SCHUR-CONVEXITY AND SCHUR-GEOMETRICALLY CONCAVITY OF GINI MEAN

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ABSTRACT. The monotonicity and the Schur-convexity with parameters (s,t) in $\mathbb{R}^2$ for fixed (x,y) and the Schur-convexity and the Schur-geometrically convexity with variables (x,y) in $\mathbb{R}^2_{++}$ for fixed (s,t) of Gini mean $G(r,s;x,y)$ are discussed. Some new inequalities are obtained.

1. INTRODUCTION

Throughout the paper we assume that the set of the real number, the nonnegative real number, the nonpositive real number and the positive real number by $\mathbb{R}, \mathbb{R}_+, \mathbb{R}_- and \mathbb{R}_{++}$ respectively.

let $(s,t) \in \mathbb{R}^2, (x,y) \in \mathbb{R}^2_{++}$. The Gini mean of (x,y) is defined in [1] and [2, p. 44] as

$$G(r,s;x,y) = \begin{cases} \left(\dfrac{x^s+y^s}{x^r+y^r}\right)^{1/(s-r)}, & r \neq s, \\ \exp\left(\dfrac{x^s \ln x + y^s \ln y}{x^r+y^r}\right), & r = s. \end{cases}$$

Clearly, $G(0,-1;x,y)$ is the harmonic mean, $G(0,0;x,y)$ is the geometric mean, $G(1,0;x,y)$ is the arithmetic mean. In 1988, Zs.Páles proved the following result on the comparison of the Gini means.

Theorem A ([3]). *Let $u,v,t,w \in \mathbb{R}$. Then the comparison inequality*

$$G(u,v;x,y) \leq G(t,w;x,y) \tag{1}$$

is valid if and if $u+v \leq t+w$, and
(*i*) $\min\{u,v\} \leq min\{t,w\}$, *if* $0 \leq \min\{u,v,t,w\}$,
(*ii*) $k(u,v) \leq k(t,w)$, *if* $\min\{u,v,t,w\} < 0 < \max\{u,v,t,w\}$,
(*iii*) $\max\{u,v\} \leq \max\{t,w\}$, *if* $\max\{u,v,t,w\} \leq 0$,
here

$$k(x,y) = \begin{cases} \frac{|x|-|y|}{x-y}, & x \neq y \\ sign(x), & x = y. \end{cases}$$

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The extended mean (or Stolarsky mean) of (x,y) is defined in [2, p. 43] as

$$E(r,s;x,y)=\begin{cases}\left(\dfrac{r}{s}\cdot\dfrac{y^s-x^s}{y^r-x^r}\right)^{1/(s-r)}, & rs(r-s)(x-y)\neq 0,\\ \left(\dfrac{1}{r}\cdot\dfrac{y^r-x^r}{\ln y-\ln x}\right)^{1/r}, & r(x-y)\neq 0;\\ \dfrac{1}{e^{1/r}}\left(\dfrac{x^{x^r}}{y^{y^r}}\right)^{1/(x^r-y^r)}, & r(x-y)\neq 0;\\ \sqrt{xy}, & x\neq y;\\ x, & x=y.\end{cases}$$

The Schur-convexities of the extended mean $E(r,s;x,y)$ with (r,s) and (x,y) were presented in [4, 5] as follows.

Theorem B ([4])**.** *For fixed (x,y) with $x>0$, $y>0$ and $x\neq y$, the extended mean $E(r,s;x,y)$ are Schur-concave on $[0,+\infty)\times[0,+\infty)$ and Schur-convex on $(-\infty,0]\times(-\infty,0]$ with (r,s).*

Theorem C ([5])**.** *For fixed $(r,s)\in\mathbb{R}^2$,*

(1) *if $2<2r<s$ or $2\leq 2s\leq r$, then the extended mean values $E(r,s;x,y)$ is Schur-convex with $(x,y)\in(0,\infty)\times(0,\infty)$,*
(2) *if $(r,s)\in\{r<s\leq 2r, 0<r\leq 1\}\cup\{s<r\leq 2s, 0<s\leq 1\}\cup\{0<s<r\leq 1\}\cup\{0<r<s\leq 1\}\cup\{s\leq 2r<0\}\cup\{r\leq 2s<0\}$, then the extended mean $E(r,s;x,y)$ is Schur-concave with $(x,y)\in(0,\infty)\times(0,\infty)$.*

For more information on the extended mean values and Gini mean, please refer to [3, 4, 5, 6, 7, 8, 9, 10] and the references therein.

In this paper, the monotonicity and the Schur-convexity with parameters (s,t) in $\mathbb{R}^2$ for fixed (x,y) and the Schur-convexity and the Schur-geometrically convexity with variables (x,y) in $\mathbb{R}^2_{++}$ for fixed (s,t) of Gini mean $G(r,s;x,y)$ are discussed. Moreover, some new inequalities are obtained.

2. Definitions and Lemmas

We need the following definitions and lemmas.

Definition 1 ([11, 12])**.** Let $\boldsymbol{x}=(x_1,\ldots,x_n)$ and $\boldsymbol{y}=(y_1,\ldots,y_n)\in\mathbb{R}^n$.

(1) $\boldsymbol{x}$ is said to be majorized by $\boldsymbol{y}$ (in symbols $\boldsymbol{x}\prec\boldsymbol{y}$) if $\sum_{i=1}^k x_{[i]}\leq\sum_{i=1}^k y_{[i]}$ for $k=1,2,\ldots,n-1$ and $\sum_{i=1}^n x_i=\sum_{i=1}^n y_i$, where $x_{[1]}\geq\cdots\geq x_{[n]}$ and $y_{[1]}\geq\cdots\geq y_{[n]}$ are rearrangements of $\boldsymbol{x}$ and $\boldsymbol{y}$ in a descending order.
(2) $\boldsymbol{x}\geq\boldsymbol{y}$ means $x_i\geq y_i$ for all $i=1,2,\ldots,n$. let $\Omega\subset\mathbb{R}^n$, φ: $\Omega\to\mathbb{R}$ is said to be increasing if $\boldsymbol{x}\geq\boldsymbol{y}$ implies $\varphi(\boldsymbol{x})\geq\varphi(\boldsymbol{y})$. φ is said to be decreasing if and only if $-\varphi$ is increasing.
(3) $\Omega\subset\mathbb{R}^n$ is called a convex set if $(\alpha x_1+\beta y_1,\ldots,\alpha x_n+\beta y_n)\in\Omega$ for any $\boldsymbol{x}$ and $\boldsymbol{y}\in\Omega$, where α and $\beta\in[0,1]$ with $\alpha+\beta=1$.
(4) let $\Omega\subset\mathbb{R}^n$, φ: $\Omega\to\mathbb{R}$ is said to be a Schur-convex function on Ω if $\boldsymbol{x}\prec\boldsymbol{y}$ on Ω implies $\varphi(\boldsymbol{x})\leq\varphi(\boldsymbol{y})$. φ is said to be a Schur-concave function on Ω if and only if $-\varphi$ is Schur-convex function.

Definition 2 ([13, 14])**.** Let $\boldsymbol{x}=(x_1,\ldots,x_n)$ and $\boldsymbol{y}=(y_1,\ldots,y_n)\in\mathbb{R}^n_{++}$.

(1) $\Omega \subset \mathbb{R}_{++}^n$ is called a geometrically convex set if $(x_1^\alpha y_1^\beta, \ldots, x_n^\alpha y_n^\beta) \in \Omega$ for any $\boldsymbol{x}$ and $\boldsymbol{y} \in \Omega$, where α and $\beta \in [0,1]$ with $\alpha + \beta = 1$.
(2) let $\Omega \subset \mathbb{R}_{++}^n$, φ: $\Omega \to \mathbb{R}_+$ is said to be a Schur-geometrically convex function on Ω if $(\ln x_1, \ldots, \ln x_n) \prec (\ln y_1, \ldots, \ln y_n)$ on Ω implies $\varphi(\boldsymbol{x}) \leq \varphi(\boldsymbol{y})$. φ is said to be a Schur-geometrically concave function on Ω if and only if $-\varphi$ is Schur-geometrically convex function.

Lemma 1 ([11, p. 38])**.** *A function $\varphi(\boldsymbol{x})$ is increasing if and only if $\nabla\varphi(\boldsymbol{x}) \geq 0$ for $\boldsymbol{x} \in \Omega$, where $\Omega \subset \mathbb{R}^n$ is an open set, $\varphi : \Omega \to R$ is differentiable, and*

$$\nabla\varphi(\boldsymbol{x}) = \left(\frac{\partial\varphi(\boldsymbol{x})}{\partial x_1}, \ldots, \frac{\partial\varphi(\boldsymbol{x})}{\partial x_n}\right) \in \mathbb{R}^n.$$

Lemma 2 ([11, p. 58])**.** *Let $\Omega \subset \mathbb{R}^n$ is symmetric and has a nonempty interior set. Ω^0 is the interior of Ω. $\varphi : \Omega \to \mathbb{R}$ is continuous on Ω and differentiable in Ω^0. Then φ is the $Schur-convex(Schur-concave)function$, if and only if φ is symmetric on Ω and*

$$(x_1 - x_2)\left(\frac{\partial\varphi}{\partial x_1} - \frac{\partial\varphi}{\partial x_2}\right) \geq 0 (\leq 0) \tag{2}$$

holds for any $\boldsymbol{x} = (x_1, x_2, \cdots, x_n) \in \Omega^0$.

Lemma 3 ([13, p. 108])**.** *Let $\Omega \subset \mathbb{R}_{++}^n$ is a symmetric and has a nonempty interior geometrically convex set. Ω^0 is the interior of Ω. $\varphi : \Omega \to \mathbb{R}_+$ is continuous on Ω and differentiable in Ω^0. If φ is symmetric on Ω and*

$$(\ln x_1 - \ln x_2)\left(x_1\frac{\partial\varphi}{\partial x_1} - x_2\frac{\partial\varphi}{\partial x_2}\right) \geq 0 (\leq 0) \tag{3}$$

holds for any $\boldsymbol{x} = (x_1, x_2, \cdots, x_n) \in \Omega^0$, then φ is the Schur-geometrically convex (Schur-geometrically concave) function.

Lemma 4 ([15, 5])**.** *Let f be a continuous function. The arithmetic mean of function f defined by*

$$F(x,y) = \begin{cases} \dfrac{1}{x-y} \int_x^y f(t)\,\mathrm{d}t, & x \neq y, \\ f(x), & x = y. \end{cases} \tag{4}$$

Then

(1) *function $F(x,y)$ is increasing (decreasing) on I^2 if and only if f is an increasing (decreasing) function on I.*
(2) *$F(x,y)$ is Schur-convex (Schur-concave) on I^2 if and only if $f(x)$is convex (concave) on I*

Lemma 5 ([16])**.** *Let $a \leq b, u(t) = tb + (1-t)a, v(t) = ta + (1-t)b$ and let $1/2 \leq t_2 \leq t_1 \leq 1$. Then*

$$(u(t_2), v(t_2)) \prec (u(t_1), v(t_1)) \prec (a, b).$$

Lemma 6 ([17])**.** *Let $l, t, p, q \in \mathbb{R}_{++}, p > q$ and $p + q \leq 3(l+t)$. Assume also that $1/3 \leq l/t \leq 3$ or $q \leq l + t$. Then*

$$G(l, t; x, y) \leq (p/q)^{1/(p-q)} E(p, q; x, y).$$

3. Main results and their proofs

In the following, we are in a position to state our main results and give proofs of them.

Theorem 1. *For fixed $(x, y) \in \mathbb{R}^2_{++}$,*

(1) *$G(r, s; x, y)$ is increasing in $\mathbb{R}^2$,*
(2) *$G(r, s; x, y)$ is Schur-concave in $\mathbb{R}^2_+$,*
(3) *$G(r, s; x, y)$ is Schur-convex in $\mathbb{R}^2_-$*

with(r, s).

Proof. Let $g(t) = \dfrac{x^t \ln x + y^t \ln y}{x^t + y^t}$. It is easy to check that

$$\ln G(r, s; x, y) = \frac{1}{r-s} \int_r^s g(t)\ dt, \quad r \neq s,$$

(see the proof of Lemma 3.1 of in [18]). By computing, we get

$$g^{'}(t) = \frac{x^t y^t}{(x^t + y^t)^2} (\ln x - \ln y)^2,$$

$$g^{''}(t) = -\frac{x^t y^t (\ln x - \ln y)^2}{(x^t - y^t)^4} (\ln x - \ln y)(x^{2t} - y^{2t}).$$

(1) Since $g^{'}(t) \geq 0$, $g(t)$ is increasing in $\mathbb{R}$. By (1) in lemma 4 , it follows that $\ln G(r, s; x, y)$ is increasing in $\mathbb{R}^2$ with (r, s), and so does $G(r, s; x, y)$.

(2) When $(r, s) \in \mathbb{R}^2_+$, $\ln u$ and u^{2t} are increasing in $\mathbb{R}_+$ with u, hence $(\ln x - \ln y)(x^{2t} - y^{2t}) \geq 0$, moreover $g^{''}(t) \leq 0$. That is, $g(t)$ is concave in $\mathbb{R}_+$. By (2) in lemma 4 , it follows that

$$F(r, s) = \begin{cases} \ln G(r, s; x, y), & r \neq s, \\ g(t), & r = s \end{cases}$$

is Schur-concave in $\mathbb{R}^2_+$ with (r, s), and so does $G(r, s; x, y)$ by Corollary 6.14 in [11, p. 64].

(3) When $(r, s) \in \mathbb{R}^2_-$, $\ln u$ is increasing and u^{2t} is decreasing in $\mathbb{R}_+$ with u, hence $(\ln x - \ln y)(x^{2t} - y^{2t}) \leq 0$, moreover $g^{''}(t) \geq 0$. That is, $g(t)$ is convex in $\mathbb{R}_-$. By (2) in lemma 4 , it follows that $F(r, s)$ is Schur-convex in $\mathbb{R}^2_-$ with (r, s), and so does $G(r, s; x, y)$.

The proof is complete. □

Remark 1. Let $0 \leq \min\{u, v, t, w\}$. If $(u, v) \prec (t, w)$, it follows that $u+v = t+w$ and $\min\{u, v\} \leq \min\{t, w\}$. By the Theorem A, we have $G(u, v; x, y) \leq G(t, w; x, y)$, but by (2) in the Theorem 1, we have $G(u, v; x, y) \geq G(t, w; x, y)$. In fact, (1) is wrong for the cases (i), for example, taking $u = 2, v = 1, t = 3, w = 0$, then (1) is reduces to

$$\frac{x^2 + y^2}{x + y} \leq \left(\frac{x^3 + y^3}{2} \right)^{\frac{1}{3}},$$

it is equivalent to

$$(x^2 + y^2)^3 \leq 3xy(x^4 + y^4) + 2x^3 y^3.$$

Taking $x > 0$ and let $y \to 0^+$, it following that $x^6 \leq 0$, this leads to a contradiction with $x > 0$.

In fact, (3) of Theorem 1 gives a sufficient condition such that the reversed inequality of (1) is valid, i.e.

$$\max\{u,v,t,w\} \le 0, \max\{u,v\} \le \max\{t,w\}, u+v = t+w.$$

Theorem 2. *If $(r,s) \in \Omega_1 = \{r \ge s \ge 1\} \cup \{s \ge r \ge 1\} \subseteq \mathbb{R}^2$, then $G(r,s;x,y)$ is the Schur-convex with (x,y) in $\mathbb{R}^2_{++}$.*

Proof. Let $\varphi(x,y) = \dfrac{x^s+y^s}{x^r+y^r}$. When $r \ne s$, for fixed $(x,y) \in \mathbb{R}^2$, we have

$$\frac{\partial \varphi}{\partial x} = \frac{sx^{s-1}(x^r+y^r) - rx^{r-1}(x^s+y^s)}{(x^r+y^r)^2},$$

$$\frac{\partial \varphi}{\partial y} = \frac{sy^{s-1}(x^r+y^r) - ry^{r-1}(x^s+y^s)}{(x^r+y^r)^2}.$$

$$\begin{aligned}\frac{\partial \varphi}{\partial x} - \frac{\partial \varphi}{\partial y} &= \frac{s(x^r+y^r)(x^{s-1}-y^{r-1}) - r(x^s+y^s)(x^{r-1}-y^{r-1})}{(x^r+y^r)^2} \\ &= \frac{s(x^{r-1}-y^{r-1})}{(x^r+y^r)}\left[\frac{s-1}{r-1}\cdot\frac{(r-1)(x^{s-1}-y^{s-1})}{(s-1)(x^{r-1}-y^{r-1})} - \frac{r}{s}\cdot\frac{x^s+y^s}{x^r+y^r}\right] \\ &= \frac{s(x^{r-1}-y^{r-1})}{(x^r+y^r)}\left[\frac{s-1}{r-1}\cdot E^{s-r}(r-1,s-1;x,y) - \frac{r}{s}\cdot G^{s-r}(r,s;x,y)\right],\end{aligned}$$

and then

$$\begin{aligned}&(x-y)\left(\frac{\partial G}{\partial x} - \frac{\partial G}{\partial y}\right) = \frac{x-y}{s-r}\left(\frac{\partial \varphi}{\partial x} - \frac{\partial \varphi}{\partial y}\right)\varphi^{\frac{1}{s-r}-1}(x,y) \\ &= \frac{s(x-y)(x^{r-1}-y^{r-1})}{(s-r)(x^r+y^r)}\left[\frac{s-1}{r-1}\cdot E^{s-r}(r-1,s-1;x,y) - \frac{r}{s}\cdot G^{s-r}(r,s;x,y)\right]\varphi^{\frac{1}{s-r}-1}(x,y)\end{aligned}$$

In lemma 6, taking $l=r, t=s, p=r-1, q=s-1$, we have

$$\begin{cases} l>0, t>0, p>0, q>0 \\ p>q \\ p+q \le 3(l+t) \\ 1/3 \le l/t \le 3 \end{cases} \Leftrightarrow \begin{cases} r>1, s>1 \\ r>s \\ r+s \ge -1 \\ s/3 \le r \le 3s \end{cases} \Leftrightarrow 3s \ge r > s > 1$$

and

$$\begin{cases} l>0, t>0, p>0, q>0 \\ p>q \\ p+q \le 3(l+t) \\ q \le l+t \end{cases} \Leftrightarrow \begin{cases} r>1, s>1 \\ r>s \\ r+s \ge -1 \\ r \ge -1 \end{cases} \Leftrightarrow r > s > 1.$$

Hence, when $r > s > 1$, we have

$$G(r,s;x,y) \le \left(\frac{r-1}{s-1}\right)^{\frac{1}{r-s}} E(r-1,s-1;x,y),$$

i.e.

$$G^{s-r}(r,s;x,y) \ge \frac{s-1}{r-1}\cdot E^{s-r}(r-1,s-1;x,y). \tag{5}$$

When $r > s > 1$, we have $s - r < 0$ and $(x-y)(x^{r-1} - y^{r-1}) \geq 0$. Combining with (5), it follows that $(x-y)\left(\frac{\partial G}{\partial x} - \frac{\partial G}{\partial y}\right) \geq 0$. By lemma 2, $G(r,s;x,y)$ is the Schur-convex with (x,y) in $\mathbb{R}^2_{++}$. Since $G(r,s;x,y)$ is symmetric with (r,s), when $s > r > 1$, $G(r,s;x,y)$ is also the Schur-convex with (x,y) in $\mathbb{R}^2_{++}$.

When $r = s$, let $\psi(x,y) = \dfrac{x^s \ln x + y^s \ln y}{x^r + y^r} = \dfrac{x^s \ln x + y^s \ln y}{x^s + y^s}$. Then

$$\frac{\partial \psi}{\partial x} = \frac{x^{s-1}h(x,y)}{(x^s+y^s)^2}, \quad \frac{\partial \psi}{\partial y} = \frac{y^{s-1}k(x,y)}{(x^s+y^s)^2},$$

where

$$h(x,y) = (s\ln x + 1)(x^s + y^s) - s(x^s \ln x + y^s \ln y),$$
$$k(x,y) = (s\ln y + 1)(x^s + y^s) - s(x^s \ln x + y^s \ln y).$$

By computing,

$$\begin{aligned}&x^{s-1}h(x,y) - y^{s-1}k(x,y)\\ &= (x^s+y^s)\left[x^{s-1}(s\ln x+1) - y^{s-1}(s\ln y + 1)\right] - s(x^s\ln x + y^s \ln y)(x^{s-1} - y^{s-1})\\ &= s^{s-1}y^{s-1}(x+y)(\ln x - \ln y) + (x^{s-1} - y^{s-1})(x^s+y^s),\end{aligned}$$

and then,

$$\begin{aligned}&(x-y)\left(\frac{\partial G}{\partial x} - \frac{\partial G}{\partial y}\right) = (x-y)\left(\frac{\partial \psi}{\partial x} - \frac{\partial \psi}{\partial y}\right)e^{\psi(x,y)}\\ &= \frac{sx^{s-1}y^{s-1}(x+y)(x-y)(\ln x - \ln y) + (x-y)(x^{s-1}-y^{s-1})(x^s+y^s)}{(x^s+y^s)^2}e^{\psi(x,y)}.\end{aligned}$$

Since $\ln t$ and t^{s-1} are increasing in $\mathbb{R}_+$ with t for $s \geq 1$, therefore $(x-y)(\ln x - \ln y) \geq 0$ and $(x-y)(x^{s-1} - y^{s-1}) \geq 0$, moreover $(x-y)\left(\frac{\partial G}{\partial x} - \frac{\partial G}{\partial y}\right) \geq 0$. That is, when $r = s \geq 1$, $G(r,s;x,y)$ is the Schur-convex with (x,y) in $\mathbb{R}^2_{++}$.

The proof is complete. □

Theorem 3. *If $(r,s) \in \Omega_2 = \{r \geq s \geq 0\} \cup \{s \geq r \geq 0\} \subseteq \mathbb{R}^2$, then $G(r,s;x,y)$ is the Schur-geometrically convex with (x,y) in $\mathbb{R}^2_{++}$.*

Proof. Let $\varphi(x,y) = \dfrac{x^s+y^s}{x^r+y^r}$. When $r \neq s$, for fixed $(x,y) \in \mathbb{R}^2$, we have

$$x\frac{\partial \varphi}{\partial x} = \frac{sx^s(x^r+y^r) - rx^r(x^s+y^s)}{(x^r+y^r)^2},$$

$$y\frac{\partial \varphi}{\partial y} = \frac{sy^s(x^r+y^r) - ry^r(x^s+y^s)}{(x^r+y^r)^2}.$$

$$\begin{aligned}x\frac{\partial \varphi}{\partial x} - y\frac{\partial \varphi}{\partial y} &= \frac{s(x^r+y^r)(x^s-y^r) - r(x^s+y^s)(x^r-y^r)}{(x^r+y^r)^2}\\ &= \frac{s(x^r-y^r)}{x^r+y^r}\left[\frac{s}{r}\cdot\frac{r(x^s-y^s)}{s(x^r-y^r)} - \frac{r}{s}\cdot\frac{x^s+y^s}{x^r+y^r}\right]\\ &= \frac{s(x^r-y^r)}{x^r+y^r}\left[\frac{s}{r}\cdot E^{s-r}(r,s;x,y) - \frac{r}{s}\cdot G^{s-r}(r,s;x,y)\right],\end{aligned}$$

and then

$$(\ln x-\ln y)\left(x\frac{\partial G}{\partial x}-y\frac{\partial G}{\partial y}\right)=\frac{\ln x-\ln y}{s-r}\left(x\frac{\partial \varphi}{\partial x}-y\frac{\partial \varphi}{\partial y}\right)\varphi^{\frac{1}{s-r}-1}(x,y)$$
$$=\frac{s(\ln x-\ln y)(x^r-y^r)}{(s-r)(x^r+y^r)}\left[\frac{s}{r}\cdot E^{s-r}(r,s;x,y)-\frac{r}{s}\cdot G^{s-r}(r,s;x,y)\right]\varphi^{\frac{1}{s-r}-1}(x,y)$$

In lemma 6, taking $l=p=r, t=q=s$, we have

$$\begin{cases} l>0, t>0, p>0, q>0\\ p>q\\ p+q\le 3(l+t)\\ 1/3\le l/t\le 3\end{cases} \Leftrightarrow \begin{cases} r>0, s>0\\ r>s\\ r+s\ge -1\\ s/3\le r\le 3s\end{cases} \Leftrightarrow 3s\ge r>s>0$$

and

$$\begin{cases} l>0, t>0, p>0, q>0\\ p>q\\ p+q\le 3(l+t)\\ q\le l+t\end{cases} \Leftrightarrow \begin{cases} r>0, s>0\\ r>s\\ r+s\ge -1\\ r\ge 0\end{cases} \Leftrightarrow r>s>0.$$

Hence, when $r>s>0$, we have

$$G(r,s;x,y)\le\left(\frac{r}{s}\right)^{\frac{1}{r-s}}E(r,s;x,y),$$

i.e.

$$G^{s-r}(r,s;x,y)\ge\frac{s}{r}\cdot E^{s-r}(r,s;x,y). \tag{6}$$

When $r>s>0$, we have $s-r<0$, and since $\ln t$ and t^r are increasing in $\mathbb{R}_+$ with t, therefore $(\ln x-\ln y)(x^r-y^r)\ge 0$. Combining with (6), it follows that $(\ln x-\ln y)\left(x\frac{\partial G}{\partial x}-y\frac{\partial G}{\partial y}\right)\ge 0$. By lemma 3, $G(r,s;x,y)$ is the Schur-geometrically convex with (x,y) in $\mathbb{R}^2_{++}$. Since $G(r,s;x,y)$ is symmetric with (r,s), when $s>r>0$, $G(r,s;x,y)$ is also the Schur-geometrically convex with (x,y) in $\mathbb{R}^2_{++}$.

When $r=s$, we have

$$\frac{\partial \psi}{\partial x}=\frac{x^{s-1}h(x,y)}{(x^s+y^s)^2},\quad \frac{\partial \psi}{\partial y}=\frac{x^{s-1}k(x,y)}{(x^s+y^s)^2},$$

where $h(x,y), k(x,y)$ and $\psi(x,y)$ are same as in Theorem 2.

By computing,

$$x^s h(x,y)-y^s k(x,y)=s^s y^s(x+y)(\ln x-\ln y)+(x^s-y^s)(x^s+y^s),$$

and then,

$$(\ln x-\ln y)\left(x\frac{\partial G}{\partial x}-y\frac{\partial G}{\partial y}\right)=(\ln x-\ln y)\left(x\frac{\partial \psi}{\partial x}-y\frac{\partial \psi}{\partial y}\right)e^{\psi(x,y)}$$
$$=\frac{sx^s y^s(x+y)(\ln x-\ln y)^2+(\ln x-\ln y)(x^s-y^s)(x^s+y^s)}{(x^s+y^s)^2}e^{\psi(x,y)}.$$

Since when $s\ge 0$, $\ln t$ and t^s are increasing in $\mathbb{R}_+$, $(\ln x-\ln y)(x^s-y^s)\ge 0$, moreover $(\ln x-\ln y)\left(x\frac{\partial G}{\partial x}-y\frac{\partial G}{\partial y}\right)\ge 0$. That is, when $r=s\ge 0$, $G(r,s;x,y)$ is the Schur-geometrically convex with (x,y) in $\mathbb{R}^2_{++}$.

The proof is complete.

□

4. Applications

Theorem 4. *Let $u(t) = ts + (1-t)r, v(t) = tr + (1-t)s$, and let $\frac{1}{2} \le t_2 \le t_1 \le 1$. If $s \ge r \ge 0$, then for fixed $(x, y) \in \mathbb{R}^2_{++}$,*

$$G\left(\frac{r+s}{2}, \frac{r+s}{2}; x, y\right) \le G\left(u(t_2), v(t_2); x, y\right)$$
$$\le G\left(u(t_1), v(t_1); x, y\right) \le G(r, s; x, y) \le G(r+s, 0; x, y), \tag{7}$$

and if $s \le r \le 0$, then inequalities in (7) are all reversed.

Proof. From lemma 5, we have

$$\left(\frac{r+s}{2}, \frac{r+s}{2}\right) \prec (u(t_2), v(t_2)) \prec (u(t_1), v(t_1)) \prec (s, r).$$

and it is clear that $(r, s) \prec (r+s, 0)$, therefore, by Theorem 1, it follows that (7) are holds. The proof is complete. □

According to Theorem 2, The following Theorem 5 can be proved in a similar way as shown Theorem 4.

Theorem 5. *Let $(x, y) \in \mathbb{R}^2_{++}$, $u(t) = ty + (1-t)x, v(t) = tx + (1-t)y$, and let $\frac{1}{2} \le t_2 \le t_1 \le 1$. If $(r, s) \in \Omega_1 = \{r \ge s \ge 1\} \cup \{s \ge r \ge 1\} \subseteq \mathbb{R}^2$, then*

$$G\left(r, s; \frac{x+y}{2}, \frac{x+y}{2}\right) \le G\left(r, s; u(t_2), v(t_2)\right)$$
$$\le G\left(r, s; u(t_1), v(t_1)\right) \le G(r, s; x, y) \le G(r, s; x+y, 0). \tag{8}$$

Theorem 6. *Let $(x, y) \in \mathbb{R}^2_{++}$. If $(r, s) \in \Omega_2 = \{r \ge s \ge 0\} \cup \{s \ge r \ge 0\} \subseteq \mathbb{R}^2$, then*

$$G\left(r, s; \sqrt{xy}, \sqrt{xy}\right) \le G(r, s; x, y). \tag{9}$$

Proof. Since $(\ln\sqrt{xy}, \ln\sqrt{xy}) \prec (\ln x, \ln y)$, by Theorem 3, it follows that (9) is holds.

The proof is complete. □

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