A note on f-minimum functions

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Abstract. For a given arithmetical function $f : \mathbb{N} \to \mathbb{N}$, let $F : \mathbb{N} \to \mathbb{N}$ be defined by $F(n) = \min\{m \geq 1 : n|f(m)\}$, if this exists. Such functions, introduced in [4], will be called as the f-minimum functions. If f satisfies the property $a \leq b \Rightarrow f(a)|f(b)$, we shall prove that $F(ab) = \max\{F(a), F(b)\}$ for $(a, b) = 1$. For a more restrictive class of functions, we will determine $F(n)$ where n is an even perfect number. These results are generalizations of theorems from [10], [1], [3], [6].

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1 Introduction

Let $\mathbb{N} = \{1, 2, \dots\}$ be the set of positive integers, and $f : \mathbb{N} \to \mathbb{N}$ a given arithmetical function, such that for each $n \in \mathbb{N}$ there exists at least an $m \in \mathbb{N}$ such that $n|f(m)$. In 1999 and 2000 [4], [5], as a common generalization of many arithmetical functions, we have defined the application $F : \mathbb{N} \to \mathbb{N}$ given by

$$F(n) = \min\{m \geq 1 : n|f(m)\}, \tag{1}$$

called as the "f-minimum function". Particularly, for $f(m) = m!$ one obtains the Smarandache function (see [10], [1])

$$S(n) = \min\{m \geq 1 : n|m!\} \tag{2}$$

Moree and Roskam [2], and independently the author [4], [5], have considered the Euler minimum function

$$E(n) = \min\{m \geq 1 : n|\varphi(n)\} \tag{3}$$

where φ is Euler's totient. Many other particular cases of (1), as well as, their "dual" or analogues functions have been studied in the literature; for a survey of concepts and results, see [9].

In 1980 Smarandache discovered the following basic property of $S(n)$ given by (2):

$$S(ab) = \max\{S(a), S(b)\} \text{ for } (a, b) = 1. \tag{4}$$

Our aim in what follows is to extend property (4) to a general class of f-minimum functions. Further, for a subclass we will be able to determine $F(n)$ for even perfect numbers n.

2 Main results

Theorem 1. *Suppose that* F *of (1) is well defined. Then for distinct primes* p_i*, and arbitrary* $\alpha_i \geq 1$ $(i = \overline{1,r})$ *one has*

$$F\left(\prod_{i=1}^{r} p_i^{\alpha_i}\right) \geq \max\{F(p_i^{\alpha_i}) : i = \overline{1,r}\}. \tag{5}$$

The second result offers a reverse inequality:

Theorem 2. *With the notations of Theorem 1 suppose that* f *satisfies the following divisibility condition:*

$$a|b \;\Rightarrow\; f(a)|f(b) \quad (a, b \geq 1) \tag{$*$}$$

Then one has

$$F\left(\prod_{i=1}^{r} p_i^{\alpha_i}\right) \leq l.c.m.\{F(p_i^{\alpha_i}) : i = \overline{1,r}\}, \tag{6}$$

where l.c.m. denotes the least common multiple.

By replacing $(*)$ with another condition, a more precise result is obtainable:

Theorem 3. *Suppose that* f *satisfies the condition:*

$$a \leq b \;\Rightarrow\; f(a)|f(b) \quad (a, b \geq 1) \tag{$**$}$$

Then

$$F(mn) = \max\{F(m), F(n)\} \text{ for } (m,n) = 1. \tag{7}$$

Finally, we shall prove the following:

Theorem 4. *Suppose that f satisfies $(**)$ and the following two assumptions:*

(i) $n|f(n)$; (ii) For each prime p and $m < p$ we have $p \nmid f(n)$. (8)

Let k be an even perfect number. Then

$$F(k) = k/2^s, \text{ where } 2^s \| k. \tag{9}$$

Remarks. 1) The function φ satisfies property $(*)$. Then relation (6) gives a result for the Euler minimum function $E(n)$ (see [7], [8]).

2) Let $f(m) = m!$. Then clearly $(**)$ holds true. Thus (7) extends relation (4). For another example, let $f(m) = l.c.m.\{1, 2, \ldots, m\}$. Then the function F given by (1) satisfies again (7), proved e.g. in [1].

3) If $f(n) = n!$, then both (i) and (ii) of (8) are satisfied. This relation (9) for $F \equiv S$ follows. This was first proved in [3] (see also [6]).

3 Proofs of theorems

Theorem 1. There is no loss of generality to prove (5) for $r = 2$. Let p^α, q^β be two distinct prime powers. Then

$$F(p^\alpha q^\beta) = \min\{n \geq 1 : p^\alpha q^\beta | f(m)\} = m_0,$$

so $p^\alpha q^\beta | f(m_0)$. This is equivalent to $p^\alpha | f(m_0)$, $q^\beta | f(m_0)$. By definition (1) we get $m_0 \geq F(p^\alpha)$ and $m_0 \geq F(q^\beta)$, i.e. $F(p^\alpha q^\beta) \geq \max\{F(p^\alpha), F(q^\beta)\}$. It is immediate that the same proof applies to $F\left(\prod p^\alpha\right) \geq \max\{F(p^\alpha)\}$, where p^α are distinct prime powers.

Theorem 2. Let $F(p^\alpha) = m_1$, $F(q^\beta) = m_2$. By definition (1) of function F it follows that $p^\alpha | F(m_1)$ and $q^\beta | F(m_2)$. Let $l.c.m.\{m_1, m_2\} = g$. Since $m_1 | g$, one has $f(m_1) | f(g)$ by $(*)$. Similarly, since $m_2 | g$, one can write $f(m_2) | f(g)$. These imply $p^\alpha | f(m_1) | f(g)$ and $q^\beta | f(m_2) | f(g)$, yielding $p^\alpha q^\beta | f(g)$. By definition (1) this gives $g \geq F(p^\alpha q^\beta)$, i.e. $l.c.m.\{F(p^\alpha), F(q^\beta)\} \geq F(p^\alpha q^\beta)$, proving the theorem for $r = 2$. The general case follows exactly by the same lines.

Theorem 3. By taking into account of (5), one needs only to show that the reverse inequality is true. For simplicity, let us consider again $r = 2$. Let $F(p^\alpha) = m$, $F(q^\beta) = n$ with $m \leq n$. By definition (1) one has $p^\alpha | f(m)$, $q^\beta | f(n)$. Now, by assumption $(**)$ we can write $f(m) | f(n)$, so $p^\alpha | f(m) | f(n)$. Therefore, one has $p^\alpha | f(n)$, $q^\beta | f(n)$. This in turn implies $p^\alpha q^\beta | f(n)$, so $n \geq F(p^\alpha q^\beta)$; i.e. $\max\{F(p^\alpha), F(q^\beta)\} \geq F(p^\alpha q^\beta)$. The general case follows exactly the same lines. Thus, we have proved essentially, that $F(p^\alpha q^\beta) = \max\{F(p^\alpha), F(q^\beta)\}$, or more generally

$$F\left(\prod_{i=1}^{r} p_i^{\alpha_i}\right) = \max\{F(p_i^{\alpha_i}) : i = \overline{1, r}\}. \tag{10}$$

Now, relation (7) is an immediate consequence of (10), for by writing

$$m = \prod_{i=1}^{r} p_i^{\alpha_i}, \quad n = \prod_{j=1}^{s} q_j^{\beta_j}, \text{ with } (p_i, q_j) = 1,$$

it follows that

$$F(mn) = \max\{F(p_i^{\alpha_i}), F(q_j^{\beta_j}) : i = \overline{1, r}, j = \overline{1, s}\}$$

$$= \max\{\max\{E(p_i^{\alpha_i}) : i = \overline{1, r}\}, \max\{E(q_j^{\beta_j}) : j = \overline{1, s}\}\}$$

$$= \max\{F(m), F(n)\},$$

by equality (10).

Theorem 4. By (i) and definition (1) we get

$$F(n) \leq n. \tag{11}$$

Now, by (i), one has $p|f(p)$ for any prime p, but by (ii), p is the least such number. This implies that

$$F(p) = p \text{ for any prime } p. \tag{12}$$

Now, let k be an even perfect number. By the Euclid-Euler theorem (see e.g. [7]) k may be written as $k = 2^{n-1}(2^n - 1)$, where $p = 2^n - 1$ is a prime ("Mersenne prime"). Since $(**)$ holds true, by Theorem 3 we can write

$$F(k) = F(2^{n-1}(2^n - 1)) = \max\{F(2^{n-1}), F(2^n - 1)\}.$$

Since $F(2^n - 1) = 2^n - 1$ (by (12)), and $F(2^{n-1}) \leq 2^{n-1}$ (by (11)), from $2^{n-1} < 2^n - 1$ for $n \geq 2$, we get $F(k) = 2^n - 1 = \dfrac{k}{2^s}$, where $s = n - 1$ and $2^s \| k$. This finishes the proof of Theorem 4.

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