THE BEESACK-DARST-POLLARD INEQUALITIES AND APPROXIMATIONS OF THE RIEMANN-STIELTJES INTEGRAL

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Abstract. Utilising the Beesack version of the Darst-Pollard inequality, some error bounds for approximating the Riemann-Stieltjes integral are given. Some applications related to the trapezoid and mid-point quadrature rules are provided.

1. Introduction

In 1970, R. Darst and H. Pollard [3] obtained the following inequality for the Riemann-Stieltjes integral:

$$\int_a^b h(t)\,dg(t) \le \inf_{t\in[a,b]} h(t)\,[g(b)-g(a)] + S(g;a,b)\bigvee_a^b(h) \tag{1.1}$$

where $\bigvee_a^b(h)$ denotes the total variation of h on $[a,b]$ and

$$S(g;a,b) := \sup_{a\le\alpha<\beta\le b}[g(\beta)-g(\alpha)] \tag{1.2}$$

under the assumption that h is of bounded variation and g is continuous on $[a,b]$.

As P.R. Beesack observed in [1] that, by replacing g with $(-g)$ in (1.1), we can also obtain the "dual" Darst-Pollard inequality

$$\int_a^b h(t)\,dg(t) \ge \inf_{t\in[a,b]} h(t)\,[g(b)-g(a)] + s(g;a,b)\bigvee_a^b(h) \tag{1.3}$$

where

$$s(g;a,b) := \inf_{a\le\alpha<\beta\le b}[g(\beta)-g(\alpha)]. \tag{1.4}$$

Beesack also showed that the inequalities (1.1) and (1.4) remain valid even if g is *not* continuous on $[a,b]$, provided only that g is bounded on $[a,b]$ and $\int_a^b h(t)\,dg(t)$ exists.

In a recent paper [6], in order to approximate the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ by the quadrature rule

$$\frac{m+M}{2}[u(b)-u(a)]$$

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where $m \leq f(t) \leq M$ for each $t \in [a,b]$, the second author defined the *error functional*

$$\Delta(f,u,m,M;a,b) := \int_a^b f(t)\,du(t) - \frac{m+M}{2}\left[u(b)-u(a)\right]$$

and showed that

$$(1.5)\qquad |\Delta(f,u,m,M;a,b)| \leq \begin{cases} \frac{1}{2}(M-m)\bigvee_a^b(u) \\ \qquad\qquad \text{if } u \text{ is of bounded variation;} \\ \\ \frac{1}{2}(M-m)L(b-a) \\ \qquad\qquad \text{if } f \text{ is } L-\text{Lipschitzian;} \\ \int_a^b \left|f(t)-\frac{m+M}{2}\right| du(t) \\ \qquad\qquad \text{if } u \text{ is monotonic nondecreasing.} \end{cases}$$

The constant $\frac{1}{2}$ is the best possible in both inequalities. The last inequality in (1.5) is also sharp.

In the same paper [6], in order to approximate the integral $\int_a^b f(t)\,du(t)$ in terms of the *generalised trapezoid rule*

$$\left[u(b)-\frac{n+N}{2}\right]f(b)+\left[\frac{n+N}{2}-u(a)\right]f(a),$$

the second author introduced the error functional

$$\nabla(f,u,n,N;a,b) := \left[u(b)-\frac{n+N}{2}\right]f(b)+\left[\frac{n+N}{2}-u(a)\right]f(a)-\int_a^b f(t)\,du(t),$$

where $-\infty < n \leq u(t) \leq N < \infty$ for $t \in [a,b]$ and showed that

$$(1.6)\qquad |\nabla(f,u,n,N;a,b)| \leq \begin{cases} \frac{1}{2}(N-n)\bigvee_a^b(f) \\ \qquad\qquad \text{if } f \text{ is of bounded variation;} \\ \\ \frac{1}{2}(N-n)K(b-a) \\ \qquad\qquad \text{if } f \text{ is } K-\text{Lipschitzian;} \\ \int_a^b \left|u(t)-\frac{n+N}{2}\right| df(t) \\ \qquad\qquad \text{if } f \text{ is monotonic nondecreasing.} \end{cases}$$

The constant $\frac{1}{2}$ is the best possible in (1.6) and the last inequality is sharp.

In this paper, by use of the Beesack-Darst-Pollard inequalities (1.1) and (1.3), we provide other error bounds for the functionals Δ and ∇. Applications for the generalised trapezoid and Ostrowski inequalities are also given.

2. The Results

We can state the following result concerning the error bounds for the error functional $\Delta(f,u,m,M;a,b)$.

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be a function of bounded variation and assume that*

$$(2.1)\qquad -\infty < m = \inf_{t\in[a,b]} f(t), \quad \sup_{t\in[a,b]} f(t) = M < \infty.$$

If u is bounded and the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ exists, then

$$|\Delta(f,u,m,M;a,b)| \le \min\left\{\bigvee_a^b(f)\cdot S(u;a,b)-\frac{1}{2}(M-m)[u(b)-u(a)],\right. \tag{2.2}$$
$$\left.\frac{1}{2}(M-m)[u(b)-u(a)]-\bigvee_a^b(f)\cdot s(u;a,b)\right\}$$
$$\le \frac{1}{2}\cdot\bigvee_a^b(f)[S(u;a,b)-s(u;a,b)].$$

The constant $\frac{1}{2}$ is the best possible and the inequalities are sharp.

Proof. If we apply the inequality (1.3) for $h(t)=f(t)$, $g(t)=u(t)$, we can write,

$$\int_a^b f(t)\,du(t) \ge m[u(b)-u(a)]+s(u;a,b)\bigvee_a^b(f). \tag{2.3}$$

If we apply the same inequality (1.3) for $h(t)=M-f(t)$ and $g(t)=u(t)$, we get

$$M[u(b)-u(a)]-\int_a^b f(t)\,du(t) \ge s(u;a,b)\bigvee_a^b(f) \tag{2.4}$$

since, obviously, $\bigvee_a^b(M-f)=\bigvee_a^b(f)$.

The inequalities (2.3) and (2.4) give the following double inequality that is of interest:

$$M[u(b)-u(a)]-s(u;a,b) \ge \int_a^b f(t)\,du(t) \tag{2.5}$$
$$\ge m[u(b)-u(a)]+s(u;a,b).$$

Now, if we subtract from all terms the same quantity

$$\frac{M+m}{2}[u(b)-u(a)]$$

we get

$$\frac{1}{2}(M-m)[u(b)-u(a)]-s(u;a,b) \tag{2.6}$$
$$\ge \int_a^b f(t)\,du(t)-\frac{M+m}{2}[u(b)-u(a)]$$
$$\ge -\frac{1}{2}(M-m)[u(b)-u(a)]+s(u;a,b),$$

which is equivalent to

$$|\Delta(f,u,m,M;a,b)| \le \frac{1}{2}(M-m)[u(b)-u(a)]-s(u;a,b). \tag{2.7}$$

On utilising (1.1) we can also prove in a similar way that

$$|\Delta(f,u,m,M;a,b)| \le \bigvee_a^b(f)\,S(u;a,b)-\frac{1}{2}(M-m)[u(b)-u(a)]. \tag{2.8}$$

These show that the first inequality in (2.2) is valid. The second part is obvious since for any $\alpha,\beta\in\mathbb{R}$, $\min(\alpha,\beta)\le\frac{\alpha+\beta}{2}$.

For the sharpness of the inequality, we assume that $u(t) = t$, $t \in [a,b]$. Since for this selection of u we have

$$S(u;a,b) = b-a \qquad \text{and} \qquad s(u;a,b) = 0,$$

hence the inequality (2.3) becomes

$$\left| \int_a^b f(t)\,du(t) - \frac{M+m}{2}(b-a) \right| \tag{2.9}$$
$$\leq \min\left\{ (b-a)\bigvee_a^b(f) - \frac{1}{2}(M-m)(b-a), \frac{1}{2}(M-m)(b-a) \right\}$$
$$\leq \frac{1}{2}\bigvee_a^b(f)(b-a).$$

If we consider the function $f_0 : [a,b] \to \mathbb{R}$,

$$f_0(t) = \begin{cases} 0 & \text{if } t \in [a,b]; \\ k & \text{if } t = b, \end{cases}$$

where $k > 0$, then obviously $m = 0$, $M = k$, $\int_a^b f_0(t)\,dt = 0$, $\bigvee_a^b(f_0) = k$ and in all parts of (2.9) we get the same quantity $\frac{1}{2}k(b-a)$. ∎

The following corollary that provides error bounds for the error functional $\nabla(f,u,n,N;a,b)$ can be stated as well.

Corollary 1. *Let $u : [a,b] \to \mathbb{R}$ be a function of bounded variation such that there exist the constants n, N with*

$$-\infty < n = \inf_{t\in[a,b]} u(t), \quad \sup_{t\in[a,b]} u(t) = N < \infty. \tag{2.10}$$

If f is bounded and the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ exists, then

$$|\nabla(f,u,n,N;a,b)| \leq \min\left\{ \bigvee_a^b(u)\,S(f;a,b) - \frac{1}{2}(N-n)[f(b)-f(a)], \right. \tag{2.11}$$
$$\left. \frac{1}{2}(N-n)[f(b)-f(a)] - \bigvee_a^b(u)\,s(f;a,b) \right\}$$
$$\leq \frac{1}{2}\bigvee_a^b(u)[S(f;a,b) - s(f;a,b)].$$

The constant $\frac{1}{2}$ is the best possible and the inequalities are sharp.

Proof. Follows by Theorem 1 on utilising the identity

$$f(b)\left[u(b) - \frac{n+N}{2}\right] + f(a)\left[\frac{n+N}{2} - u(a)\right] - \int_a^b f(t)\,du(t)$$
$$= \int_a^b \left[u(t) - \frac{n+N}{2}\right] df(t)$$
$$= \int_a^b u(t)\,df(t) - \frac{n+N}{2}[f(b)-f(a)].$$

The details are omitted. ∎

The following particular cases of Theorem 1 may be of interest in applications.

Corollary 2. *Assume that $f:[a,b]\to\mathbb{R}$ is as in Theorem 1. If $u:[a,b]\to\mathbb{R}$ is of the $r-H-$Hölder type, i.e.,*

$$|u(t)-u(s)|\le H|t-s|^r \quad \text{for any} \quad t,s\in[a,b], \tag{2.12}$$

where $H>0$ and $s\in(0,1]$ are given, then

$$\begin{aligned}(2.13)\qquad &|\Delta(f,u,m,M;a,b)| \\ &\le \min\left\{H(b-a)^r\bigvee_a^b(f)-\frac{1}{2}(M-m)[u(b)-u(a)],\right. \\ &\qquad\left.\frac{1}{2}(M-m)[u(b)-u(a)]+H(b-a)^r\bigvee_a^b(f)\right\} \\ &\le H(b-a)^r\bigvee_a^b(f).\end{aligned}$$

Proof. For any $a\le\alpha<\beta\le b$ we have, by (2.12), that

$$-H(\beta-\alpha)^r\le u(\beta)-u(s)\le H(\beta-\alpha)^r.$$

This implies that

$$S(u;a,b)\le\sup_{a\le\alpha<\beta\le b}[H(\beta-\alpha)^r]=H(b-a)^r$$

and

$$\begin{aligned}s(u;a,b)&\ge\inf_{a\le\alpha<\beta\le b}[-H(\beta-\alpha)^r]=-\sup_{a\le\alpha<\beta\le b}[H(\beta-\alpha)^r] \\ &=-H(b-a)^r.\end{aligned}$$

Utilising (2.2) we deduce the desired inequality (2.13). ∎

Corollary 3. *Assume that f is as in Theorem 1. If $u:[a,b]\to\mathbb{R}$ is monotonic non-decreasing and such that the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ exists, then*

$$\begin{aligned}(2.14)\qquad &|\Delta(f,u,m,M;a,b)| \\ &\le\min\left\{\bigvee_a^b(f)-\frac{1}{2}(M-m),\frac{1}{2}(M-m)\right\}[u(b)-u(a)] \\ &\le\frac{1}{2}[u(b)-u(a)]\bigvee_a^b(f).\end{aligned}$$

The proof is obvious by Theorem 1 on taking into account that for the monotonic nondecreasing function $u:[a,b]\to\mathbb{R}$ we have:

$$S(u;a,b)=u(b)-u(a)$$

and

$$s(u;a,b)=0.$$

3. Applications

The following inequality obtained in [2] is known as the trapezoid inequality for functions of bounded variation:

$$\left| \frac{1}{b-a} \int_a^b f(t)\,dt - \frac{f(a)+f(b)}{2} \right| \le \frac{1}{2} \bigvee_a^b (f), \tag{3.1}$$

where the constant $\frac{1}{2}$ is best possible in the sense that it cannot be replaced by a smaller constant.

The following trapezoid inequality for the larger class of Riemann integrable functions can be stated:

Proposition 1. *Let $f : [a,b] \to \mathbb{R}$ be Riemann integrable on $[a,b]$, then:*

$$\begin{aligned} &\left| \frac{1}{b-a} \int_a^b f(t)\,dt - \frac{f(a)+f(b)}{2} \right| \\ &\le \min\left\{ S(f;a,b) - \frac{1}{2}[f(b)-f(a)], \frac{1}{2}[f(b)-f(a)] - s(f;a,b) \right\} \\ &\le \frac{1}{2}[S(f;a,b) - s(f;a,b)]. \end{aligned} \tag{3.2}$$

Proof. We use the following identity holding for the Riemann integrable function $f : [a,b] \to \mathbb{R}$:

$$f(b)(b-x) + f(a)(x-a) - \int_a^b f(t)\,dt = \int_a^b (t-x)\,df(t) \tag{3.3}$$

for any $x \in [a,b]$, see [2].

We observe that $\sup_{t\in[a,b]}(t-x) = b-a$, $\inf_{t\in[a,b]}(t-x) = a-x$, for $x \in [a,b]$ and, applying Theorem 1 for the Stieltjes integral $\int_a^b (t-x)\,df(t)$, $x \in [a,b]$, we obtain:

$$\begin{aligned} &\left| \int_a^b (t-x)\,df(t) - \left(\frac{a+b}{2} - x\right)[f(b)-f(a)] \right| \\ &\le \min\left\{ \bigvee_a^b (\cdot - x)\, S(f;a,b) - \frac{1}{2}(b-a)[f(b)-f(a)], \right. \\ &\qquad \left. \frac{1}{2}(b-a)[f(b)-f(a)] - \bigvee_a^b (\cdot - x)\, s(f;a,b) \right\} \\ &\le \frac{1}{2} \bigvee_a^b (\cdot - x)[S(f;a,b) - s(f;a,b)]. \end{aligned} \tag{3.4}$$

On utilising the identity (3.3) and the fact that $\bigvee_a^b (\cdot - x) = b-a$, we deduce from (3.4) the desired result (3.2). ∎

In [5], S.S. Dragomir obtained the following Ostrowski type inequality for functions of bounded variation:

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \le \left[\frac{1}{2} + \frac{\left| x - \frac{a+b}{2} \right|}{b-a} \right] \bigvee_a^b (f), \tag{3.5}$$

for any $x \in [a,b]$. The constant $\frac{1}{2}$ is the best possible in (3.5).

The best inequality one can obtain from (3.5) is the *mid-point inequality*:

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \frac{1}{2}\bigvee_a^b (f), \tag{3.6}$$

for which $\frac{1}{2}$ is also the best possible constant.

In order to extend (3.5) to the larger class of Riemann integrable functions, we can state:

Proposition 2. *Let $f:[a,b]\to\mathbb{R}$ be Riemann integrable on $[a,b]$, then:*

$$\begin{aligned} &\left| f(x) - \left(x - \frac{a+b}{2}\right)\cdot\frac{f(b)-f(a)}{b-a} - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \\ &\leq \min\left\{ S(f;a,b) - \frac{1}{2}[f(b)-f(a)], \frac{1}{2}[f(b)-f(a)] - s(f;a,b) \right\} \\ &\leq \frac{1}{2}[S(f;a,b) - s(f;a,b)]. \end{aligned} \tag{3.7}$$

Proof. We use the Montgomery type identity [5] for the Riemann integrable function $f:[a,b]\to\mathbb{R}$:

$$\int_a^b p(t,x)\,df(t) = f(x)(b-a) - \int_a^b f(t)\,dt$$

for any $x\in[a,b]$, where the kernel $p:[a,b]^2\to\mathbb{R}$ is defined by

$$p(t,x) := \begin{cases} t-a & \text{if } t\in[a,x], \\ t-b & \text{if } t\in(x,b]. \end{cases}$$

For any fixed $x\in[a,b]$, the function $p(\cdot,x)$ is of bounded variation, and

$$\begin{aligned} \bigvee_a^b p(\cdot,x) &= \bigvee_a^x p(\cdot,x) + \bigvee_x^b p(\cdot,x) \\ &= x-a+b-x = b-a. \end{aligned}$$

Also, observe that

$$\sup_{t\in[a,b]} p(t,x) = x-a \qquad \text{and} \qquad \inf_{t\in[a,b]} p(t,x) = x-b$$

for any $x\in[a,b]$.

Now, applying Theorem 1 for the Riemann-Stieltjes integral $\int_a^b p(t,x)\,df(t)$, we can write that

$$\begin{aligned} &\left| \int_a^b p(t,x)\,df(t) - \left(x-\frac{a+b}{2}\right)\cdot[f(b)-f(a)] \right| \\ &\leq (b-a)\min\left\{ S(f;a,b) - \frac{1}{2}[f(b)-f(a)], \frac{1}{2}[f(b)-f(a)] - s(f;a,b) \right\} \\ &\leq \frac{1}{2}[S(f;a,b) - s(f;a,b)], \end{aligned}$$

which is clearly equivalent to (3.2). ∎

The following mid-point inequality holds.

Corollary 4. *Let f be as in Proposition 2, then*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{3.8}$$
$$\leq \min\left\{S(f;a,b)-\frac{1}{2}[f(b)-f(a)],\frac{1}{2}[f(b)-f(a)]-s(f;a,b)\right\}$$
$$\leq \frac{1}{2}[S(f;a,b)-s(f;a,b)].$$

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