

# A CLASS OF LOGARITHMICALLY COMPLETELY MONOTONIC FUNCTIONS AND THE BEST BOUNDS IN THE SECOND KERSHAW'S DOUBLE INEQUALITY

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ABSTRACT. In the article, the sufficient and necessary conditions such that a class of functions which involve the psi function  $\psi$  and the ratio  $\frac{\Gamma(x+t)}{\Gamma(x+s)}$  are logarithmically completely monotonic are established, the best bounds for the ratio  $\frac{\Gamma(x+t)}{\Gamma(x+s)}$  are given, and some comparisons with known results are carried out, where  $s$  and  $t$  are two real numbers and  $x > -\min\{s, t\}$ .

## 1. INTRODUCTION

Recall [30, 32, 52, 54] that a function  $f$  is said to be completely monotonic on an interval  $I$  if  $f$  has derivatives of all orders on  $I$  such that  $(-1)^k f^{(k)}(x) \geq 0$  for  $x \in I$  and  $k \geq 0$ . Recall also [3, 30, 39, 41, 42, 43] that a positive function  $f$  is said to be logarithmically completely monotonic on an interval  $I$  if its logarithm  $\ln f$  satisfies  $(-1)^k [\ln f(x)]^{(k)} \geq 0$  for  $k \in \mathbb{N}$  on  $I$ . For our own convenience, the sets of the completely monotonic functions and the logarithmically completely monotonic functions on  $I$  are denoted by  $\mathcal{C}[I]$  and  $\mathcal{C}_{\mathcal{L}}[I]$  respectively.

The famous Bernstein-Widder's Theorem [54, p. 161] states that  $f \in \mathcal{C}[(0, \infty)]$  if and only if there exists a bounded and nondecreasing function  $\mu(t)$  such that

$$f(x) = \int_0^{\infty} e^{-xt} d\mu(t) \quad (1)$$

converges for  $0 < x < \infty$ .

In [5, 29, 39, 41, 42, 43, 52] and many other references, the inclusions  $\mathcal{C}_{\mathcal{L}}[I] \subset \mathcal{C}[I]$  and  $\mathcal{S} \subset \mathcal{C}_{\mathcal{L}}[(0, \infty)]$  were revealed implicitly or explicitly, where  $\mathcal{S}$  denotes the class of Stieltjes transforms [5, 54]. There are three different proofs in [5], [39, 41] and [29, 45] for the inclusion  $\mathcal{C}_{\mathcal{L}}[I] \subset \mathcal{C}[I]$ . The class  $\mathcal{C}_{\mathcal{L}}[(0, \infty)]$  is characterized in [5, Theorem 1.1] implicitly and in [19, Theorem 4.4] explicitly:  $f \in \mathcal{C}_{\mathcal{L}}[(0, \infty)] \iff f^\alpha \in \mathcal{C}$  for all  $\alpha > 0 \iff \sqrt[n]{f} \in \mathcal{C}$  for all  $n \in \mathbb{N}$ . In other words, the functions in  $\mathcal{C}_{\mathcal{L}}[(0, \infty)]$  are those completely monotonic functions for which the representing measure  $\mu$  in (1) is infinitely divisible in the convolution sense: For each  $n \in \mathbb{N}$  there exists a positive measure  $\nu$  on  $[0, \infty)$  with  $n$ -th convolution power equal to  $\mu$ .

By the way, recall [30, 32, 48, 52, 54] that a function  $f$  is said to be absolutely monotonic on an interval  $I$  if it has derivatives of all orders and  $f^{(k-1)}(t) \geq 0$  for  $t \in I$  and  $k \in \mathbb{N}$ . In [29, 45], it was defined that a positive function  $f$  is said

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to be logarithmically absolutely monotonic on an interval  $I$  if it has derivatives of all orders and  $[\ln f(t)]^{(k)} \geq 0$  for  $t \in I$  and  $k \in \mathbb{N}$  and it was showed that a logarithmically absolutely monotonic function on an interval  $I$  is also absolutely monotonic on  $I$ , but not conversely.

In recent years, the logarithmically completely monotonic functions and their properties have been investigated extensively and explicitly in [3, 5, 9, 10, 11, 16, 17, 18, 23, 25, 26, 29, 36, 38, 39, 40, 41, 42, 43, 44, 45, 46, 50] and the references therein.

Let  $\Gamma$  and  $\psi = \frac{\Gamma'}{\Gamma}$  stand for the classical Euler's gamma function and the psi function respectively. The first and second Kershaw's inequalities [21] state that

$$\left(x + \frac{s}{2}\right)^{1-s} < \frac{\Gamma(x+1)}{\Gamma(x+s)} < \left(x - \frac{1}{2} + \sqrt{s + \frac{1}{4}}\right)^{1-s} \quad (2)$$

and

$$\exp[(1-s)\psi(x + \sqrt{s})] < \frac{\Gamma(x+1)}{\Gamma(x+s)} < \exp\left[(1-s)\psi\left(x + \frac{s+1}{2}\right)\right] \quad (3)$$

for  $s \in (0, 1)$  and  $x \geq 1$ . There have been a lot of literature on these two double inequalities, for example, [4, 8, 12, 14, 15, 20, 21, 22, 23, 25, 26, 27, 28, 33, 34, 36, 38, 44, 53] and the references therein.

For real numbers  $a, b, c$  and  $\rho = \min\{a, b, c\}$ , let  $H_{a,b,c}(x) = (x+c)^{b-a} \frac{\Gamma(x+a)}{\Gamma(x+b)}$  in  $(-\rho, \infty)$ . Recently, the following sufficient and necessary conditions are established elegantly in [37]:  $H_{a,b,c}(x) \in \mathcal{C}_{\mathcal{L}}[(-\rho, \infty)]$  if and only if  $(a, b, c) \in \{(a, b, c) : (b-a)(1-a-b+2c) \geq 0\} \cap \{(a, b, c) : (b-a)(|a-b|-a-b+2c) \geq 0\} \setminus \{(a, b, c) : a = c+1 = b+1\} \setminus \{(a, b, c) : b = c+1 = a+1\}$  and  $H_{b,a,c}(x) \in \mathcal{C}_{\mathcal{L}}[(-\rho, \infty)]$  if and only if  $(a, b, c) \in \{(a, b, c) : (b-a)(1-a-b+2c) \leq 0\} \cap \{(a, b, c) : (b-a)(|a-b|-a-b+2c) \leq 0\} \setminus \{(a, b, c) : b = c+1 = a+1\} \setminus \{(a, b, c) : a = c+1 = b+1\}$ . These conclusions can be used to extend, generalize, refine and sharpen [25, Theorem 1], inequality (2) and some other known results.

It is easy to see that inequality (3) can be rewritten for  $s \in (0, 1)$  and  $x \geq 1$  as

$$\exp[\psi(x + \sqrt{s})] < \left[\frac{\Gamma(x+1)}{\Gamma(x+s)}\right]^{1/(1-s)} < \exp\left[\psi\left(x + \frac{s+1}{2}\right)\right]. \quad (4)$$

Now it is natural to ask: What are the best constants  $\delta_1(s, t)$  and  $\delta_2(s, t)$  such that

$$\exp[\psi(x + \delta_1(s, t))] \leq \left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)} \leq \exp[\psi(x + \delta_2(s, t))] \quad (5)$$

holds for  $x > -\min\{s, t, \delta_1(s, t), \delta_2(s, t)\}$ , where  $s$  and  $t$  are two real numbers? In order to give an answer to this problem, we would like to establish the logarithmically complete monotonicity of the function

$$\nu_{s,t}(x) = \frac{1}{\exp[\psi(x + \theta(s, t))]} \left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)}. \quad (6)$$

Our first main result is the following Theorem 1.

**Theorem 1.** *Let  $s$  and  $t$  be two real numbers with  $s \neq t$  and  $\theta(s, t)$  a constant depending on  $s$  and  $t$ .*

- (1) *If  $\theta(s, t) \leq \min\{s, t\}$ , then  $\nu_{s,t}(x) \in \mathcal{C}_{\mathcal{L}}[(-\theta(s, t), \infty)]$ .*

(2)  $\frac{1}{\nu_{s,t}(x)} \in \mathcal{C}_{\mathcal{L}}[(-\min\{s, t\}, \infty)]$  if and only if  $\theta(s, t) \geq \frac{s+t}{2}$ .

Our second main result, as a straightforward consequence of Theorem 1, is the following Theorem 2.

**Theorem 2.** *Let  $s$  and  $t$  be two real numbers with  $s \neq t$ .*

(1) *Inequality*

$$\left[ \frac{\Gamma(x+t)}{\Gamma(x+s)} \right]^{1/(t-s)} < \exp \left[ \psi \left( x + \frac{s+t}{2} \right) \right] \quad (7)$$

*is valid in  $(-\min\{s, t\}, \infty)$ . The constant  $\frac{s+t}{2}$  in (7) is the best possible.*

(2) *Inequality*

$$\left[ \frac{\Gamma(x+t)}{\Gamma(x+s)} \right]^{1/(t-s)} \geq \left[ \frac{\Gamma(\delta+t)}{\Gamma(\delta+s)} \right]^{1/(t-s)} \quad (8)$$

*validates for  $x \geq \delta > -\min\{s, t\}$ .*

(3) *Inequality*

$$\left[ \frac{\Gamma(x+t)}{\Gamma(x+s)} \right]^{1/(t-s)} \geq \exp(-\psi(x + \theta(s, t))) \quad (9)$$

*holds for  $x > -\theta(s, t) > -\min\{s, t\}$ .*

(4) *Inequality*

$$\left[ \frac{\Gamma(x+t)}{\Gamma(x+s)} \right]^{1/(t-s)} < \left[ \frac{\Gamma(\tau+t)}{\Gamma(\tau+s)} \right]^{1/(t-s)} \exp[\psi(\tau + \theta(s, t)) - \psi(x + \theta(s, t))] \quad (10)$$

*sounds for  $x > \tau \geq -\theta(s, t) > -\min\{s, t\}$ .*

Before proving Theorem 1 and Theorem 2 in Section 3, we would like to compare them with some recent known results and to give several remarks in Section 2.

## 2. COMPARISONS OF THEOREMS WITH SOME KNOWN RESULTS

2.1. In order to refine and extend the first Kershaw's double inequality (2), the logarithmically complete monotonicity of the function  $(x+c)^{b-a} \frac{\Gamma(x+a)}{\Gamma(x+b)}$  for  $x \in (-\rho, \infty)$  was studied in [25], where  $a, b$  and  $c$  are real numbers and  $\rho = \min\{a, b, c\}$ .

2.2. It is clear that inequality (7) extends the ranges of variables of the right hand side inequality in (4) which is a rearranged form of (3).

2.3. Taking  $t = \delta = 1$  and  $s \in (0, 1)$  in (8) gives

$$\left[ \frac{\Gamma(x+1)}{\Gamma(x+s)} \right]^{1/(1-s)} \geq \frac{1}{[\Gamma(1+s)]^{1/(1-s)}}. \quad (11)$$

When

$$1 \leq x \leq \psi^{-1}((s-1) \ln \Gamma(1+s)) - \sqrt{s} \quad (12)$$

inequality (11) is better than the left hand side inequality in (4), where  $\psi^{-1}$  stands for the inverse function of  $\psi$ . This can be realized since  $\lim_{s \rightarrow 0^+} [\psi^{-1}((s-1) \ln \Gamma(1+s)) - \sqrt{s}]$  equals the unique zero 1.4626... of  $\psi(x)$  in  $(0, \infty)$  clearly.

2.4. Inequality (9) for the case of  $t = -\theta(s, 1) = 1$  and  $s \in (0, 1)$  is better than the lower bound in (4) when  $\psi(x + \sqrt{s}) + \psi(x - 1) \leq 0$  which can be rewritten as  $0 < s \leq [\psi^{-1}(-\psi(x-1)) - x]^2 < 1$ . This can be realized since  $\lim_{x \rightarrow 1^+} [\psi(x + \sqrt{s}) + \psi(x - 1)] = -\infty$  obviously.

2.5. Inequality (10) for the case of  $\tau = t = 1$ ,  $s \in (0, 1)$  and  $-1 < \theta(s, 1) < s = \min\{s, t\}$  is better than the right hand side inequality in (4) when  $x > 1$  and

$$\frac{\ln \Gamma(1+s)}{s-1} \leq \psi\left(x + \frac{s+1}{2}\right) - \psi(1+\theta) + \psi(x+\theta). \quad (13)$$

This can be realized since  $\lim_{x \rightarrow \infty} [\psi(x + \frac{s+1}{2}) - \psi(1+\theta) + \psi(x+\theta)] = \infty$  for any given  $s$  and  $\theta(s, 1)$  apparently.

2.6. Inequality (8) can also be deduced from a fact obtained in [44, Proposition 3]: The function  $\left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(s-t)}$  is logarithmically completely monotonic in the interval  $(-\min\{s, t\}, \infty)$  with  $s \neq t$ .

2.7. Let  $a, b$  and  $c$  be real numbers and  $\rho = \min\{a, b, c\}$ . Define

$$F_{a,b,c}(x) = \begin{cases} \left[\frac{\Gamma(x+b)}{\Gamma(x+a)}\right]^{1/(a-b)} \exp[\psi(x+c)], & a \neq b \\ \exp[\psi(x+c) - \psi(x+a)], & a = b \neq c \end{cases} \quad (14)$$

for  $x \in (-\rho, \infty)$ . Furthermore, let  $\theta(t)$  be an implicit function defined by equation

$$e^t - t = e^{\theta(t)} - \theta(t) \quad (15)$$

with  $\theta(t) \neq t$  for  $t \neq 0$  and let  $p(t) = t - \theta(t - 1)$  in  $(-\infty, \infty)$ , where  $p^{-1}$  stands for the inverse function of  $p$ . In [26], the following conclusions are proved:

(1)  $F_{a,b,c}(x) \in \mathcal{C}_L[(-\rho, \infty)]$  if  $(a, b, c) \in D_1(a, b, c)$ , where

$$D_1(a, b, c) = \{c \geq a, c \geq b\} \cup \{c \geq a, 0 \geq c - b \geq \theta(c - a)\} \\ \cup \{c \leq a, c - b \geq \theta(c - a)\} \setminus \{a = b = c\}; \quad (16)$$

(2)  $[F_{a,b,c}(x)]^{-1} \in \mathcal{C}_L[(-\rho, \infty)]$  if  $(a, b, c) \in D_2(a, b, c)$ , where

$$D_2(a, b, c) = \{c \leq a, c \leq b\} \cup \{c \geq a, c - b \leq \theta(c - a)\} \\ \cup \{c \leq a, 0 \leq c - b \leq \theta(c - a)\} \setminus \{a = b = c\}; \quad (17)$$

(3) If  $(a, b, c) \in D_1(a, b, c)$ , then

$$\left[\frac{\Gamma(x+b)}{\Gamma(x+a)}\right]^{1/(b-a)} < \exp[\psi(x+c)] \quad (18)$$

for  $x \in (-\rho, \infty)$  and

$$\left[\frac{\Gamma(x+b)}{\Gamma(x+a)}\right]^{1/(b-a)} \geq \left[\frac{\Gamma(\delta+b)}{\Gamma(\delta+a)}\right]^{1/(b-a)} \exp[\psi(x+c) - \psi(\delta+c)] \quad (19)$$

for  $x \in [\delta, \infty)$  are valid, where  $\delta$  is a constant greater than  $-\rho$ ;

(4) If  $(a, b, c) \in D_2(a, b, c)$ , inequalities (18) and (19) are reversed.

As special cases of inequalities (18) and (19), inequalities

$$\frac{\Gamma(x+1)}{\Gamma(x+s)} < \exp[(1-s)\psi(x+p^{-1}(s))] \tag{20}$$

for  $x \in (-s, \infty)$  and

$$\frac{\Gamma(x+1)}{\Gamma(x+s)} \geq \frac{\Gamma(\delta+1)}{\Gamma(\delta+s)} \exp[\psi(x+p^{-1}(s)) - \psi(\delta+p^{-1}(s))] \tag{21}$$

for  $x \in (\delta, \infty)$  are valid, where  $s \in (0, 1)$ ,  $\delta > -s$  and  $s \leq p^{-1}(s) \leq 1$ .

Since the function  $e^t - t$  is increasing in  $(0, \infty)$  and decreasing in  $(-\infty, 0)$ , as showed by Figure 1, then  $t\theta(t) < 0$  for  $\theta(t) \neq t$ . A ready differentiation on both

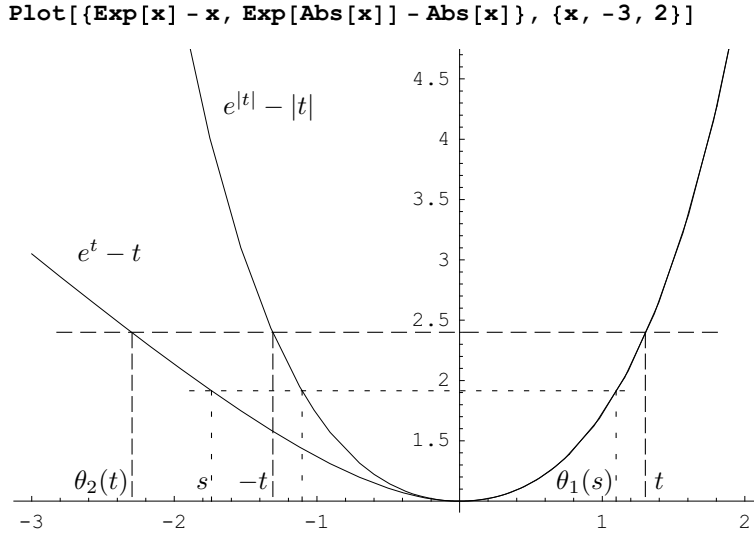


FIGURE 1. Graphs of the functions  $e^t - t$  and  $e^{|t|} - |t|$  by MATHEMATICA 5.2

sides of equation (15) yields  $\theta'(t) = \frac{e^t - 1}{e^{\theta(t)} - 1} < 0$ , and then  $\theta(t)$  is decreasing and  $p(t)$  is increasing for  $t \in (-\infty, \infty)$ .

It is claimed that  $t + \theta(t) < 0$  for  $\theta(t) \neq t$ , as showed by Figure 1. This claim can be verified as follows. Let  $\phi_1(t) = e^{|t|} - |t|$  and  $\phi_2(t) = e^t - t$  in  $(-\infty, \infty)$ . If  $t \in (-\infty, 0]$ , then  $\phi_1(t) = e^{-t} + t$ ; if  $t \in [0, \infty)$ , then  $\phi_1(t) = \phi_2(t)$ . It is clear that  $\phi_1(0) = \phi_2(0) = 0$  and  $\lim_{t \rightarrow -\infty} \phi_1(t) = \lim_{t \rightarrow -\infty} \phi_2(t) = \lim_{t \rightarrow \infty} \phi_1(t) = \lim_{t \rightarrow \infty} \phi_2(t) = \infty$ . An easy calculation gives  $\phi_1'(t) = -e^{-t} + 1$  and  $\phi_2'(t) = e^t - 1$  in  $(-\infty, 0]$ . It is obvious that  $\phi_1'(t) < \phi_2'(t) < 0$  in  $(-\infty, 0)$ . This implies that the functions  $\phi_1(t)$  and  $\phi_2(t)$  are decreasing with  $0 < \phi_2(t) < \phi_1(t)$  in  $(-\infty, 0)$ . Accordingly, since the function  $\phi_1(t)$  is even in  $(-\infty, \infty)$ , for any given negative number  $s < 0$ , there exists a unique point  $\theta_1(s) > 0$  such that  $s < -\theta_1(s) < 0$  and  $\phi_2(s) = \phi_1(-\theta_1(s)) = \phi_1(\theta_1(s))$ ; for any given positive number  $t > 0$ , there exists a unique point  $\theta_2(t) < 0$  such that  $\theta_2(t) < -t < 0$  and  $\phi_2(\theta_2(t)) = \phi_1(-t) = \phi_1(t)$ . In conclusion, for any given  $t \in (-\infty, \infty) \setminus \{0\}$ , there exists a unique point  $\theta(t) \neq t$  such that  $t + \theta(t) < 0$  and  $\phi_1(t) = \phi_2(\theta(t))$  which is equivalent to equation (15). In other words, if  $t$  and  $\theta(t)$  with  $t \neq \theta(t)$  satisfy equation (15), then  $t + \theta(t) < 0$ .

Now we can claim that, for  $x \geq 1$  and  $s \in (0, 1)$ , inequality (20) is better than the right hand side inequality in (3), since

$$\begin{aligned} \psi\left(x + \frac{1+s}{2}\right) > \psi\left(x + p^{-1}(s)\right) &\iff p\left(\frac{1+s}{2}\right) > s \\ \iff \frac{1+s}{2} - \theta\left(\frac{1+s}{2} - 1\right) > s &\iff \theta\left(\frac{s-1}{2}\right) < \frac{1-s}{2} \end{aligned}$$

is valid, where the monotonicities of  $\psi$  and  $p$  and the fact that  $t + \theta(t) < 0$  for  $t\theta(t) < 0$  are used.

2.8. In [4, Theorem 2.4], the following double inequality was obtained:

$$\exp\left[(x-y)\psi\left(\frac{x-y}{\ln(x+1) - \ln(y+1)} - 1\right)\right] \leq \frac{\Gamma(x)}{\Gamma(y)} \leq \exp\left[(x-y)\psi\left(\frac{x+y}{2}\right)\right], \quad (22)$$

where  $x$  and  $y$  are positive real numbers.

The right hand side inequality in (22) is the same as (7) essentially.

It is noted that a more strengthened conclusion than the right hand side inequality in (22) has been established in [12, p. 250] and [44, Proposition 4]: Let  $s$  and  $t$  be two real numbers and  $\alpha = \min\{s, t\}$ . Then the function

$$\exp\left[\psi\left(x + \frac{s+t}{2}\right)\right] \left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(s-t)} \in \mathcal{C}_{\mathcal{L}}[(-\alpha, \infty)]. \quad (23)$$

Consequently, inequality (7) follows.

In the left hand side inequality of (22), substituting  $x$  by  $x+s$  and  $y$  by  $x+t$  for two real numbers  $s$  and  $t$  and  $x \in (-\min\{s, t\}, \infty)$  leads to

$$\exp\left[\psi\left(\frac{s-t}{\ln(x+s+1) - \ln(x+t+1)} - 1\right)\right] \leq \left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)}. \quad (24)$$

It was proved in [12, p. 248] that

$$\exp\left(\psi\left(x + \psi^{-1}\left(\frac{1}{t-s} \int_s^t \psi(u) \, du\right)\right)\right) \leq \left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)}, \quad (25)$$

where  $x \geq 0$ ,  $s > 0$ ,  $t > 0$ , and  $\psi^{-1}$  denotes the inverse function of  $\psi$ . The lower bounds in (24) and (25) do not contain each other, since a simple numerical computation by the well known software MATHEMATICA 5.2 shows that

$$\psi\left(\frac{s-t}{\ln(x+s+1) - \ln(x+t+1)} - x - 1\right) - \frac{1}{t-s} \int_s^t \psi(u) \, du$$

equals  $0.21728 \dots$  if  $(x, s, t) = (191, 1, 92)$  and  $-0.10331 \dots$  if  $(x, s, t) = (11, 1, 92)$ .

2.9. In [6] the following complete monotonicity were established:

(1) The functions

$$\frac{\Gamma(x+s)}{\Gamma(x+1)} \exp\left[(1-s)\psi\left(x + \frac{s+1}{2}\right)\right] \quad \text{and} \quad \frac{\Gamma(x+1)}{\Gamma(x+s)} \left(x + \frac{s}{2}\right)^{s-1} \quad (26)$$

are completely monotonic on  $(0, \infty)$  for  $0 \leq s \leq 1$ . When  $0 < s < 1$ , the functions in (26) satisfy  $(-1)^n f^{(n)}(x) > 0$  for  $x > 0$ .

(2) Let  $0 < s < 1$  and  $x > 0$ . Then both

$$\frac{\Gamma(x+1)}{\Gamma(x+s)} \exp[(s-1)\psi(x+\sqrt{s})] \quad \text{and} \quad \frac{\Gamma(x+s)}{\Gamma(x+1)} \left[ x - \frac{1}{2} + \sqrt{s + \frac{1}{4}} \right]^{1-s} \quad (27)$$

are strictly decreasing functions.

The complete monotonicities of the second functions in (26) and (27) are generalized in [25] to logarithmically complete monotonicities.

It is clear that the complete monotonicities of the first functions in (26) and (27) are included in Theorem 1 of this paper.

### 3. PROOFS OF THEOREMS

In order to prove our main result, the following more general proposition than our need are presented.

**Proposition 1.** *Let  $\psi$  be the psi function defined by  $\frac{\Gamma'}{\Gamma}$ , and  $s$  and  $t$  two positive numbers.*

(1) *If  $m > n \geq 0$  are two integers, then*

$$\left(\psi^{(m)}\right)^{-1} \left( \frac{1}{t-s} \int_s^t \psi^{(m)}(v) \, dv \right) \leq \left(\psi^{(n)}\right)^{-1} \left( \frac{1}{t-s} \int_s^t \psi^{(n)}(v) \, dv \right). \quad (28)$$

(2) *Inequality*

$$\psi^{(i)} \left( \frac{t-s}{\ln t - \ln s} \right) \leq \frac{1}{t-s} \int_s^t \psi^{(i)}(u) \, du \quad (29)$$

*is valid for  $i$  being positive odd number or zero and reversed for  $i$  being nonnegative even number.*

(3) *The function*

$$\left(\psi^{(\ell)}\right)^{-1} \left( \frac{1}{t-s} \int_s^t \psi^{(\ell)}(x+v) \, dv \right) - x \quad (30)$$

*for  $\ell \geq 0$  is increasing and concave in  $x > -\min\{s, t\}$  and has a sharp upper bound  $\frac{s+t}{2}$ .*

*Proof.* It was presented in [13, Theorem 3] that if the second derivative of  $f$  is continuous on an interval  $I$  such that  $f$  is increasingly concave and  $\frac{f''}{f'}$  is increasing then

$$(f')^{-1} \left( \frac{1}{t-s} \int_s^t f'(u) \, du \right) \leq f^{-1} \left( \frac{1}{t-s} \int_s^t f(u) \, du \right) \quad (31)$$

holds for  $s, t \in I$ , where  $(f')^{-1}$  and  $f^{-1}$  stand for the inverse functions of  $f'$  and  $f$ .

It was presented in [24, p. 366, Theorem 1] and [54, p. 167] that if  $w(x) \in \mathcal{C}[I]$  then

$$w^{(k+1)}(x)w^{(k-1)}(x) \geq [w^{(k)}(x)]^2 \quad (32)$$

for  $k \in \mathbb{N}$  and  $x \in I$ . This means that

$$\left[ \frac{w^{(k)}(x)}{w^{(k-1)}(x)} \right]' = \frac{w^{(k+1)}(x)w^{(k-1)}(x) - [w^{(k)}(x)]^2}{[w^{(k-1)}(x)]^2} \geq 0 \quad (33)$$

and the function  $\frac{w^{(k)}(x)}{w^{(k-1)}(x)}$  is increasing.

It is easy to see that an inverse function has the property that

$$(af(x))^{-1} = f^{-1}\left(\frac{x}{a}\right) \quad (34)$$

for  $a \neq 0$ , where  $[af(x)]^{-1}$  denotes the inverse function of  $af(x)$ .

It is well known that  $\psi'(x) \in \mathcal{C}[(0, \infty)]$  and  $(-1)^i[\psi'(x)]^{(i)} \geq 0$  for nonnegative integer  $i$ . This implies  $\psi^{(2k-1)}(x) \in \mathcal{C}[(0, \infty)]$ ,  $-\psi^{(2k)}(x) \in \mathcal{C}[(0, \infty)]$  and

$$\psi^{(k+2)}(x)\psi^{(k)}(x) \geq [\psi^{(k+1)}(x)]^2 \quad (35)$$

for  $k \in \mathbb{N}$ . Hence, the functions  $-\psi^{(2i+1)}(x)$  and  $\psi^{(2i)}(x)$  are increasingly concave in  $(0, \infty)$  and

$$\begin{aligned} \left\{ \frac{[-\psi^{(2i+1)}(x)]''}{[-\psi^{(2i+1)}(x)]'} \right\}' &= \left[ \frac{\psi^{(2i+3)}(x)}{\psi^{(2i+2)}(x)} \right]' \\ &= \frac{\psi^{(2i+4)}(x)\psi^{(2i+2)}(x) - [\psi^{(2i+3)}(x)]^2}{[\psi^{(2i+2)}(x)]^2} \geq 0, \end{aligned}$$

$$\left\{ \frac{[\psi^{(2i)}(x)]''}{[\psi^{(2i)}(x)]'} \right\}' = \left[ \frac{\psi^{(2i+2)}(x)}{\psi^{(2i+1)}(x)} \right]' = \frac{\psi^{(2i+3)}(x)\psi^{(2i+1)}(x) - [\psi^{(2i+2)}(x)]^2}{[\psi^{(2i+1)}(x)]^2} \geq 0,$$

which are equivalent to the functions  $\frac{[-\psi^{(2i+1)}(x)]''}{[-\psi^{(2i+1)}(x)]'}$  and  $\frac{[\psi^{(2i)}(x)]''}{[\psi^{(2i)}(x)]'}$  are increasing in  $(0, \infty)$  for given nonnegative integer  $i \geq 0$ . Accordingly, substituting  $-\psi^{(2i+1)}(x)$  and  $\psi^{(2i)}(x)$  into (31) and utilizing (34) yields

$$\begin{aligned} \left(\psi^{(2i+2)}\right)^{-1} \left( \frac{1}{t-s} \int_s^t \psi^{(2i+2)}(u) \, du \right) \\ \leq \left(\psi^{(2i+1)}\right)^{-1} \left( \frac{1}{t-s} \int_s^t \psi^{(2i+1)}(u) \, du \right) \quad (36) \end{aligned}$$

and

$$\left(\psi^{(2i+1)}\right)^{-1} \left( \frac{1}{t-s} \int_s^t \psi^{(2i+1)}(u) \, du \right) \leq \left(\psi^{(2i)}\right)^{-1} \left( \frac{1}{t-s} \int_s^t \psi^{(2i)}(u) \, du \right) \quad (37)$$

for positive real numbers  $s$  and  $t$  and nonnegative integer  $i \geq 0$ . As a result, by induction, inequality (28) follows.

By using Jensen's inequality, it was obtained in [7] that if  $g$  is strictly monotonic,  $f$  is strictly increasing and  $f \circ g^{-1}$  is convex (or concave, respectively) on an interval  $I$ , then

$$g^{-1} \left( \frac{1}{t-s} \int_s^t g(u) \, du \right) \leq f^{-1} \left( \frac{1}{t-s} \int_s^t f(u) \, du \right) \quad (38)$$

holds (or reverses, respectively) for  $s, t \in I$ . It is apparent that  $f(x) = (-1)^i \psi^{(i)}(x)$  for  $i \geq 0$  is increasing strictly and  $g(x) = \frac{1}{x}$  is decreasing strictly and  $g^{-1}(x) = g(x)$ . Direct computation gives

$$g^{-1} \left( \frac{1}{t-s} \int_s^t g(u) \, du \right) = \frac{t-s}{\ln t - \ln s}, \quad (39)$$

$$h(x) \triangleq f \circ g^{-1}(x) = (-1)^i \psi^{(i)} \left( \frac{1}{x} \right) \quad (40)$$



and

$$\begin{aligned} h''(x) &= \frac{(-1)^i \left[ 2x\psi^{(i+1)}\left(\frac{1}{x}\right) + \psi^{(i+2)}\left(\frac{1}{x}\right) \right]}{x^4} \\ &= (-1)^i u^3 \left[ 2\psi^{(i+1)}(u) + u\psi^{(i+2)}(u) \right]. \end{aligned}$$

It was proved in [2] that the function  $\frac{x\psi^{(k+1)}(x)}{\psi^{(k)}(x)}$  is strictly increasing from  $[0, \infty)$  onto  $[-(k+1), -k]$  for  $k \in \mathbb{N}$ . This means that

$$(-1)^k(k+1)\psi^{(k)}(x) \leq (-1)^{k+1}x\psi^{(k+1)}(x) < (-1)^k k\psi^{(k)}(x) \quad (41)$$

holds in  $(0, \infty)$  for  $k \in \mathbb{N}$ , which can be rewritten as

$$\begin{aligned} (-i)[(-1)^i\psi^{(i+1)}(x)] &\leq (-1)^i[2\psi^{(i+1)}(u) + x\psi^{(i+2)}(x)] \\ &< (1-i)[(-1)^i\psi^{(i+1)}(x)] \end{aligned} \quad (42)$$

in  $(0, \infty)$  for given nonnegative integer  $i$ . Consequently, the function  $h(x)$  is convex if  $i = 0$  or concave if  $i \geq 1$ . So, the conditions of inequality (38) (or reversed inequality of (38), respectively) are satisfied by  $f(x) = (-1)^i\psi^{(i)}(x)$  and  $g(x) = \frac{1}{x}$  for  $i = 0$  (or for  $i \geq 1$ , respectively). The case of  $i = 0$  in (38) is just inequality (29) for  $i = 0$ . For  $i \geq 1$ , this leads to

$$\begin{aligned} \frac{t-s}{\ln t - \ln s} &\geq \left( (-1)^i\psi^{(i)} \right)^{-1} \left( \frac{1}{t-s} \int_s^t (-1)^i\psi^{(i)}(u) \, du \right) \\ &= \left( \psi^{(i)} \right)^{-1} \left( \frac{1}{t-s} \int_s^t \psi^{(i)}(u) \, du \right). \end{aligned} \quad (43)$$

Since  $\psi^{(2i)}(x)$  is increasing and  $\psi^{(2i-1)}(x)$  for  $i \in \mathbb{N}$ , inequality (29) or its reversed form is deduced from (43).

Let  $\phi_{s,t;\ell}(x)$  denote the function (30). It is said in [13, p. 194, Corollary 1] that if  $f$  is an increasing function such that  $f'$  is completely monotonic on an interval  $I$ , then the function  $h_{f;s,t}(x) = f^{-1}\left(\frac{1}{t-s}\int_s^t f(x+v) \, dv\right) - x$  is increasing and concave for  $s, t \in I$  and  $x > -\min\{s, t\}$ . It is clear that the functions  $\psi^{(2i)}(x)$  is increasing such that  $\psi^{(2i+1)}(x) \in \mathcal{C}[(0, \infty)]$  for  $i \geq 0$ , so do the functions  $-\psi^{(2i+1)}(x)$  for  $i \geq 0$ . From (34) it is easy to deduce that  $h_{a f;s,t}(x) = h_{f;s,t}(x)$  holds for any given nonzero constant  $a$ . Consequently, the increasing concavity of the functions  $h_{\psi^{(\ell)};s,t}(x) = \phi_{s,t;\ell}(x)$  for  $\ell \geq 0$  is proved.

Since the function  $(-1)^{\ell+1}\psi^{(\ell)}(x)$  for  $\ell \geq 0$  is decreasingly convex in  $(0, \infty)$ , by Hermite-Hadamard-Jensen's integral inequality [47, 49] and (34), it is deduced that

$$\begin{aligned} &\left( \psi^{(\ell)} \right)^{-1} \left( \frac{1}{t-s} \int_s^t \psi^{(\ell)}(x+v) \, dv \right) \\ &= \left( (-1)^{\ell+1}\psi^{(\ell)} \right)^{-1} \left( \frac{1}{t-s} \int_s^t \left[ (-1)^{\ell+1}\psi^{(\ell)}(x+v) \right] \, dv \right) \\ &\leq \left( (-1)^{\ell+1}\psi^{(\ell)} \right)^{-1} \left( (-1)^{\ell+1}\psi^{(\ell)} \left( x + \frac{s+t}{2} \right) \right) \\ &= x + \frac{s+t}{2}. \end{aligned} \quad (44)$$

Combining this with inequality (29) yields

$$\frac{t-s}{\ln(x+t) - \ln(x+s)} - x \leq \phi_{s,t;\ell}(x) \leq \frac{s+t}{2}. \quad (45)$$

Since

$$\lim_{x \rightarrow \infty} \left[ \frac{t-s}{\ln(x+t) - \ln(x+s)} - x \right] = \frac{s+t}{2}$$

by L'Hôpital's rule, then the function  $\phi_{s,t;\ell}(x)$  has a sharp upper bound  $\frac{s+t}{2}$ . The proof of Proposition 1 is complete.  $\square$

Now we are in a position to prove Theorem 1 and Theorem 2.

*Proof of Theorem 1.* It is well known [1, 6.1.50 and 6.3.21] that

$$\ln \Gamma(x) = \int_0^\infty \frac{1}{u} \left[ (x-1)e^{-u} - \frac{e^{-u} - e^{-xu}}{1 - e^{-u}} \right] du, \quad (46)$$

$$\psi(x) = \int_0^\infty \left( \frac{e^{-u}}{u} - \frac{e^{-xu}}{1 - e^{-u}} \right) du. \quad (47)$$

Straightforward calculation gives

$$\begin{aligned} \ln \nu_{s,t}(x) &= \frac{1}{t-s} [\ln \Gamma(x+t) - \ln \Gamma(x+s)] - \psi(x + \theta(s,t)) \\ &= \int_0^\infty \frac{e^{-xu}}{1 - e^{-u}} \left[ \frac{e^{-tu} - e^{-su}}{(t-s)u} + e^{-u\theta(s,t)} \right] du \\ &\triangleq \int_0^\infty \frac{e^{-[x+\theta(s,t)]u}}{1 - e^{-u}} [q_{s,t}(u) + 1] du, \end{aligned}$$

where

$$\begin{aligned} q_{s,t}(u) &= \frac{e^{-tu} - e^{-su}}{(t-s)u} e^{u\theta(s,t)} \\ &= -e^{u\theta(s,t)} \left( \frac{1}{t-s} \int_s^t e^{-uv} dv \right) \\ &= -\exp \left\{ u \left[ \theta(s,t) + \ln \left( \frac{1}{t-s} \int_s^t e^{-uv} dv \right)^{1/u} \right] \right\} \\ &\triangleq -\exp \{ u[\theta(s,t) + \ln p_{s,t}(u)] \} \end{aligned}$$

and, by using [31, p. 2], [32, Theorem 3.3] or [51, Theorem 1.1], see also [35], the function  $p_{s,t}(u)$  is increasing in  $u \geq 0$  with

$$\lim_{u \rightarrow 0} p_{s,t}(u) = e^{-(s+t)/2} \quad \text{and} \quad \lim_{u \rightarrow \infty} p_{s,t}(u) = e^{-\min\{s,t\}}.$$

Accordingly, if  $\theta(s,t) \leq \min\{s,t\}$  then  $h_{s,t}(u) \geq 0$ , if  $\theta(s,t) \geq \frac{s+t}{2}$  then  $h_{s,t}(u) \leq 0$ .

This means  $(-1)^k [\ln \nu_{s,t}(x)]^{(k)} \begin{cases} \geq 0, & \theta(s,t) \leq \min\{s,t\} \\ \leq 0, & \theta(s,t) \geq \frac{s+t}{2} \end{cases}$  for  $k \in \mathbb{N}$ .

Conversely, if  $\frac{1}{\nu_{s,t}(x)}$  is logarithmically completely monotonic, then  $[\ln \nu_{s,t}(x)]' \geq 0$  which can be rearranged as

$$\frac{\psi(x+t) - \psi(x+s)}{t-s} \geq \psi'(x + \theta(s,t)). \quad (48)$$

Since  $\psi'$  is decreasing, thus

$$\begin{aligned} \theta(s, t) &\geq (\psi')^{-1} \left( \frac{\psi(x+t) - \psi(x+s)}{t-s} \right) - x \\ &= (\psi')^{-1} \left( \frac{1}{t-s} \int_s^t \psi'(x+v) \, dv \right) - x = \phi_{s,t;1}(x), \end{aligned} \quad (49)$$

where  $(\psi')^{-1}$  denotes the inverse function of  $\psi'$  and  $\phi_{s,t;1}(x)$  is defined by (30). Proposition 1 tells us that the function  $\phi_{s,t;1}(x)$  has a sharp upper bound  $\frac{s+t}{2}$ , thus, it holds that  $\theta(s, t) \geq \frac{s+t}{2}$ . The proof of Theorem 1 is complete.  $\square$

*Proof of Theorem 2.* If  $\theta(s, t) \geq \frac{s+t}{2}$ , then the function  $\nu_{s,t}(x)$  defined by (6) is increasing by Theorem 1. Hence, for any given  $\delta > -\min\{s, t\}$  and  $\theta(s, t) \geq \frac{s+t}{2}$ , inequality

$$\nu_{s,t}(\delta) \leq \nu_{s,t}(x) \quad (50)$$

holds in  $[\delta, \infty)$  and

$$\nu_{s,t}(x) < \lim_{x \rightarrow \infty} \nu_{s,t}(x) \quad (51)$$

is valid in  $(-\min\{s, t\}, \infty)$ .

For  $a$  and  $b$  being two constants, as  $x \rightarrow \infty$ , the following asymptotic formula is given in [1, p. 261, 6.1.47]:

$$\begin{aligned} x^{b-a} \frac{\Gamma(x+a)}{\Gamma(x+b)} &= 1 + \frac{(a-b)(a+b-1)}{2x} \\ &\quad + \frac{1}{12} \binom{a-b}{2} \frac{3(a+b-1)^2 - a + b - 1}{x^2} + O\left(\frac{1}{x^3}\right) = 1 + O\left(\frac{1}{x}\right). \end{aligned} \quad (52)$$

In [36], it was proved that  $\psi(x) - \ln x + \frac{\alpha}{x} \in \mathcal{C}[(0, \infty)]$  if and only if  $\alpha \geq 1$  and  $\ln x - \frac{\alpha}{x} - \psi(x) \in \mathcal{C}[(0, \infty)]$  if and only if  $\alpha \leq \frac{1}{2}$ . From this, it is deduced that

$$\ln x - \frac{1}{x} < \psi(x) < \ln x - \frac{1}{2x} \quad (53)$$

in  $(0, \infty)$ . Utilization of (52) and (53) leads to

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{1}{\nu_{s,t}(x)} &= \lim_{x \rightarrow \infty} \left\{ \frac{\exp[\psi(x + \theta(s, t))]}{x} \left[ 1 + O\left(\frac{1}{x}\right) \right]^{1/(t-s)} \right\} \\ &= \lim_{x \rightarrow \infty} \frac{\exp[\psi(x + \theta(s, t))]}{x} \leq \lim_{x \rightarrow \infty} \left\{ \frac{x + \theta(s, t)}{x} \exp \left[ -\frac{1}{2(x + \theta(s, t))} \right] \right\} = 1 \end{aligned}$$

and

$$\lim_{x \rightarrow \infty} \frac{1}{\nu_{s,t}(x)} \geq \lim_{x \rightarrow \infty} \left\{ \frac{x + \theta(s, t)}{x} \exp \left[ -\frac{1}{x + \theta(s, t)} \right] \right\} = 1,$$

thus  $\lim_{x \rightarrow \infty} \nu_{s,t}(x) = 1$  and inequality (51) is reduced to

$$\left[ \frac{\Gamma(x+t)}{\Gamma(x+s)} \right]^{1/(t-s)} < \exp[\psi(x + \theta(s, t))] \quad (54)$$

for  $x > -\min\{s, t\}$  and  $\theta(s, t) \geq \frac{s+t}{2}$ . From the increasing monotonicity of  $\psi$ , inequality (7) is proved.

By standard calculation, inequality (50) can be rearranged as

$$\left[ \frac{\Gamma(x+t)}{\Gamma(x+s)} \right]^{1/(t-s)} \geq \left[ \frac{\Gamma(\delta+t)}{\Gamma(\delta+s)} \right]^{1/(t-s)} \exp[\psi(x + \theta(s, t)) - \psi(\delta + \theta(s, t))] \quad (55)$$

for  $x \in [\delta, \infty)$  and  $\theta(s, t) \geq \frac{s+t}{2}$ . From the decreasing monotonicity in  $y$  of the function  $\psi(x+y) - \psi(\delta+y)$  and  $\lim_{y \rightarrow \infty} [\psi(x+y) - \psi(\delta+y)] = 0$  for  $x \geq \delta$ , inequality (8) is concluded.

Combination of the conclusion  $\nu_{s,t}(x) \in \mathcal{C}_{\mathcal{L}}[(-\theta(s, t), \infty)]$  for  $\theta(s, t) \leq \min\{s, t\}$  in Theorem 1 with  $\lim_{x \rightarrow \infty} \nu_{s,t}(x) = 1$  and discussion by standard argument yields inequalities (9) and (10). The proof of Theorem 2 is complete.  $\square$

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