

**ON SOME INEQUALITIES OF
CAUCHY-BUNYAKOVSKY-SCHWARZ TYPE AND
APPLICATIONS**

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ABSTRACT. Some discrete inequalities of Cauchy-Bunyakovsky-Schwarz type for complex numbers with applications for the maximal deviation of a sequence from its weighted mean are given.

1. INTRODUCTION

The following result for complex numbers $a_k, b_k, k \in \{1, \dots, n\}$ is well known in the literature as the *Cauchy-Bunyakovsky-Schwarz (CBS) inequality*:

$$(1.1) \quad \left| \sum_{k=1}^n a_k b_k \right|^2 \leq \sum_{k=1}^n |a_k|^2 \sum_{k=1}^n |b_k|^2,$$

with equality if and only if there is a complex number $c \in \mathbb{C}$ such that $a_k = c \bar{b}_k$ for each $k \in \{1, \dots, n\}$, and \bar{b}_k is the complex conjugate of b_k .

A simple proof of this statement can be achieved by utilising the following *Lagrange identity* for complex numbers (see [2, p. 3])

$$\sum_{k=1}^n |a_k|^2 \sum_{k=1}^n |b_k|^2 - \left| \sum_{k=1}^n a_k b_k \right|^2 = \frac{1}{2} \sum_{k,l=1}^n |\bar{a}_k b_l - \bar{a}_l b_k|^2.$$

If $p_k, k \in \{1, \dots, n\}$ are positive weights, then the weighted version of (1.1) can be stated as

$$(1.2) \quad \left| \sum_{k=1}^n p_k a_k b_k \right|^2 \leq \sum_{k=1}^n p_k |a_k|^2 \sum_{k=1}^n p_k |b_k|^2.$$

In [4], the following result connecting the unweighted version of the (CBS) inequality with the weighted one has been established (see also [2, p. 67 – 69]):

$$(1.3) \quad \left(\sum_{k=1}^n |x_k|^2 \sum_{k=1}^n |y_k|^2 \right)^{\frac{1}{2}} - \left| \sum_{k=1}^n x_k y_k \right| \\ = \sup_{\mathbf{p} \in S_n(\mathbf{1})} \left\{ \sum_{k=1}^n p_k |x_k|^2 \sum_{k=1}^n p_k |y_k|^2 - \left| \sum_{k=1}^n p_k x_k y_k \right| \right\},$$

where $S_n(\mathbf{1}) = \{\mathbf{p} = (p_1, \dots, p_n) \mid 0 \leq p_k \leq 1 \text{ for each } k \in \{1, \dots, n\}\}$.

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In the same paper the authors also established the following result concerning the length of summation in the CBS inequality:

$$(1.4) \quad \left(\sum_{k=1}^n p_k |x_k|^2 \sum_{k=1}^n p_k |y_k|^2 \right)^{\frac{1}{2}} - \left| \sum_{k=1}^n p_k x_k y_k \right| \\ = \sup_{I \subseteq \{1, \dots, n\}} \left[\left(\sum_{k \in I} p_k |x_k|^2 \sum_{k \in I} p_k |y_k|^2 \right)^{\frac{1}{2}} - \left| \sum_{k=1}^n p_k x_k y_k \right| \right]$$

and

$$(1.5) \quad \left(\sum_{k=1}^n p_k |x_k|^2 \sum_{k=1}^n p_k |y_k|^2 \right)^{\frac{1}{2}} - \left| \sum_{k=1}^n p_k x_k y_k \right| \\ \geq \max_{1 \leq k < l \leq n} \left\{ \left[p_k |x_k|^2 + p_l |x_l|^2 \right]^{\frac{1}{2}} \left[p_k |y_k|^2 + p_l |y_l|^2 \right]^{\frac{1}{2}} - |p_k x_k y_k + p_l x_l y_l| \right\},$$

for any $x_k, y_k \in \mathbb{C}$, $k \in \{1, \dots, n\}$.

For some historical facts on CBS inequality, see [9] and [2]. Refinements of this inequality are provided in [1], [6], [8] and in the Chapter 2 of [2]. Other results related to CBS inequality may be found in [5] and [7].

The aim of the present paper is to establish some inequalities of CBS type under the supplementary assumption that either $\sum_{k=1}^n x_k y_k = 0$ or $\sum_{k=1}^n p_k x_k y_k = 0$, when the weighted version is considered. Applications that provide upper bounds for the maximal deviation of a sequence x_k from the weighted mean $\sum_{j=1}^n p_j x_j$, namely, for the quantity

$$(1.6) \quad \max_{k \in \{1, \dots, n\}} \left| x_k - \sum_{j=1}^n p_j x_j \right|,$$

where $x_k \in \mathbb{C}$, $p_k \geq 0$, $k \in \{1, \dots, n\}$, $\sum_{k=1}^n p_k = 1$, are also given.

2. THE RESULTS

The following result holds:

Theorem 1. *Let $a_k, b_k \in \mathbb{C}$, $k \in \{1, \dots, n\}$, $n \geq 2$ with the property that*

$$(2.1) \quad \sum_{k=1}^n a_k b_k = 0.$$

Then

$$(2.2) \quad \max_{i \in \{1, \dots, n\}} \{|a_i b_i|\} \leq \frac{1}{2} \left(\sum_{k=1}^n |a_k|^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^n |b_k|^2 \right)^{\frac{1}{2}}.$$

The constant $\frac{1}{2}$ in (2.2) is best possible in the sense that it cannot be replaced by a smaller constant.

Proof. For any $i \in \{1, \dots, n\}$, we have

$$(2.3) \quad a_i b_i = - \sum_{\substack{k=1 \\ k \neq i}}^n a_k b_k.$$

Taking the modulus in (2.3) we have

$$(2.4) \quad \begin{aligned} |a_i b_i| &= \left| \sum_{\substack{k=1 \\ k \neq i}}^n a_k b_k \right| \leq \left(\sum_{\substack{k=1 \\ k \neq i}}^n |a_k|^2 \right)^{\frac{1}{2}} \left(\sum_{\substack{k=1 \\ k \neq i}}^n |b_k|^2 \right)^{\frac{1}{2}} \\ &= \left(\sum_{k=1}^n |a_k|^2 - |a_i|^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^n |b_k|^2 - |b_i|^2 \right)^{\frac{1}{2}}, \end{aligned}$$

for any $i \in \{1, \dots, n\}$, where we used the Cauchy-Bunyakovsky-Schwarz inequality to state the required inequality in (2.4).

Utilising the elementary inequality for real numbers

$$(\alpha^2 - \beta^2)^{\frac{1}{2}} (\gamma^2 - \delta^2)^{\frac{1}{2}} \leq \alpha\gamma - \beta\delta,$$

provided $\alpha, \beta, \gamma, \delta > 0$ and $\alpha \geq \beta, \gamma \geq \delta$, we have

$$(2.5) \quad \begin{aligned} &\left(\sum_{k=1}^n |a_k|^2 - |a_i|^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^n |b_k|^2 - |b_i|^2 \right)^{\frac{1}{2}} \\ &= \left\{ \left[\left(\sum_{k=1}^n |a_k|^2 \right)^{\frac{1}{2}} \right]^2 - |a_i|^2 \right\}^{\frac{1}{2}} \left\{ \left[\left(\sum_{k=1}^n |b_k|^2 \right)^{\frac{1}{2}} \right]^2 - |b_i|^2 \right\}^{\frac{1}{2}} \\ &\leq \left(\sum_{k=1}^n |a_k|^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^n |b_k|^2 \right)^{\frac{1}{2}} - |a_i b_i|, \end{aligned}$$

for each $i \in \{1, \dots, n\}$.

Now, on making use of (2.4) and (2.5) we get the desired inequality (2.2).

To prove the sharpness of the constant, we assume that the inequality (2.2) holds true for a constant $C > 0$, i.e.,

$$(2.6) \quad \max_{i \in \{1, \dots, n\}} |a_i b_i| \leq C \left(\sum_{k=1}^n |a_k|^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^n |b_k|^2 \right)^{\frac{1}{2}},$$

provided $a_k, b_k, k \in \{1, \dots, n\}$ ($n \geq 2$) are complex numbers such that $\sum_{k=1}^n a_k b_k = 0$.

Now, for $n = 2$, choose $a_1 = a, a_2 = -b, b_1 = b, b_2 = -a$ with $a, b > 0$. Then $a_1 b_1 + a_2 b_2 = 0, |a_1 b_1| = |a_2 b_2| = ab$ and by (2.6) we get

$$(2.7) \quad ab \leq C (a^2 + b^2) \quad \text{for } a, b > 0.$$

Choosing in (2.7) $a = b = 1$, we deduce $C \geq \frac{1}{2}$ and the proof is complete. ■

The following corollary is of interest.

Corollary 1. *Let $x_k \in \mathbb{C}, k \in \{1, \dots, n\}$ and $p_k, k \in \{1, \dots, n\}$ be a probability sequence, i.e., $p_k \geq 0, k \in \{1, \dots, n\}$ and $\sum_{k=1}^n p_k = 1$. Then we have the*

inequality:

$$\begin{aligned}
(2.8) \quad \max_{i \in \{1, \dots, n\}} \left\{ p_i \left| x_i - \sum_{j=1}^n p_j x_j \right| \right\} &\leq \frac{1}{2} \left(\sum_{k=1}^n p_k^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^n \left| x_k - \sum_{j=1}^n p_j x_j \right|^2 \right)^{\frac{1}{2}} \\
&= \frac{1}{2} \left(\sum_{k=1}^n p_k^2 \right)^{\frac{1}{2}} \left\{ \sum_{k=1}^n |x_k|^2 + n \left| \sum_{j=1}^n p_j x_j \right|^2 \right. \\
&\quad \left. - 2 \operatorname{Re} \left[\left(\sum_{k=1}^n x_k \right) \left(\sum_{j=1}^n p_j \bar{x}_j \right) \right] \right\}^{\frac{1}{2}}.
\end{aligned}$$

Proof. If we choose $a_k = p_k$, $b_k := x_k - \sum_{j=1}^n p_j x_j$, then

$$\sum_{k=1}^n a_k b_k = \sum_{k=1}^n p_k \left(x_k - \sum_{j=1}^n p_j x_j \right) = 0$$

and the condition (2.1) is satisfied.

Applying the inequality (2.2), we obtain

$$\begin{aligned}
&\max_{i \in \{1, \dots, n\}} \left\{ p_i \left| x_i - \sum_{j=1}^n p_j x_j \right| \right\} \\
&\leq \frac{1}{2} \left(\sum_{k=1}^n p_k^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^n \left| x_k - \sum_{j=1}^n p_j x_j \right|^2 \right)^{\frac{1}{2}} \\
&= \frac{1}{2} \left(\sum_{k=1}^n p_k^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^n |x_k|^2 - 2 \operatorname{Re} \left(\sum_{k=1}^n x_k \cdot \sum_{j=1}^n p_j \bar{x}_j \right) + n \left| \sum_{j=1}^n p_j x_j \right|^2 \right)^{\frac{1}{2}}
\end{aligned}$$

and the inequality (2.8) is obtained. ■

Remark 1. If $\min_{i \in \{1, \dots, n\}} p_i = p_m > 0$, then from (2.8) we can obtain a coarser and perhaps more useful inequality, providing some upper bounds for the maximal deviation of x_k from the weighted mean $\sum_{j=1}^n p_j x_j$, namely,

$$(2.9) \quad \max_{k \in \{1, \dots, n\}} \left| x_k - \sum_{j=1}^n p_j x_j \right| \leq \frac{1}{2p_m} \left(\sum_{k=1}^n p_k^2 \right)^{\frac{1}{2}} \left(\left| x_k - \sum_{j=1}^n p_j x_j \right|^2 \right)^{\frac{1}{2}}.$$

The following weighted version of Theorem 1 may be stated as well:

Theorem 2. Let $x_k, y_k \in \mathbb{C}$, $k \in \{1, \dots, n\}$ and p_k , $k \in \{1, \dots, n\}$ be a probability sequence with the property that

$$(2.10) \quad \sum_{k=1}^n p_k x_k y_k = 0.$$

Then

$$(2.11) \quad \max_{i \in \{1, \dots, n\}} \{p_i |x_i y_i|\} \leq \frac{1}{2} \left(\sum_{k=1}^n p_k |x_k|^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^n p_k |y_k|^2 \right)^{\frac{1}{2}}.$$

The constant $\frac{1}{2}$ in (2.11) is best possible in (2.11).

Proof. It follows from Theorem 1 on choosing $a_k = \sqrt{p_k} x_k$, $b_k = \sqrt{p_k} y_k$. ■

Remark 2. One should notice that Theorem 1 and Theorem 2 are equivalent in the sense that one implies the other.

The above result provides the opportunity to obtain a different bound for the maximal deviation of x_k from the weighted mean.

Corollary 2. With the assumptions in Corollary 1, we have the inequality:

$$(2.12) \quad \max_{i \in \{1, \dots, n\}} \left\{ p_i \left| x_i - \sum_{j=1}^n p_j x_j \right| \right\} \leq \frac{1}{2} \left(\sum_{k=1}^n p_k \left| x_k - \sum_{j=1}^n p_j x_j \right|^2 \right)^{\frac{1}{2}} \\ = \frac{1}{2} \left[\sum_{k=1}^n p_k |x_k|^2 - \left| \sum_{j=1}^n p_j x_j \right|^2 \right]^{\frac{1}{2}}.$$

Proof. Follows by Theorem 2 on choosing $y_k = 1$, $k \in \{1, \dots, n\}$. ■

Remark 3. If $\min_{i \in \{1, \dots, n\}} p_i = p_m > 0$, then

$$(2.13) \quad \max_{i \in \{1, \dots, n\}} \left| x_i - \sum_{j=1}^n p_j x_j \right| \leq \frac{1}{2p_m} \left(\sum_{k=1}^n p_k \left| x_k - \sum_{j=1}^n p_j x_j \right|^2 \right)^{\frac{1}{2}}.$$

Remark 4. It is natural to ask which of the bounds for the maximal deviation

$$\max_{i \in \{1, \dots, n\}} \left\{ p_i \left| x_i - \sum_{j=1}^n p_j x_j \right| \right\}$$

provided by (2.8) and (2.12) are better and when, respectively?

For $n = 2$, let $p_1 = p$, $p_2 = 1 - p$, $p \in [0, 1]$, $x_1 = x$, $x_2 = y$, then we have the specific case of

$$B_1(p, x, y) := \frac{1}{2} \left[p^2 + (1-p)^2 \right]^{\frac{1}{2}} \left[(x - px - (1-p)y)^2 + (y - px - (1-p)y)^2 \right]^{\frac{1}{2}} \\ = \frac{1}{2} \left[p^2 + (1-p)^2 \right]^{\frac{1}{2}} \left[(1-p)^2 (x-y)^2 + p^2 (x-y)^2 \right]^{\frac{1}{2}} \\ = \frac{1}{2} \cdot \left[p^2 + (1-p)^2 \right] |x-y|$$

and

$$\begin{aligned} B_2(p, x, y) &:= \frac{1}{2} \left[p(x - px - (1-p)y)^2 + (1-p)(y - px - (1-p)y)^2 \right]^{\frac{1}{2}} \\ &= \frac{1}{2} \left[p(1-p)^2(x-y)^2 + (1-p)p^2(x-y)^2 \right]^{\frac{1}{2}} \\ &= \frac{1}{2} \cdot \sqrt{p(1-p)} |x-y|. \end{aligned}$$

Since $p^2 + (1-p)^2 \geq \sqrt{p(1-p)}$ for $p \in [0, 1]$, we have that the bound (2.12) is always better than (2.8) for $n = 2$.

Remark 5. For $n = 3$, $p_1 = p$, $p_2 = q$, $p_3 = r$, $x_1 = x$, $x_2 = y$, $x_3 = z$, we should compare the bounds

$$\begin{aligned} B_1(p, q, r, x, y, z) &= \frac{1}{2} (p^2 + q^2 + r^2)^{\frac{1}{2}} \times \left[p(x - px - qy - rz)^2 \right. \\ &\quad \left. + q(y - px - qy - rz)^2 + r(z - px - qy - rz)^2 \right]^{\frac{1}{2}} \end{aligned}$$

and

$$\begin{aligned} B_1(p, q, r, x, y, z) &= \frac{1}{2} \left[p(x - px - qy - rz)^2 \right. \\ &\quad \left. + q(y - px - qy - rz)^2 + r(z - px - qy - rz)^2 \right]^{\frac{1}{2}}. \end{aligned}$$

The plot of the function

$$\Delta(0.1, 0.5, 0.4, x, y, -4) = B_1(0.1, 0.5, 0.4, x, y, -4) - B_2(0.1, 0.5, 0.4, x, y, -4)$$

on the box $[0, 6] \times [8, 10]$ shows that one bound is not always better the other (see Figure 1):

Remark 6. In the case of uniform distribution, i.e., when $p_i = \frac{1}{n}$, $i \in \{1, \dots, n\}$, we obtain from both inequalities (2.8) and (2.12) the same result:

$$\begin{aligned} (2.14) \quad \max_{k \in \{1, \dots, n\}} \left| x_k - \frac{1}{n} \sum_{j=1}^n x_j \right| &\leq \frac{1}{2} \sqrt{n} \sum_{k=1}^n \left| x_k - \frac{1}{n} \sum_{j=1}^n x_j \right|^2 \\ &= \frac{1}{2} \left[n \sum_{k=1}^n |x_k|^2 - \left| \sum_{k=1}^n x_k \right|^2 \right]^{\frac{1}{2}}. \end{aligned}$$

3. RELATED RESULTS

The following result may be stated as well.

Theorem 3. Let $a_k, b_k \in \mathbb{C} \setminus \{0\}$, $k \in \{1, \dots, n\}$ so that $\sum_{k=1}^n a_k b_k = 0$. Then for any probability sequence p_k , $k \in \{1, \dots, n\}$, we have:

$$(3.1) \quad \frac{\sum_{j=1}^n p_j |a_j|^2}{\sum_{k=1}^n |a_k|^2} + \frac{\sum_{j=1}^n p_j |b_j|^2}{\sum_{k=1}^n |b_k|^2} \leq 1.$$

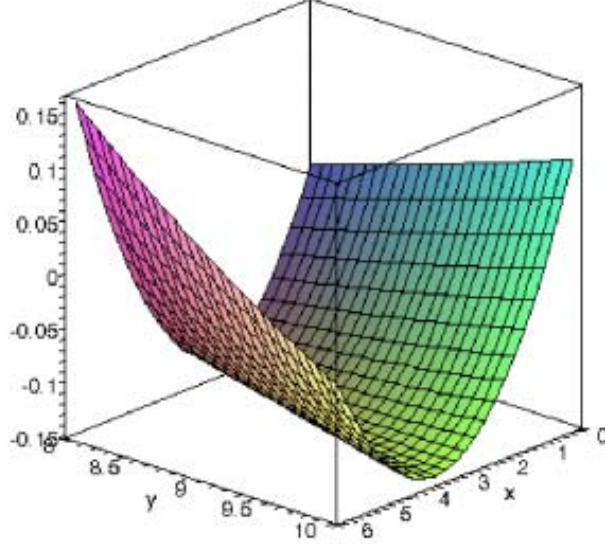


FIGURE 1. Plot of the difference $\Delta(0.1, 0.5, 0.4, x, y, -4)$ showing a transition from positive to negative

Proof. We know, from the proof of Theorem 1, that

$$\begin{aligned} |a_i b_i|^2 &\leq \left(\sum_{k=1}^n |a_k|^2 - |a_i|^2 \right) \left(\sum_{k=1}^n |b_k|^2 - |b_i|^2 \right) \\ &= \sum_{k=1}^n |a_k|^2 \sum_{k=1}^n |b_k|^2 - |a_i|^2 |b_i|^2 - |a_i|^2 \sum_{k=1}^n |b_k|^2 - |b_i|^2 \sum_{k=1}^n |a_k|^2, \end{aligned}$$

which is clearly equivalent with

$$(3.2) \quad |a_i|^2 \sum_{k=1}^n |b_k|^2 + |b_i|^2 \sum_{k=1}^n |a_k|^2 \leq \sum_{k=1}^n |a_k|^2 \sum_{k=1}^n |b_k|^2$$

for each $i \in \{1, \dots, n\}$.

Now, if we multiply (3.2) by $p_i \geq 0$ and sum over $i \in \{1, \dots, n\}$, we deduce:

$$(3.3) \quad \sum_{i=1}^n p_i |a_i|^2 \sum_{k=1}^n |b_k|^2 + \sum_{i=1}^n p_i |b_i|^2 \sum_{k=1}^n |a_k|^2 \leq \sum_{k=1}^n |a_k|^2 \sum_{k=1}^n |b_k|^2$$

which is clearly equivalent with (3.1). ■

Corollary 3. *With the assumptions of the above theorem, we have:*

$$(3.4) \quad \sum_{i=1}^n p_i |a_i|^2 \sum_{i=1}^n p_i |b_i|^2 \leq \frac{1}{4} \sum_{k=1}^n |a_k|^2 \sum_{k=1}^n |b_k|^2.$$

The constant $\frac{1}{4}$ is best possible in (3.4).

Proof. On utilising the inequality $\alpha^2 + \beta^2 \geq 2\alpha\beta$, $\alpha, \beta \in \mathbb{R}_+$, we have

$$(3.5) \quad \sum_{j=1}^n p_j |a_j|^2 \sum_{k=1}^n |b_k|^2 + \sum_{j=1}^n p_j |b_j|^2 \sum_{k=1}^n |a_k|^2 \\ \geq 2 \left(\sum_{j=1}^n p_j |a_j|^2 \sum_{j=1}^n p_j |b_j|^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^n |a_k|^2 \sum_{k=1}^n |b_k|^2 \right)^{\frac{1}{2}}.$$

Now, by (3.3) and (3.5) we deduce the desired inequality (3.4).

To prove the sharpness of the constant, we assume that (3.4) holds true with a $D > 0$, i.e.,

$$\sum_{j=1}^n p_j |a_j|^2 \sum_{j=1}^n p_j |b_j|^2 \leq D \sum_{k=1}^n |a_k|^2 \sum_{k=1}^n |b_k|^2,$$

provided $\sum_{k=1}^n a_k b_k = 0$, $n \geq 2$.

For $n = 2$, we choose $a_1 = a$, $a_2 = -b$, $b_1 = b$, $b_2 = -a$ and $p_1 = p$, $p_2 = 1 - p$ to get:

$$(3.6) \quad [pa^2 + (1-p)b^2] [pb^2 + (1-p)a^2] \leq D [a^2 + b^2]^2.$$

If in (3.6) we choose $p = \frac{1}{2}$, then we get

$$\frac{1}{4} (a^2 + b^2)^2 \leq D (a^2 + b^2)^2,$$

which shows that $D \geq \frac{1}{4}$. ■

Corollary 4. Let $x_k \in \mathbb{C}$, $k \in \{1, \dots, n\}$ and p_k , $k \in \{1, \dots, n\}$ be a probability sequence. Then:

$$\sum_{k=1}^n p_k |x_k|^2 - \left| \sum_{k=1}^n p_k x_k \right|^2 = \sum_{j=1}^n p_j \left| x_j - \sum_{l=1}^n p_l x_l \right|^2 \\ \leq \frac{1}{4} \cdot \frac{\sum_{k=1}^n p_k^2}{\sum_{k=1}^n p_k^3} \sum_{k=1}^n \left| x_k - \sum_{l=1}^n p_l x_l \right|^2.$$

Proof. It is obvious by (3.4) on choosing $a_k = p_k$ and $b_k = x_k - \sum_{l=1}^n p_l x_l$, $k \in \{1, \dots, n\}$. ■

The following result that provides a refinement of Theorem 2 should be noted.

Theorem 4. Let $x_k, y_k \in \mathbb{C}$, $k \in \{1, \dots, n\}$ and p_k , $k \in \{1, \dots, n\}$ be a probability sequence with the property that

$$(3.7) \quad \sum_{k=1}^n p_k x_k y_k = 0.$$

Then

$$\begin{aligned}
(3.8) \quad & \max_{i \in \{1, \dots, n\}} \{p_i |x_i y_i|\} \\
& \leq \frac{1}{2} \cdot \frac{\max_{i \in \{1, \dots, n\}} \left[p_i |x_i|^2 \sum_{k=1}^n p_k |y_k|^2 + p_i |y_i|^2 \sum_{k=1}^n p_k |x_k|^2 \right]}{\left(\sum_{k=1}^n p_k |x_k|^2 \sum_{k=1}^n p_k |y_k|^2 \right)^{\frac{1}{2}}} \\
& \leq \frac{1}{2} \cdot \left(\sum_{k=1}^n p_k |x_k|^2 \sum_{k=1}^n p_k |y_k|^2 \right)^{\frac{1}{2}}.
\end{aligned}$$

Proof. As in the proof of Theorem 1, we have

$$p_i |x_i y_i| \leq \left(\sum_{k=1}^n p_k |x_k|^2 - p_i |x_i|^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^n p_k |y_k|^2 - p_i |y_i|^2 \right)^{\frac{1}{2}},$$

which gives

$$\begin{aligned}
p_i^2 |x_i y_i|^2 & \leq \left(\sum_{k=1}^n p_k |x_k|^2 - p_i |x_i|^2 \right) \left(\sum_{k=1}^n p_k |y_k|^2 - p_i |y_i|^2 \right) \\
& = \sum_{k=1}^n p_k |x_k|^2 \sum_{k=1}^n p_k |y_k|^2 + p_i^2 |x_i|^2 |y_i|^2 - p_i |x_i|^2 \sum_{k=1}^n p_k |y_k|^2 \\
& \quad - p_i |y_i|^2 \sum_{k=1}^n p_k |x_k|^2,
\end{aligned}$$

i.e.,

$$(3.9) \quad p_i |x_i|^2 \sum_{k=1}^n p_k |y_k|^2 + p_i |y_i|^2 \sum_{k=1}^n p_k |x_k|^2 \leq \sum_{k=1}^n p_k |x_k|^2 \sum_{k=1}^n p_k |y_k|^2$$

for each $i \in \{1, \dots, n\}$.

Taking the maximum in (3.9) over $i \in \{1, \dots, n\}$, we get the second inequality in (3.8).

The first inequality follows by the elementary fact that

$$\begin{aligned}
p_i |x_i|^2 \sum_{k=1}^n p_k |y_k|^2 + p_i |y_i|^2 \sum_{k=1}^n p_k |x_k|^2 \\
\geq 2p_i |x_i| |y_i| \left(\sum_{k=1}^n p_k |x_k|^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^n p_k |y_k|^2 \right)^{\frac{1}{2}},
\end{aligned}$$

for each $i \in \{1, \dots, n\}$. ■

Remark 7. The inequality (3.8) is obviously a refinement of the inequality (2.11) in Theorem 2. However, the inequality (3.8) is not apparently useful in deriving upper bounds for the maximal deviation of x_k from its weighted mean $\sum_{j=1}^n p_j x_j$, as the inequality (2.11).

REFERENCES

- [1] N.G. de BRUIJN, Problem 12, *Wisk. Opgaven*, **21**(1960), 12-14.
- [2] S.S. DRAGOMIR, *Discrete Inequalities of the Cauchy-Bunykovsky-Schwarz Type*, Nova Science Publishers Inc., N.Y., 2004.
- [3] S.S. DRAGOMIR, Reverses of the Cauchy-Bunyakovsky-Schwarz inequality for n -tuples of complex numbers. *Bull. Austral. Math. Soc.* **69** (2004), no. 3, 465–480.
- [4] S.S. DRAGOMIR and B. MOND, On the superadditivity and monotonicity of Schwarz's inequality in inner product spaces, *Contributions Ser. Math. Sci. Macedonian Acad. Sci.*, **15**(1) (1994), 5-22.
- [5] S.S. DRAGOMIR and B. MOND, Some mappings associated with Cauchy-Buniakowski-Schwarz's inequality in inner product spaces. *Soochow J. Math.* **21** (1995), no. 4, 413–426.
- [6] J. W. HOVENIER, Sharpening Cauchy's inequality. *J. Math. Anal. Appl.* **186** (1994), no. 1, 156–160.
- [7] Y.-C. LI, ;S.-Y. SHAW, A proof of Hölder's inequality using the Cauchy-Schwarz inequality. *J. Inequal. Pure Appl. Math.* **7** (2006), no. 2, Article 62, 3 pp.
- [8] Z. LIU, Remark on a refinement of the Cauchy-Schwarz inequality. *J. Math. Anal. Appl.* **218** (1998), no. 1, 13–21.
- [9] P. SCHREIBER, The Cauchy-Bunyakovsky-Schwarz inequality, Hermann Graßmann (Lieschow, 1994), 64–70, Ernst-Moritz-Arndt Univ., Greifswald, 1995.

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