then

$$(3.2) \quad \begin{aligned} & \frac{np}{(b-a)^p} \left[\int_a^b (b-t)^{p-1} f(t) dt - \int_a^b (t-a)^{p-1} f(t) dt \right] \\ & \quad - (N-n) [f(b) - f(a)] \\ & \leq F(f, u, p; a, b) \\ & \leq \frac{Np}{(b-a)^p} \left[\int_a^b (b-t)^{p-1} f(t) dt - \int_a^b (t-a)^{p-1} f(t) dt \right] \\ & \quad - (N-n) [f(b) - f(a)], \end{aligned}$$

where $F(f, u, p; a, b)$ is given by (2.4).

Proof. From (3.1) we obviously have:

$$\begin{aligned} n(t-a)^p &\leq u(b) \leq N(t-a)^p, \\ n(b-t)^p &\leq u(a) \leq N(b-t)^p, \\ -N(b-a)^p &\leq -u(t)(b-a)^p \leq -n(b-a)^p \end{aligned}$$

for any $t \in [a, b]$. Summing the above three inequalities, we have that

$$n \cdot \frac{(t-a)^p + (b-t)^p}{(b-a)^p} - N \leq \Delta_p(u; t, a, b) \leq N \cdot \frac{(t-a)^p + (b-t)^p}{(b-a)^p} - n$$

for any $t \in [a, b]$.

Now, integrating over the monotonic nondecreasing function f we have

$$(3.3) \quad \begin{aligned} & \frac{n}{(b-a)^p} \left[\int_a^b (t-a)^p df(t) + \int_a^b (b-t)^p df(t) \right] - N [f(b) - f(a)] \\ & \leq F(f, u, p; a, b) \\ & \leq \frac{N}{(b-a)^p} \left[\int_a^b (t-a)^p df(t) + \int_a^b (b-t)^p df(t) \right] - n [f(b) - f(a)]. \end{aligned}$$

Integrating by parts, we also have

$$\int_a^b (t-a)^p df(t) = (b-a)^p f(b) - p \int_a^b (t-a)^{p-1} f(t) dt$$

and

$$\int_a^b (b-t)^p df(t) = -(b-a)^p f(a) + p \int_a^b (b-t)^{p-1} f(t) dt.$$

Then

$$\begin{aligned}
(3.4) \quad & \frac{n}{(b-a)^p} \left[\int_a^b (t-a)^p df(t) + \int_a^b (b-t)^p df(t) \right] - N[f(b) - f(a)] \\
&= \frac{n}{(b-a)^p} \left\{ (b-a)^p [f(b) - f(a)] + p \left[\int_a^b (b-t)^{p-1} f(t) dt \right. \right. \\
&\quad \left. \left. - \int_a^b (t-a)^{p-1} f(t) dt \right] \right\} - N[f(b) - f(a)] \\
&= \frac{np}{(b-a)^p} \left[\int_a^b (b-t)^{p-1} f(t) dt - \int_a^b (t-a)^{p-1} f(t) dt \right] \\
&\quad - (N-n)[f(b) - f(a)]
\end{aligned}$$

and

$$\begin{aligned}
(3.5) \quad & \frac{N}{(b-a)^p} \left[\int_a^b (t-a)^p df(t) + \int_a^b (b-t)^p df(t) \right] - n[f(b) - f(a)] \\
&= \frac{N}{(b-a)^p} \left\{ (b-a)^p [f(b) - f(a)] + p \left[\int_a^b (b-t)^{p-1} f(t) dt \right. \right. \\
&\quad \left. \left. - \int_a^b (t-a)^{p-1} f(t) dt \right] \right\} - n[f(b) - f(a)] \\
&= \frac{Np}{(b-a)^p} \left[\int_a^b (b-t)^{p-1} f(t) dt - \int_a^b (t-a)^{p-1} f(t) dt \right] \\
&\quad + (N-n)[f(b) - f(a)].
\end{aligned}$$

Now, on utilising (3.3) – (3.5) we deduce the desired result (3.2). \square

Remark 4. In the particular case when $p = 1$, the inequality (3.2) reduces to

$$(3.6) \quad |F(f, u, p; a, b)| \leq (N-n)[f(b) - f(a)].$$

4. AN INEQUALITY FOR INTEGRATORS THAT ARE s -CONVEX IN THE SECOND SENSE

Following Hudzik and Maligranda [12] (see also [11, p. 286]) we say that the function $g : \mathbb{R}_+ \rightarrow \mathbb{R}$ is p -convex in the second sense, where $p > 0$ is fixed, if:

$$(4.1) \quad g(tu + (1-t)v) \leq t^p g(u) + (1-t)^p g(v)$$

for any $u, v \geq 0$ and $t \in [0, 1]$.

For different properties of this class of functions, see [12] and [11, pp. 286 – 293].

The following inequality of Hermite-Hadamard type is due to Dragomir and Fitzpatrick [10]:

Theorem 3. Let g be a p -convex function in the second sense on an interval $I \subset [0, \infty)$ with $p \in (0, 1]$ and let $a, b \in I$ with $a < b$. Then:

$$(4.2) \quad 2^{p-1} g\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b g(t) dt \leq \frac{g(a) + g(b)}{p+1}.$$

We can state and prove now the following result about the Riemann-Stieltjes integral:

Theorem 4. *Let u be p -convex with $p > 0$, f be monotonic nondecreasing on $[a, b]$ and such that the Riemann-Stieltjes integral $\int_a^b f(t) du(t)$ and the Riemann integrals $\int_a^b (t-a)^{p-1} f(t) dt$, $\int_a^b (b-t)^{p-1} f(t) dt$ exist. Then*

$$(4.3) \quad \int_a^b f(t) du(t) \geq \frac{p}{(b-a)^p} \left[u(b) \int_a^b (t-a)^{p-1} f(t) dt - u(a) \int_a^b (b-t)^{p-1} f(t) dt \right].$$

Proof. Since u is p -convex, then:

$$u(t) = u\left(\frac{t-a}{b-a} \cdot b + \frac{b-t}{b-a} \cdot a\right) \leq \left(\frac{t-a}{b-a}\right)^p u(b) + \left(\frac{b-t}{b-a}\right)^p u(a),$$

which shows, upon using the notations of (2.5), that

$$\Delta_p(u, t; a, b) \geq 0 \quad \text{for any } t \in [a, b].$$

Since f is monotonic nondecreasing on $[a, b]$, we have then

$$\int_a^b \Delta_p(u, t; a, b) df(t) \geq 0,$$

which, via the representation (2.1), is equivalent with the desired inequality (4.3). \square

Remark 5. *The case $p = 1$, i.e., where the function u is convex in the usual sense, produces the following inequality*

$$(4.4) \quad \int_a^b f(t) du(t) \geq \frac{u(b) - u(a)}{b-a} \int_a^b f(t) dt,$$

where f is monotonic nondecreasing, which has been obtained in [5], see also [6].

5. APPROXIMATING THE FINITE FOURIER TRANSFORM

The *Fourier Transform* is one of the most important mathematical tools in a wide variety of fields in science and engineering [1, p. xi].

Throughout this section $f : [a, b] \rightarrow \mathbb{R}$ will be a Riemann integrable function defined on the finite interval $[a, b]$ and $\mathcal{F}(g)$ will be its *Finite Fourier Transform*. That is,

$$\mathcal{F}(f)(t) := \int_a^b f(s) e^{-2\pi its} ds.$$

Consider also the *exponential mean* of two complex numbers z, w defined by

$$E(z, w) := \begin{cases} \frac{e^z - e^w}{z - w} & \text{if } z \neq w, \\ \exp(w) & \text{if } z = w, \end{cases} \quad z, w \in \mathbb{C}.$$

Now, for $w(s) = \exp(-2\pi its)$, on applying the identity (2.9), which holds for complex-valued functions as well, we get the following representation of the Finite Fourier Transform

$$(5.1) \quad \mathcal{F}(f)(t) = \frac{p}{(b-a)^{p-1}} \cdot E(-2\pi itb, -2\pi ita) \cdot \int_a^b (s-a)^{p-1} f(s) ds + \mathcal{R}(f)(t),$$

where the remainder $\mathcal{R}(f)$ has the representation

$$\begin{aligned} \mathcal{R}(f)(t) = & \frac{1}{(b-a)^{p-1}} \int_a^b (s-a) \left[(s-a)^{p-1} E(-2\pi itb, -2\pi ita) \right. \\ & \left. - (b-a)^{p-1} E(-2\pi its, -2\pi ita) \right] df(s). \end{aligned}$$

In order to provide a composite rule in approximating the Finite Fourier Transform in terms of moments for the function f , we consider a division $I_n : a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$ and the quadrature rule

$$(5.2) \quad A(I_n, t) : = \sum_{i=0}^{n-1} \frac{p}{(x_{i+1} - x_i)^{p-1}} \cdot E(-2\pi itx_{i+1}, -2\pi itx_i) \times \int_{x_i}^{x_{i+1}} (s-x_i)^{p-1} f(s) ds$$

that has been obtained from (5.1) applied on each subinterval $[x_i, x_{i+1}]$ and the results were summed over i from 0 to $n-1$. It is an open question as to whether or not $A(I_n, t)$ is uniformly convergent to $\mathcal{F}(f)(t)$ on $[a, b]$ and what the order of convergence is?

The following numerical experiment obtained by implementing the quadrature (5.2) for the function $f(s) = s + 1$ and $p = 2$ shows the behavior of the absolute error value

$$(5.3) \quad E_n(t) := |\mathcal{F}(f)(t) - A(I_n, t)|, t \in [a, b],$$

for a division with 10 points (Figure 1) respectively 100 points (Figure 2).

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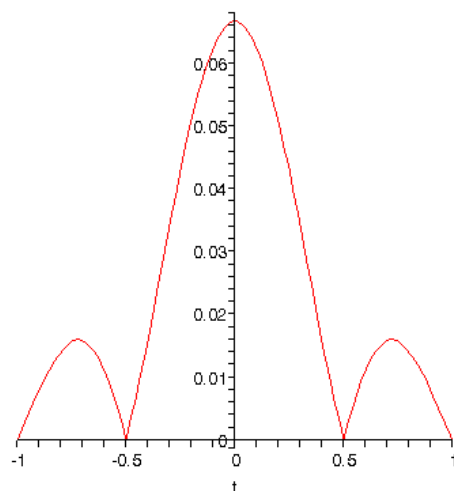


FIGURE 1. The behaviour of $E_n(t)$ from (5.3) for $n = 10$.

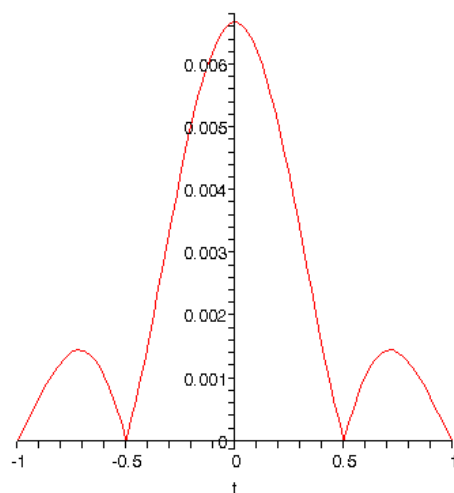


FIGURE 2. The behaviour of $E_n(t)$ from (5.3) for $n = 100$.

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