APPROXIMATING THE STIELTJES INTEGRAL FOR (φ,Φ)-LIPSCHITZIAN INTEGRATORS AND APPLICATIONS

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Abstract. Approximations for the Stieltjes integral with (φ,Φ) $-$Lipschitzian integrators are given. Applications for the Riemann integral of a product and for the generalised trapezoid and Ostrowski inequalities are also provided.

1. Introduction

One can approximate the *Stieltjes integral* $\int_a^b f(t)\,du(t)$ with the following simpler quantities:

$$\frac{1}{b-a}[u(b)-u(a)]\cdot\int_a^b f(t)\,dt \qquad ([17],\ [18]) \tag{1.1}$$

$$f(x)[u(b)-u(a)] \qquad ([10],\ [11]) \tag{1.2}$$

or with

$$[u(b)-u(x)]f(b)+[u(x)-u(a)]f(a) \qquad ([16]), \tag{1.3}$$

where $x\in[a,b]$.

In order to provide *a priori* sharp bounds for the *approximation error*, consider the functionals:

$$D(f,u;a,b):=\int_a^b f(t)\,du(t)-\frac{1}{b-a}[u(b)-u(a)]\cdot\int_a^b f(t)\,dt,$$

$$\Theta(f,u;a,b,x):=\int_a^b f(t)\,du(t)-f(x)[u(b)-u(a)]$$

and

$$T(f,u;a,b,x):=\int_a^b f(t)\,du(t)-[u(b)-u(x)]f(b)-[u(x)-u(a)]f(a).$$

If the *integrand* f is *Riemann integrable* on $[a,b]$ and the *integrator* $u:[a,b]\to\mathbb{R}$ is $L-$*Lipschitzian*, i.e.,

$$|u(t)-u(s)|\le L|t-s| \qquad \text{for each } t,s\in[a,b], \tag{1.4}$$

then the Stieltjes integral $\int_a^b f(t)\,du(t)$ exists and, as pointed out in [17],

$$|D(f,u;a,b)|\le L\int_a^b\left|f(t)-\int_a^b\frac{1}{b-a}f(s)\,ds\right|dt. \tag{1.5}$$

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The inequality (1.5) is sharp in the sense that the multiplicative constant $C=1$ in front of L cannot be replaced by a smaller quantity. Moreover, if there exists the constants $m,M\in\mathbb{R}$ such that $m\leq f(t)\leq M$ for a.e. $t\in[a,b]$, then [17]

$$|D(f,u;a,b)|\leq\frac{1}{2}L(M-m)(b-a). \tag{1.6}$$

The constant $\frac{1}{2}$ is best possible in (1.6).

A different approach in the case of integrands of bounded variation were considered by the same authors in 2001, [18], where they showed that

$$|D(f,u;a,b)|\leq\max_{t\in[a,b]}\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|\bigvee_a^b(u), \tag{1.7}$$

provided that f is continuous and u is of bounded variation. Here $\bigvee_a^b(u)$ denotes the total variation of u on $[a,b]$. The inequality (1.7) is sharp.

If we assume that f is $K-$Lipschitzian, then [18]

$$|D(f,u;a,b)|\leq\frac{1}{2}K(b-a)\bigvee_a^b(u), \tag{1.8}$$

with $\frac{1}{2}$ the best possible constant in (1.8).

For various bounds on the error functional $D(f,u;a,b)$ where f and u belong to different classes of function for which the Stieltjes integral exists, see [15], [14], [13], and [7] and the references therein.

For the functional $\theta(f,u;a,b,x)$ we have the bound [10]:

$$\begin{aligned}
&|\theta(f,u;a,b,x)| \\
&\leq H\left[(x-a)^r\bigvee_a^x(f)+(b-x)^r\bigvee_x^b(f)\right] \\
&\leq H\times\begin{cases}
[(x-a)^r+(b-x)^r]\left[\frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right]; \\
[(x-a)^{qr}+(b-x)^{qr}]^{\frac{1}{q}}\left[\left(\bigvee_a^x(f)\right)^p+\left(\bigvee_x^b(f)\right)^p\right]^{\frac{1}{p}} \\
\qquad\qquad\text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\
\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^r\bigvee_a^b(f),
\end{cases}
\end{aligned} \tag{1.9}$$

provided f is of bounded variation and u is of $r-H-$*Hölder type*, i.e.,

$$|u(t)-u(s)|\leq H|t-s|^r \qquad \text{for each } \ t,s\in[a,b], \tag{1.10}$$

with given $H>0$ and $r\in(0,1]$.

If f is of $q-K-$Hölder type and u is of bounded variation, then [11]

$$|\theta(f,u;a,b,x)|\leq K\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^q\bigvee_a^b(u), \tag{1.11}$$

for any $x\in[a,b]$.

If u is monotonic nondecreasing and f of $q-K-$Hölder type, then the following refinement of (1.11) also holds [7]:

$$|\theta(f,u;a,b,x)| \leq K\Bigg[(b-x)^q u(b)-(x-a)^q u(a) + q\left\{\int_a^x \frac{u(t)\,dt}{(x-t)^{1-q}} - \int_x^b \frac{u(t)\,dt}{(t-x)^{1-q}}\right\}\Bigg] \leq K\left[(b-x)^q[u(b)-u(x)]+(x-a)^q[u(x)-u(a)]\right] \leq K\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^q [u(b)-u(a)], \tag{1.12}$$

for any $x\in[a,b]$.

If f is monotonic nondecreasing and u is of $r-H-$Hölder type, then [7]:

$$|\theta(f,u;a,b,x)| \leq H\Bigg[[(x-a)^r-(b-x)^r]f(x) + r\left\{\int_a^x \frac{f(t)\,dt}{(b-t)^{1-r}} - \int_x^b \frac{f(t)\,dt}{(t-r)^{1-r}}\right\}\Bigg] \leq H\left\{(b-x)^r[f(b)-f(x)]+(x-a)^r[f(x)-f(a)]\right\} \leq H\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^r [f(b)-f(a)], \tag{1.13}$$

for any $x\in[a,b]$.

The error functional $T(f,u;a,b,x)$ satisfies similar bounds, see [16], [7], [2] and [1] and the details are omitted.

The main aim of this paper is to provide a different approximation of the Stieltjes integral $\int_a^b f(t)\,du(t)$ in terms of the simpler quantity

$$\frac{\varphi+\Phi}{2}\int_a^b f(t)\,dt$$

provided that the integrator u is $(\varphi,\Phi)-$Lipschitzian on $[a,b]$.

Applications for the Riemann integral of a product of two functions and for the generalised trapezoid and Ostrowski inequalities are also provided.

2. $(\varphi,\Phi)-$Lipschitzian Functions

We say that the function $v:[a,b]\to\mathbb{R}$ is $L-$*Lipschitzian* on $[a,b]$ if

$$|v(t)-v(s)|\leq L|t-s| \quad \text{for any } t,s\in[a,b], \tag{2.1}$$

where $L>0$ is a given constant.

The following lemma may be stated.

Lemma 1. *Let $u:[a,b]\to\mathbb{R}$ and $\varphi,\Phi\in\mathbb{R}$ with $\Phi>\varphi$. The following statements are equivalent:*

(i) *The function $u-\frac{\varphi+\Phi}{2}\cdot e$, where $e(t)=t$, $t\in[a,b]$, is $\frac{1}{2}(\Phi-\varphi)-$Lipschitzian;*

(ii) *We have the inequality:*

$$\varphi \leq \frac{u(t)-u(s)}{t-s} \leq \Phi \quad \text{for each} \quad t,s \in [a,b] \quad \text{with } t \neq s; \tag{2.2}$$

(iii) *We have the inequality:*

$$\varphi(t-s) \leq u(t)-u(s) \leq \Phi(t-s) \quad \text{for each} \quad t,s \in [a,b] \quad \text{with } t > s. \tag{2.3}$$

Following [19], we can introduce the concept:

Definition 1. *The function $u : [a,b] \to \mathbb{R}$ which satisfies one of the equivalent conditions (i) – (iii) is said to be $(\varphi, \Phi)-$Lipschitzian on $[a,b]$.*

Notice that in [19], the definition was introduced on utilising the statement (iii) and only the equivalence (i) ⇔ (iii) was considered.

Utilising *Lagrange's mean value theorem*, we can state the following result that provides practical examples of $(\varphi, \Phi)-$Lipschitzian functions.

Proposition 1. *Let $u : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). If*

$$-\infty < \gamma := \inf_{t\in(a,b)} u'(t), \qquad \sup_{t\in(a,b)} u'(t) =: \Gamma < \infty \tag{2.4}$$

then u is $(\gamma, \Gamma)-$Lipschitzian on $[a,b]$.

3. Inequalities for Stieltjes Integrals

The following result may be stated.

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ be Riemann integrable on $[a,b]$, $\varphi, \Phi \in \mathbb{R}$ with $\Phi > \varphi$ and $u : [a,b] \to \mathbb{R}$ a $(\varphi, \Phi)-$Lipschitzian function on $[a,b]$. Then the Stieltjes integral $\int_a^b f(t)\,du(t)$ exists and defining the functional*

$$\Sigma(f,u,\varphi,\Phi;a,b) := \int_a^b f(t)\,du(t) - \frac{\varphi+\Phi}{2}\cdot\int_a^b f(t)\,dt$$

we have

$$|\Sigma(f,u,\varphi,\Phi;a,b)| \leq \frac{1}{2}(\Phi-\varphi)\int_a^b |f(t)|\,dt. \tag{3.1}$$

The constant $\frac{1}{2}$ is best possible in (3.1).

Proof. It is known that if $p : [a,b] \to \mathbb{R}$ is a Riemann integrable function and $v : [a,b] \to \mathbb{R}$ is $L-$Lipschitzian, then the Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and

$$\left|\int_a^b p(t)\,dv(t)\right| \leq L\int_a^b |p(t)|\,dt. \tag{3.2}$$

Since φ, Φ are finite, we can find a positive L such that $-L < \varphi < \Phi < L$ and by (2.2) we deduce that u is $L-$Lipschitzian. Therefore the Stieltjes integral exists and by (3.2) we have

$$\left|\int_a^b f(t)\,d\left(u(t) - \frac{\varphi+\Phi}{2}\cdot t\right)\right| \leq \frac{1}{2}(\Phi-\varphi)\int_a^b |f(t)|\,dt. \tag{3.3}$$

Since

$$\int_a^b f(t)\,d\left(u(t)-\frac{\varphi+\Phi}{2}\cdot t\right)=\int_a^b f(t)\,du(t)-\frac{\varphi+\Phi}{2}\int_a^b f(t)\,dt,$$

hence by (3.3) we deduce (3.1).

To prove the sharpness of the constant $\frac{1}{2}$, assume that the inequality (3.1) holds with a constant $C>0$, i.e.,

$$|\Sigma(f,u,\varphi,\Phi;a,b)|\leq C(\Phi-\varphi)\int_a^b |f(t)|\,dt, \tag{3.4}$$

provided f is Riemann integrable on $[a,b]$ and u is (φ,Φ) $-$Lipschitzian.

Consider the function $u(t):=\left|t-\frac{a+b}{2}\right|$. By the triangle inequality we have

$$|u(t)-u(s)|=\left|\left|t-\frac{a+b}{2}\right|-\left|\frac{a+b}{2}-s\right|\right|\leq |t-s| \quad \text{for each} \quad t,s\in[a,b],$$

which shows that u is $L-$Lipschitzian with $L=1$ or (φ,Φ) $-$Lipschitzian with $\varphi=-1,\Phi=1$.

For a Riemann integrable function $f:[a,b]\to\mathbb{R}$ we then have

$$\begin{aligned}\int_a^b f(t)\,du(t)&=\int_a^{\frac{a+b}{2}} f(t)\,d\left(\frac{a+b}{2}-t\right)+\int_{\frac{a+b}{2}}^b f(t)\,d\left(t-\frac{a+b}{2}\right)\\&=-\int_a^{\frac{a+b}{2}} f(t)\,dt+\int_{\frac{a+b}{2}}^b f(t)\,dt\\&=\int_a^b \operatorname{sgn}\left(t-\frac{a+b}{2}\right)f(t)\,dt.\end{aligned}$$

If $g:[a,b]\to\mathbb{R}$ is Riemann integrable and nonnegative a.e. on $[a,b]$ and if we choose $f(t)=\operatorname{sgn}\left(t-\frac{a+b}{2}\right)g(t)$, $t\in[a,b]$, then

$$\int_a^b f(t)\,du(t)=\int_a^b g(t)\,dt>0,$$
$$\int_a^b |f(t)|\,dt=\int_a^b g(t)\,dt$$

and by (3.4) we deduce that

$$\int_a^b g(t)\,dt\leq 2C\int_a^b g(t)\,dt,$$

which implies that $C\geq\frac{1}{2}$. ∎

Corollary 1. *Let $g:[a,b]\to\mathbb{R}$ be a Riemann integrable function on $[a,b]$ and $u:[a,b]\to\mathbb{R}$ a (φ,Φ) $-$Lipschitzian function on $[a,b]$. Then*

$$|D(f,u;a,b)|\leq\frac{1}{2}(\Phi-\varphi)\int_a^b\left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|dt. \tag{3.5}$$

The constant $\frac{1}{2}$ is best possible in (3.5).

Remark 1. *The inequality (3.5) has been obtained by Z. Liu in* [19]*, from which, in the case of usual Lipschitzian functions, one recaptures the result of Dragomir and Fedotov from* [17]*:*

$$|D(f,u;a,b)| \leq L\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| dt. \tag{3.6}$$

The following particular case of Theorem 1 is also of interest.

Corollary 2. *Let $g:[a,b]\to\mathbb{R}$ be a Riemann integrable function on $[a,b]$ such that*

$$-\infty < m \leq g(t) \leq M < \infty \quad \textit{for a.e.} \quad t\in[a,b]. \tag{3.7}$$

If $u:[a,b]\to\mathbb{R}$ is (φ,Φ) −Lipschitzian on $[a,b]$, then

$$\begin{aligned} &\left| \int_a^b g(t)\,du(t) - \frac{\varphi+\Phi}{2}\int_a^b g(t)\,dt \right. \\ &\qquad \left. - \frac{m+M}{2}[u(b)-u(a)] + \frac{(\varphi+\Phi)(m+M)}{4}(b-a) \right| \\ &\leq \frac{1}{2}(\Phi-\varphi)\int_a^b \left| g(t) - \frac{m+M}{2}\right| dt \\ &\leq \frac{1}{4}(M-m)(\Phi-\varphi)(b-a). \end{aligned} \tag{3.8}$$

The constants $\frac{1}{2}$ and $\frac{1}{4}$ are best possible in (3.8).

Proof. The first inequality in (3.8) follows directly from Theorem 1 on choosing $f(t) = g(t) - \frac{m+M}{2}$, $t\in[a,b]$.

The second inequality in (3.8) is obvious by the fact that

$$\left| g(t) - \frac{m+M}{2} \right| \leq \frac{1}{2}(M-m) \quad \text{for a.e.} \quad t\in[a,b].$$

Now, for the sharpness on the constants, if we choose $u(t) = \left|t-\frac{a+b}{2}\right|$, $t\in[a,b]$, then u is $(-1,1)$ −Lipschitzian on $[a,b]$, $u(a)=u(b)=(b-a)/2$ and the left side of (3.8) reduces to

$$\left| \int_a^b g(t)\,du(t) \right| = \left| \int_a^b \operatorname{sgn}\left(t-\frac{a+b}{2}\right) g(t)\,dt \right|.$$

If we choose $g(t) = \operatorname{sgn}\left(t-\frac{a+b}{2}\right)h(t)$ with $h:[a,b]\to\mathbb{R}$ a Riemann integrable function with the properties:

$$0\leq h(t)\leq 1 \quad \text{for a.e.} \quad t\in[a,b] \quad \text{and} \quad \int_a^b h(t)\,dt = b-a$$

(for instance $h(t)=1$, $t\in[a,b]$), then g is bounded above by $M=1$ and below by $m=-1$,

$$\int_a^b g(t)\,du(t) = \int_a^b h(t)\,dt = b-a,$$

$$\int_a^b \left| g(t) - \frac{m+M}{2} \right| dt = \int_a^b h(t)\,dt = b-a$$

and in both sides of (3.8) we get the same quantity $b-a$. ∎

The following result of Ostrowski type can be stated as well:

Corollary 3. *Let $g : [a,b] \to \mathbb{R}$ be a Riemann integrable function and $u : [a,b] \to \mathbb{R}$ a (φ, Φ) $-$Lipschitzian function on $[a,b]$. Then for each $x \in [a,b]$, we have the inequality:*

$$\left| \int_a^b g(t)\, du(t) - \frac{\varphi + \Phi}{2} \int_a^b g(t)\, dt - g(x) \left[u(b) - u(a) - \frac{\varphi + \Phi}{2}(b-a) \right] \right| \le \frac{1}{2}(\Phi - \varphi) \int_a^b |g(t) - g(x)|\, dt. \tag{3.9}$$

The constant $\frac{1}{2}$ is best possible in (3.9).

Proof. The inequality follows from (1.7) on choosing $f(t) = g(t) - g(x)$. For $x \in (a,b)$, define $u(t) = |t - x|$, $t \in [a,b]$. Then u is $(-1,1)$ $-$Lipschitzian and

$$\int_a^b g(t)\, du(t) = \int_a^x g(t)\, d(x-t) + \int_x^b g(t)\, d(t-x) = \int_a^b \operatorname{sgn}(t-x)\, g(t)\, dt.$$

Now, if we choose $g(t) = \operatorname{sgn}(t-x)\, h(t)$ with $h : [a,b] \to [0,\infty)$ a Riemann integrable function, then the left side of (3.9) reduces to

$$\left| \int_a^b g(t)\, du(t) \right| = \left| \int_a^b \operatorname{sgn}(t-x)\, g(t)\, dt \right| = \int_a^b h(t)\, dt.$$

Since

$$\int_a^b |g(t) - g(x)|\, dt = \int_a^b h(t)\, dt,$$

hence on both sides of (3.9) we have the same quantity $\int_a^b h(t)\, dt$. ∎

Remark 2. *If we define the function $B : [a,b] \to \mathbb{R}$ by*

$$B(x) := \int_a^b |g(t) - g(x)|\, dt,$$

then we can provide various bounds for B depending on the classes of functions g considered.

For instance, if $g : [a,b] \to \mathbb{R}$ is of $r-H-$Hölder type, where $H > 0$ and $r \in (0,1]$ are given, then

$$B(x) \le H \int_a^b |t-x|^r\, dt = \frac{H}{r+1}\left[(b-x)^{r+1} + (x-a)^{r+1} \right]. \tag{3.10}$$

If g is absolutely continuous, then $g(t) - g(x) = \int_x^t g'(s)\, ds$ and since

$$|g(t) - g(x)| = \left| \int_x^t g'(s)\, ds \right| \le \begin{cases} |t-x|\, \|g'\|_\infty & \text{if } g' \in L_\infty[a,b]; \\ |t-x|^{\frac{1}{q}}\, \|g'\|_p & \text{if } g' \in L_p[a,b], \\ & p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \|g'\|_1 \end{cases}$$

where

$$\|g'\|_{\infty} := ess \sup_{s\in[a,b]} |g'(s)|, \qquad \|g'\|_{p} := \left(\int_a^b |g'(s)|^p \, ds\right)^{\frac{1}{p}}, \quad p \geq 1,$$

hence:

$$B(x) \leq \begin{cases} \frac{1}{2}\|g'\|_{\infty}\left[(x-a)^2+(b-x)^2\right] & \text{if } g' \in L_{\infty}[a,b]; \\ \\ \frac{q}{q+1}\|g'\|_{p}\left[(b-x)^{\frac{q+1}{q}}+(x-a)^{\frac{q+1}{q}}\right] & \text{if } g' \in L_{p}[a,b], \\ & p>1, \ \frac{1}{p}+\frac{1}{q}=1; \\ \|g'\|_{1}(b-a). \end{cases} \tag{3.11}$$

If g is monotonic nondecreasing, then

$$\begin{aligned} B(x) &= \int_a^x (g(x)-g(t))\,dt + \int_x^b (g(t)-g(x))\,dt \\ &= (x-a)\,g(x) - (b-x)\,g(x) + \int_x^b g(t)\,dt - \int_a^x g(t)\,dt \\ &= [2x-(a+b)]\,g(x) + \int_a^b \operatorname{sgn}(t-x)\,g(t)\,dt. \end{aligned} \tag{3.12}$$

Also, by the monotonicity of g on $[a,b]$, we have

$$\int_x^b g(t)\,dt \leq g(b)(b-x) \quad \text{and} \quad -\int_a^x g(t)\,dt \leq -g(a)(x-a)$$

for each $x \in [a,b]$, implying that

$$\begin{aligned} B(x) &\leq (x-a)\,g(x) - (b-x)\,g(x) + g(b)(b-x) - g(a)(x-a) \\ &= (x-a)[g(x)-g(a)] + (b-x)[g(b)-g(x)] \\ &\leq \max(x-a, b-x)[g(b)-g(a)] \\ &= \left[\frac{1}{2}(b-a) + \left|x-\frac{a+b}{2}\right|\right][g(b)-g(x)]. \end{aligned} \tag{3.13}$$

Utilising the result incorporated in the equations (3.10) – (3.13), we can provide the following proposition that provides upper bounds for the absolute value of the functional

$$\begin{aligned} &\Psi(g,u;a,b,x) \\ &:= \int_a^b g(t)\,du(t) - \frac{\varphi+\Phi}{2}\int_a^b g(t)\,dt - g(x)\left[u(b)-u(a)-\frac{\varphi+\Phi}{2}(b-a)\right] \end{aligned}$$

that are coarser than the one in (3.9) but, perhaps, more useful in applications.

Proposition 2. *Let $u : [a,b] \to \mathbb{R}$ be a $(\varphi,\Phi)-$Lipschitzian function and g a Riemann integrable function on $[a,b]$.*

(i) *If $g:[a,b]\to\mathbb{R}$ is of $r-H-$Hölder type ($K-$Lipschitzian) then*

$$|\Psi(g,u;a,b,x)|\leq\frac{1}{2}(\Phi-\varphi)\frac{H}{r+1}\left[(b-x)^{r+1}+(x-a)^{r+1}\right] \tag{3.14}$$
$$\left(\leq\frac{1}{2}(\Phi-\varphi)K\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]\right),$$

for any $x\in[a,b]$;

(ii) *If $g:[a,b]\to\mathbb{R}$ is absolutely continuous on $[a,b]$, then*

$$|\Psi(g,u;a,b,x)| \tag{3.15}$$
$$\leq\frac{1}{2}(\Phi-\varphi)\times\begin{cases}\|g'\|_\infty\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right] & \text{if } g'\in L_\infty[a,b];\\ \frac{q}{q+1}\|g'\|_p\left[(b-x)^{\frac{q+1}{q}}+(x-a)^{\frac{q+1}{q}}\right] & \text{if } g'\in L_p[a,b],\\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ (b-a)\|g'\|_1, & \end{cases}$$

for any $x\in[a,b]$;

(iii) *If $g:[a,b]\to\mathbb{R}$ is monotonic nondecreasing on $[a,b]$, then*

$$\begin{aligned}|\Psi(g,u;a,b,x)| & \\ &\leq\frac{1}{2}(\Phi-\varphi)\left\{[2x-(a+b)]+\int_a^b\operatorname{sgn}(t-x)\,g(t)\,dt\right\}\\ &\leq\frac{1}{2}(\Phi-\varphi)\{(x-a)[g(x)-g(a)]+(b-x)[g(b)-g(x)]\}\\ &\leq\frac{1}{2}(\Phi-\varphi)\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right][g(b)-g(a)],\end{aligned} \tag{3.16}$$

for any $x\in[a,b]$.

In practical applications dealing with the approximation of the Stieltjes integral $\int_a^b g(t)\,du(t)$, the case $x=\frac{a+b}{2}$ is of special interest.

If we introduce the functional

$$M(g,u;a,b):=\int_a^b g(t)\,du(t)-\frac{\varphi+\Phi}{2}\int_a^b g(t)\,dt$$
$$-g\left(\frac{a+b}{2}\right)\left[u(b)-u(a)-\frac{\varphi+\Phi}{2}(b-a)\right],$$

then the following particular case of Proposition 2 can be stated.

Corollary 4. *Assume that g and u are as in Proposition 2.*

(i) *If g is of $r-H-$Hölder type ($K-$Lipschitzian), then*

$$|M(g,u;a,b)|\leq\frac{H(\Phi-\varphi)}{2^{r+1}(r+1)}(b-a)^{r+1} \tag{3.17}$$
$$\left(\leq\frac{1}{8}(\Phi-\varphi)K(b-a)^2\right);$$

(ii) *If g is absolutely continuous on $[a,b]$, then*

$$
\left|M\left(g,u;a,b\right)\right| \leq \begin{cases} \frac{1}{8}\left(\Phi-\varphi\right)\left(b-a\right)^{2}\left\|g'\right\|_{\infty} & \text{if } g'\in L_{\infty}\left[a,b\right]; \\ \\ \frac{q\left(\Phi-\varphi\right)}{\left(q+1\right)2^{\frac{q+1}{q}}}\left\|g'\right\|_{p}\left(b-a\right)^{\frac{q+1}{q}} & \text{if } g'\in L_{p}\left[a,b\right], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{2}\left(\Phi-\varphi\right)\left\|g'\right\|_{1}\left(b-a\right). \end{cases} \tag{3.18}
$$

(iii) *If g is monotonic nondecreasing on $[a,b]$, then*

$$
\begin{aligned} \left|M\left(g,u;a,b\right)\right| &\leq \frac{1}{2}\left(\Phi-\varphi\right)\int_{a}^{b}\operatorname{sgn}\left(t-\frac{a+b}{2}\right)g\left(t\right)dt \\ &\leq \frac{1}{4}\left(\Phi-\varphi\right)\left[g\left(b\right)-g\left(a\right)\right]. \end{aligned} \tag{3.19}
$$

4. Inequalities for the Weighted Riemann Integral

If $h:[a,b]\rightarrow\mathbb{R}$ is Riemann integrable on $[a,b]$, then $u(t):=\int_a^t f(s)\,ds$ is absolutely continuous on $[a,b]$ and for a Riemann integrable function $f:[a,b]\rightarrow\mathbb{R}$ we have

$$
\int_{a}^{b}f\left(t\right)du\left(t\right)=\int_{a}^{b}f\left(t\right)h\left(t\right)dt. \tag{4.1}
$$

If n,N are real numbers with $N>n$ and

$$
n\leq h\left(t\right)\leq N \quad \text{for a.e.} \quad t\in\left[a,b\right], \tag{4.2}
$$

then

$$
n\leq\frac{u\left(t\right)-u\left(s\right)}{t-s}=\frac{\int_{s}^{t}h\left(z\right)dz}{t-s}\leq N
$$

for any $t>s$, showing that $u(t)=\int_a^t h(z)\,dz$ is (n,N)–Lipschitzian on $[a,b]$.

Utilising Theorem 1, we can state the following result for weighted integrals.

Proposition 3. *Let $f,h:[a,b]\rightarrow\mathbb{R}$ be two Riemann integrable functions such that h satisfies (4.1). Then*

$$
\left|\int_{a}^{b}f\left(t\right)h\left(t\right)dt-\frac{n+N}{2}\int_{a}^{b}f\left(t\right)dt\right|\leq\frac{1}{2}\left(N-n\right)\int_{a}^{b}\left|f\left(t\right)\right|dt. \tag{4.3}
$$

The constant $\frac{1}{2}$ is best possible.

Proof. The inequality follows from (3.1) for $u(t)=\int_a^t h(s)\,ds$.

For the best constant, we choose $f(t)=t-\frac{a+b}{2}$ and $h(t)=\operatorname{sgn}\left(t-\frac{a+b}{2}\right)$. Then $n=-1,N=1$ and

$$
\begin{aligned} \int_{a}^{b}f\left(t\right)h\left(t\right)dt &= \int_{a}^{b}\left(t-\frac{a+b}{2}\right)\operatorname{sgn}\left(t-\frac{a+b}{2}\right)dt \\ &= \int_{a}^{b}\left|t-\frac{a+b}{2}\right|dt=\frac{\left(b-a\right)^{2}}{4}, \end{aligned}
$$

$$
\int_{a}^{b}f\left(t\right)dt=0 \quad \text{and} \quad \int_{a}^{b}\left|f\left(t\right)\right|dt=\frac{\left(b-a\right)^{2}}{4},
$$

which produces the same quantity on both parts of (4.3). ∎

Corollary 5. *Let g and h be Riemann integrable on $[a,b]$ and h satisfy the condition (4.2). Then*

$$\left|\int_a^b g(t)\,h(t)\,dt - \frac{1}{b-a}\int_a^b h(t)\,dt\cdot\int_a^b g(t)\,dt\right| \leq \frac{1}{2}(N-n)\int_a^b \left|g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds\right| dt. \tag{4.4}$$

The constant $\frac{1}{2}$ is best possible.

Remark 3. *This result has been obtained by Cheng and Sun in* [6]*. The natural extension to abstract Lebesgue integrals and the sharpness of the constant have been established by Cerone and Dragomir in* [4].

Corollary 6. *Let g and h be Riemann integrable functions satisfying the boundedness conditions (3.7) and (4.2). Then*

$$\begin{aligned}\Bigg|\int_a^b g(t)\,h(t)\,dt &- \frac{n+N}{2}\int_a^b g(t)\,dt \\ &- \frac{m+M}{2}\int_a^b h(t)\,dt + \frac{(n+N)(m+M)}{4}(b-a)\Bigg| \\ &\leq \frac{1}{2}(N-n)\int_a^b \left|g(t) - \frac{m+M}{2}\right| dt \\ &\leq \frac{1}{4}(M-m)(N-n)(b-a).\end{aligned} \tag{4.5}$$

The constants $\frac{1}{2}$ and $\frac{1}{4}$ are best possible in (4.5).

Remark 4. *The inequality between the first and the last term in (4.5) has been obtained in* [12]*. A generalisation for the abstract Lebesgue integral has been given as well.*

Corollary 7. *Let g, h be Riemann integrable functions and let h satisfy the boundedness condition (4.2). Then*

$$\begin{aligned}\Bigg|\int_a^b g(t)\,h(t)\,dt &- \frac{n+N}{2}\int_a^b g(t)\,dt \\ &- g(x)\left[\int_a^b h(t)\,dt - \frac{n+N}{2}(b-a)\right]\Bigg| \\ &\leq \frac{1}{2}(N-n)\int_a^b |g(t) - g(x)|\,dt,\end{aligned} \tag{4.6}$$

for any $x \in [a,b]$.

The constant $\frac{1}{2}$ is best possible in (4.6).

If we introduce the operator

$$
\begin{aligned}
(4.7)\qquad \tilde{\Psi}(g,h;a,b,x) &:= \Psi\left(g,\int_a^{\cdot} h(s)\,ds;a,b,x\right)\\
&= \int_a^b g(t)h(t)\,dt-\frac{n+N}{2}\int_a^b g(t)\,dt\\
&\quad -g(x)\left[\int_a^b h(t)\,dt-\frac{n+N}{2}(b-a)\right],
\end{aligned}
$$

then the following may be stated as well.

Proposition 4. *Let $g,h:[a,b]\to\mathbb{R}$ be Riemann integrable on $[a,b]$ and let h satisfy the boundedness condition (4.2).*

(i) *If $g:[a,b]\to\mathbb{R}$ is of $r-H-$Hölder type ($K-$Lipschitzian), then $\tilde{\Psi}(g,h;a,b,x)$ satisfies the inequality*

$$
\begin{aligned}
(4.8)\qquad \left|\tilde{\Psi}(g,h;a,b,x)\right| &\le \frac{1}{2}(N-n)\cdot\frac{H}{r+1}\left[(b-x)^{r+1}+(x-a)^{r+1}\right]\\
&\left(\le \frac{1}{2}(N-n)K\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]\right)
\end{aligned}
$$

for any $x\in[a,b]$;

(ii) *If g is absolutely continuous on $[a,b]$, then*

$$
(4.9)\qquad \left|\tilde{\Psi}(g,h;a,b,x)\right|
\le \frac{1}{2}(N-n)\times\begin{cases}
\|g'\|_\infty\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right] & \text{if } g'\in L_\infty[a,b];\\[2mm]
\frac{q}{q+1}\|g'\|_p\left[(b-x)^{\frac{q+1}{q}}+(x-a)^{\frac{q+1}{q}}\right] & \text{if } g'\in L_p[a,b],\\
& p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2mm]
(b-a)\|g'\|_1,
\end{cases}
$$

for any $x\in[a,b]$;

(iii) *If $g:[a,b]\to\mathbb{R}$ is monotonic nondecreasing on $[a,b]$, then*

$$
\begin{aligned}
(4.10)\qquad &\left|\tilde{\Psi}(g,h;a,b,x)\right|\\
&\le \frac{1}{2}(N-n)\left\{[2x-(a+b)]+\int_a^b \operatorname{sgn}(t-x)\,g(t)\,dt\right\}\\
&\le \frac{1}{2}(N-n)\left\{(x-a)[g(x)-g(a)]+(b-x)[g(b)-g(x)]\right\}\\
&\le \frac{1}{2}(N-n)\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right][g(b)-g(a)],
\end{aligned}
$$

for any $x\in[a,b]$.

Finally, on defining

$$\tilde{M}(g,h;a,b) = M\left(g,\int_a^{\cdot} h(s)\,ds;a,b\right) = \int_a^b g(t)\,du(t) - \frac{n+N}{2}\int_a^b g(t)\,dt - g\left(\frac{a+b}{2}\right)\left[\int_a^b h(t)\,dt - \frac{n+N}{2}(b-a)\right], \tag{4.11}$$

then $\tilde{M}(g,h;a,b)$ satisfies the inequalities (3.17) – (3.19) with n and N replacing φ and Φ.

5. Applications for the Generalised Trapezoid Formula

The following natural application for the generalised trapezoid formula can be stated.

Proposition 5. *Let $f:[a,b]\to\mathbb{R}$ be a $(\varphi,\Phi)-$Lipschitzian function. Then*

$$\left|\int_a^b f(t)\,dt - \left[f(b)(b-x)+f(a)(x-a)+\frac{\varphi+\Phi}{2}(b-a)\left(x-\frac{a+b}{2}\right)\right]\right| \le \frac{1}{2}(\Phi-\varphi)\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right], \tag{5.1}$$

for each $x\in[a,b]$.

The multiplicative constant $\frac{1}{2}$ is best possible.

Proof. For any $x\in[a,b]$ we have the identity (see [5])

$$\int_a^b (t-x)\,df(t) = f(b)(b-x)+f(a)(x-a)-\int_a^b f(t)\,dt. \tag{5.2}$$

Since f is assumed to be $(\varphi,\Phi)-$Lipschitzian, then, on applying Theorem 1, we have from (5.2) that

$$\left|\int_a^b (t-x)\,df(t) - \frac{\varphi+\Phi}{2}\int_a^b (t-x)\,dt\right| \le \frac{1}{2}(\Phi-\varphi)\int_a^b |t-x|\,dt. \tag{5.3}$$

Since

$$\int_a^b (t-x)\,dt = (b-a)\left(\frac{a+b}{2}-x\right), \qquad x\in[a,b]$$

and

$$\int_a^b |t-x|\,dt = \frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2, \qquad x\in[a,b],$$

hence (5.3) provides the desired inequality (5.1). ∎

Remark 5. *For $x=a$, we get the "right rectangle" inequality*

$$\left|\int_a^b f(t)\,dt - f(b)(b-a)+\frac{\varphi+\Phi}{4}(b-a)^2\right| \le \frac{1}{4}(\Phi-\varphi)(b-a)^2, \tag{5.4}$$

while for $x = b$ we obtain the "left rectangle" inequality

$$\left|\int_a^b f(t)\,dt - f(a)(b-a) - \frac{\varphi+\Phi}{4}(b-a)^2\right| \le \frac{1}{4}(\Phi-\varphi)(b-a)^2. \tag{5.5}$$

The case $x = \frac{a+b}{2}$ provides the best possible inequality in (5.1), the "trapezoid inequality":

$$\left|\int_a^b f(t)\,dt - \frac{f(a)+f(b)}{2}(b-a)\right| \le \frac{1}{8}(\Phi-\varphi)(b-a)^2. \tag{5.6}$$

The constant $\frac{1}{8}$ is best possible.

This inequality has been obtained by Z. Liu in [19] as a particular case of Corollary 1.

Remark 6. *If f is $L-$Lipschitzian, i.e., $\varphi = -L$, $\Phi = L$, then from (5.1) we get the inequality*

$$\left|\int_a^b f(t)\,dt - [f(b)(b-x) + f(a)(x-a)]\right| \le L\left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2}\right)^2\right] \tag{5.7}$$

for any $x \in [a,b]$, that has been obtained in [3].

6. Applications for Ostrowski Type Inequalities

The following particular case of Theorem 1 in connection with the celebrated Ostrowski inequality [20] can be stated as well:

Proposition 6. *Let $f : [a,b] \to \mathbb{R}$ be a $(\varphi,\Phi)-$Lipschitzian function. Then*

$$\left|\int_a^b f(t)\,dt - f(x)(b-a) + \frac{\varphi+\Phi}{2}(b-a)\left(x - \frac{a+b}{2}\right)\right| \le \frac{1}{2}(\Phi-\varphi)\left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2}\right)^2\right] \tag{6.1}$$

for each $x \in [a,b]$.

The multiplicative constant $\frac{1}{2}$ is best possible.

Proof. For any $x \in [a,b]$, we have the Montgomery type identity [8]

$$\int_a^b p(x,t)\,df(t) = f(x)(b-a) - \int_a^b f(t)\,dt \tag{6.2}$$

for any $x \in [a,b]$, where the kernel $p : [a,b]^2 \to \mathbb{R}$ is defined by

$$p(t,x) := \begin{cases} t-a & \text{if } t \in [a,x]; \\ t-b & \text{if } t \in (x,b]. \end{cases}$$

Since f is assumed to be a (φ,Φ) $-$Lipschitzian function, then, on applying Theorem 1, we have

$$\left|\int_a^b p(x,t)\,df(t)-\frac{\varphi+\Phi}{2}\int_a^b p(x,t)\,dt\right|\le\frac{1}{2}(\Phi-\varphi)\int_a^b|p(x,t)|\,dt. \tag{6.3}$$

Since

$$\int_a^b p(x,t)\,dt=\int_a^x(t-a)\,dt+\int_x^b(t-b)\,dt=(b-a)\left(x-\frac{a+b}{2}\right)$$

and

$$\int_a^b|p(x,t)|\,dt=\int_a^x(t-a)\,dt+\int_x^b(b-t)\,dt=\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2,$$

hence by (6.2) and (6.3) we get the desired inequality (6.1). ∎

Remark 7. *The cases $x=a$ and $x=b$ provide the rectangle inequalities stated in the previous section.*

The case $x=\frac{a+b}{2}$ provides the best possible inequality in (5.1), the "midpoint" inequality:

$$\left|\int_a^b f(t)\,dt-(b-a)f\left(\frac{a+b}{2}\right)\right|\le\frac{1}{8}(\Phi-\varphi)(b-a)^2. \tag{6.4}$$

The constant $\frac{1}{8}$ is best possible in (6.4).

This inequality has been obtained by Z. Liu in [19] as a particular case of Corollary 1.

Remark 8. *If f is $L-$Lipschitzian, i.e., $\varphi=-L$, $\Phi=L$, then from (6.1) we get the inequality:*

$$\left|\int_a^b f(t)\,dt-f(x)(b-a)\right|\le L\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right] \tag{6.5}$$

for any $x\in[a,b]$, which has been obtained in [10].

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