SCHUR-CONCAVITY AND SCHUR-GEOMETRICALLY CONVEXITY OF DUAL FORM FOR ELEMENTARY SYMMETRIC FUNCTION WITH APPLICATIONS

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Abstract. The Schur-concavity and the Schur-geometrically convexity of dual form for the elementary symmetric function are discussed and some relevant inequalities are established, moreover inequalities for the simplex are also established by above inequalities.

1. Definitions and Lemmas

Throughout the paper we assume that the set of n-dimensional row vector on real number field by $\mathbb{R}^n$.

$$\mathbb{R}^n_+ = \{\boldsymbol{x} = (x_1, \dots, x_n) \in \mathbb{R}^n : x_i \geq 0, i = 1, \dots, n\},$$

$$\mathbb{R}^n_{++} = \{\boldsymbol{x} = (x_1, \dots, x_n) \in \mathbb{R}^n : x_i > 0, i = 1, \dots, n\}.$$

Let $\boldsymbol{x} = (x_1, \dots, a_n) \in \mathbb{R}^n$. Its elementary symmetric functions are

$$E_k(\boldsymbol{x}) = E_k(x_1, \dots, x_n) = \sum_{1 \leq i_1 < \dots < i_k \leq n} \prod_{j=1}^k x_{i_j}, \ k = 1, \dots, n.$$

The dual form of the elementary symmetric functions are

$$E_k^*(\boldsymbol{x}) = E_k^*(x_1, \dots, x_n) = \prod_{1 \leq i_1 < \dots < i_k \leq n} \sum_{j=1}^k x_{i_j}, \ k = 1, \dots, n,$$

and defined $E_0^*(\boldsymbol{x}) = 1$, and $E_k^*(\boldsymbol{x}) = 0$ for $k < 0$ or $k > n$.

We known that the elementary symmetric function $E_k(\boldsymbol{x})$ is an increasing and Schur-concave function on $\mathbb{R}^n$[1]. The aim of this paper is to discuss the Schur-concave and the Schur-geometrically convex properties of $E_k^*(\boldsymbol{x})$ and to establish some relevant inequalities. We need the following definitions and lemmas.

Definition 1. [1, 2] Let $\boldsymbol{x} = (x_1, \dots, x_n)$ and $\boldsymbol{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$.

(1) $\boldsymbol{x}$ is said to be majorized by $\boldsymbol{y}$ (in symbols $\boldsymbol{x} \prec \boldsymbol{y}$) if $\sum_{i=1}^k x_{[i]} \leq \sum_{i=1}^k y_{[i]}$ for $k = 1, 2, \dots, n-1$ and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$, where $x_{[1]} \geq \dots \geq x_{[n]}$ and $y_{[1]} \geq \dots \geq y_{[n]}$ are rearrangements of $\boldsymbol{x}$ and $\boldsymbol{y}$ in a descending order, and $\boldsymbol{x}$ is said to strictly majors by $\boldsymbol{y}$ (written $\boldsymbol{x} \prec\prec \boldsymbol{y}$) if $\boldsymbol{x}$ is not permutation of $\boldsymbol{y}$.

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(2) $\boldsymbol{x} \geq \boldsymbol{y}$ means $x_i \geq y_i$ for all $i = 1, 2, \ldots, n$. let $\Omega \subset \mathbb{R}^n$, φ: $\Omega \to \mathbb{R}$ is said to be increasing if $\boldsymbol{x} \geq \boldsymbol{y}$ implies $\varphi(\boldsymbol{x}) \geq \varphi(\boldsymbol{y})$. φ is said to be decreasing if and only if $-\varphi$ is increasing.

(3) let $\Omega \subset \mathbb{R}^n$, φ: $\Omega \to \mathbb{R}$ is said to be a Schur-convex function on Ω if $\boldsymbol{x} \prec \boldsymbol{y}$ on Ω implies $\varphi(\boldsymbol{x}) \leq \varphi(\boldsymbol{y})$. φ is said to be a Schur-concave function on Ω if and only if $-\varphi$ is Schur-convex function on Ω. φ is said to be a strictly Schur-convex function on Ω if $\boldsymbol{x} \prec\prec \boldsymbol{y}$ on Ω implies $\varphi(\boldsymbol{x}) < \varphi(\boldsymbol{y})$. φ is said to be a strictly Schur-concave function on Ω if and only if $-\varphi$ is strictly Schur-convex function on Ω.

Definition 2. [3] let $\Omega \subset \mathbb{R}^n_{++}$, φ: $\Omega \to \mathbb{R}_+$ is said to be a Schur-geometrically convex function on Ω if $(\ln x_1, \ldots, \ln x_n) \prec (\ln y_1, \ldots, \ln y_n)$ on Ω implies $\varphi(\boldsymbol{x}) \leq \varphi(\boldsymbol{y})$. φ is said to be a Schur-geometrically concave function on Ω if and only $-\varphi$ is Schur-geometrically convex function.

Definition 3. [3] Let set $\Omega \subseteq \mathbb{R}^n$.

(1) Ω is said to be a convex set if $\boldsymbol{x}, \boldsymbol{y} \in \Omega, 0 \leq \alpha \leq 1$ implies $\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y} \in \Omega$.

(2) Ω is said to be a geometrically convex set if $\boldsymbol{x}, \boldsymbol{y} \in \Omega, 0 \leq \alpha \leq 1$ implies $\boldsymbol{x}^\alpha \boldsymbol{y}^{1-\alpha} \in \Omega$.

Lemma 1 ([1, p. 7])**.** *A function $\varphi(\boldsymbol{x})$ is increasing if and only if $\nabla\varphi(\boldsymbol{x}) \geq 0$ for $\boldsymbol{x} \in \Omega$, where $\Omega \subset \mathbb{R}^n$ is an open set, $\varphi : \Omega \to R$ is differentiable, and*

$$\nabla\varphi(\boldsymbol{x}) = \left(\frac{\partial\varphi(\boldsymbol{x})}{\partial x_1}, \ldots, \frac{\partial\varphi(\boldsymbol{x})}{\partial x_n}\right) \in \mathbb{R}^n.$$

Lemma 2 ([1, p. 5])**.** *Let $\Omega \subset \mathbb{R}^n$ is symmetric and has a nonempty interior convex set. Ω^0 is the interior of Ω. $\varphi : \Omega \to \mathbb{R}$ is continuous on Ω and differentiable in Ω^0. Then φ is the $Schur-convex(Schur-concave)function$, if and only if φ is symmetric on Ω and*

$$(x_1 - x_2)\left(\frac{\partial\varphi}{\partial x_1} - \frac{\partial\varphi}{\partial x_2}\right) \geq 0(\leq 0) \tag{1}$$

holds for any $\boldsymbol{x} \in \Omega^0$.

Lemma 3 ([3, p. 108])**.** *Let $\Omega \subset \mathbb{R}^n_+$ is a symmetric and has a nonempty interior geometrically convex set. Ω^0 is the interior of Ω. $\varphi : \Omega \to \mathbb{R}_+$ is continuous on Ω and differentiable in Ω^0. If φ is symmetric on Ω and*

$$(\ln x_1 - \ln x_2)\left(x_1\frac{\partial\varphi}{\partial x_1} - x_2\frac{\partial\varphi}{\partial x_2}\right) \geq 0(\leq 0) \tag{2}$$

holds for any $x = (x_1, x_2, \cdots, x_n) \in \Omega^0$, then φ is the Schur-geometrically convex (Schur-geometrically concave) function.

2. Main results and their proofs

In the following, we are in a position to state our main results and give proofs of them.

Theorem 1. *For $k = 1, \ldots, n, n \geq 2$, $E^*_k(\boldsymbol{x})$ is an increasing and Schur-concave function on $\mathbb{R}^n_+$; $E^*_k(\boldsymbol{x})$ is a strictly increasing and Schur-concave function and Schur-geometrically convex function on $\mathbb{R}^n_{++}$.*

Proof. It is easy to see that

$$E_k^*(\boldsymbol{x}) = E_k^*(x_1,\ldots,x_n) = E_k^*(x_2,\ldots,x_n)\cdot \prod_{2\le i_1<\ldots<i_k\le n}\left(x_1+\sum\nolimits_{j=1}^{k-1}x_{i_j}\right),$$

$$\ln E_k^*(\boldsymbol{x}) = \ln E_k^*(x_2,\ldots,x_n) + \sum_{2\le i_1<\ldots<i_k\le n}\ln\left(x_1+\sum\nolimits_{j=1}^{k-1}x_{i_j}\right),$$

and then

$$\frac{\partial E_k^*(\boldsymbol{x})}{\partial x_1} = E_k^*(\boldsymbol{x})\sum_{2\le i_1<\ldots<i_{k-1}\le n}\left(x_1+\sum\nolimits_{j=1}^{k-1}x_{i_j}\right)^{-1}$$

$$= E_k^*(\boldsymbol{x})\left[\sum_{3\le i_1<\ldots<i_{k-1}\le n}\left(x_1+\sum\nolimits_{j=1}^{k-1}x_{i_j}\right)^{-1} + \sum_{3\le i_1<\ldots<i_{k-2}\le n}\left(x_1+x_2+\sum\nolimits_{j=1}^{k-2}x_{i_j}\right)^{-1}\right]\ge 0,$$

$$\frac{\partial E_k^*(\boldsymbol{x})}{\partial x_2} = E_k^*(\boldsymbol{x})\sum_{2\le i_1<\ldots<i_{k-1}\le n}\left(x_2+\sum\nolimits_{j=1}^{k-1}x_{i_j}\right)^{-1}$$

$$= E_k^*(\boldsymbol{x})\left[\sum_{3\le i_1<\ldots<i_{k-1}\le n}\left(x_2+\sum\nolimits_{j=1}^{k-1}x_{i_j}\right)^{-1} + \sum_{3\le i_1<\ldots<i_{k-2}\le n}\left(x_1+x_2+\sum\nolimits_{j=1}^{k-2}x_{i_j}\right)^{-1}\right]\ge 0.$$

From the Lemmas 1, $E_k^*(\boldsymbol{x})$ is an increasing function on $\mathbb{R}_+^n$.

For any $\boldsymbol{x}\in(\mathbb{R}_+^n)^0$, we have

$$(x_1-x_2)\left(\frac{\partial E_k^*(\boldsymbol{x})}{\partial x_1}-\frac{\partial E_k^*(\boldsymbol{x})}{\partial x_2}\right)$$

$$= (x_1-x_2)\,E_k^*(\boldsymbol{x})\sum_{3\le i_1<\ldots<i_{k-1}\le n}\left[\left(x_1+\sum\nolimits_{j=1}^{k-1}x_{i_j}\right)^{-1} - \left(x_2+\sum\nolimits_{j=1}^{k-1}x_{i_j}\right)^{-1}\right]$$

$$= -\left(x_1-x_2\right)^2 E_k^*(\boldsymbol{x})\sum_{3\le i_1<\ldots<i_{k-1}\le n}\left(x_1+\sum\nolimits_{j=1}^{k-1}x_{i_j}\right)^{-1}\cdot\left(x_2+\sum\nolimits_{j=1}^{k-1}x_{i_j}\right)^{-1} < 0.$$

So from the Lemmas 2, $E_k^*(\boldsymbol{x})$ is a strictly Schur-concave function on $\mathbb{R}_{++}^n$.

$$x_1\frac{\partial E_k^*(\boldsymbol{x})}{\partial x_1} = x_1E_k^*(\boldsymbol{x})\sum_{2\le i_1<\ldots<i_{k-1}\le n}\left(x_1+\sum\nolimits_{j=1}^{k-1}x_{i_j}\right)^{-1}$$

$$= E_k^*(\boldsymbol{x})\left[\sum_{3\le i_1<\ldots<i_{k-1}\le n}x_1\left(x_1+\sum\nolimits_{j=1}^{k-1}x_{i_j}\right)^{-1} + \sum_{3\le i_1<\ldots<i_{k-2}\le n}x_1\left(x_1+x_2+\sum\nolimits_{j=1}^{k-2}x_{i_j}\right)^{-1}\right],$$

$$x_2\frac{\partial E_k^*(\boldsymbol{x})}{\partial x_1} = x_2E_k^*(\boldsymbol{x})\sum\nolimits_{2\le i_1<\ldots<i_{k-1}\le n}\left(x_1+\sum\nolimits_{j=1}^{k-1}x_{i_j}\right)^{-1}$$

$$= E_k^*(\boldsymbol{x})\left[\sum_{3\le i_1<\ldots<i_{k-1}\le n}x_2\left(x_1+\sum\nolimits_{j=1}^{k-1}x_{i_j}\right)^{-1} + \sum_{3\le i_1<\ldots<i_{k-2}\le n}x_2\left(x_1+x_2+\sum\nolimits_{j=1}^{k-2}x_{i_j}\right)^{-1}\right],$$

For any $\boldsymbol{x} \in (\mathbb{R}^n_{++})^0$, we have

$$(\ln x_1 - \ln x_2)\left(x_1 \frac{\partial E_k^*(\boldsymbol{x})}{\partial x_1} - x_2 \frac{\partial E_k^*(\boldsymbol{x})}{\partial x_2}\right)$$

$$= (\ln x_1 - \ln x_2)\, E_k^*(\boldsymbol{x}) \Bigg\{ \sum_{3 \le i_1 < \ldots < i_{k-1} \le n} \left[x_1 \left(x_1 + \sum\nolimits_{j=1}^{k-1} x_{i_j} \right)^{-1} - x_2 \left(x_2 + \sum\nolimits_{j=1}^{k-1} x_{i_j} \right)^{-1} \right]$$

$$+ \sum_{3 \le i_1 < \ldots < i_{k-2} \le n} \left[x_1 \left(x_1 + x_2 + \sum\nolimits_{j=1}^{k-2} x_{i_j} \right)^{-1} - x_2 \left(x_1 + x_2 + \sum\nolimits_{j=1}^{k-2} x_{i_j} \right)^{-1} \right] \Bigg\}$$

$$= (\ln x_1 - \ln x_2)\,(x_1 - x_2)\, E_k^*(\boldsymbol{x}) \Bigg[\sum_{3 \le i_1 < \ldots < i_{k-1} \le n} \left(x_1 + \sum\nolimits_{j=1}^{k-1} x_{i_j} \right)^{-1} \cdot \left(x_2 + \sum\nolimits_{j=1}^{k-1} x_{i_j} \right)^{-1}$$

$$\cdot \left(\sum\nolimits_{j=1}^{k-1} x_{i_j} \right) + \sum_{3 \le i_1 < \ldots < i_{k-2} \le n} \left(x_1 + x_2 + \sum\nolimits_{j=1}^{k-2} x_{i_j} \right)^{-1} \Bigg] > 0.$$

(Notice that$(\ln x_1 - \ln x_2)\,(x_1 - x_2) > 0$)

So from the Lemmas 2, $E_k^*(\boldsymbol{x})$ is a Schur-geometrically concave function on $\mathbb{R}^n_{++}$. The proof of Theorem 1 is completed. □

Corollary 1. *Let $\boldsymbol{x} \in \mathbb{R}^n_+, n \ge 2$ with $\sum_{i=1}^n x_i = s > 0$, and let $f(x)$ is a nonnegative concave function on $\mathbb{R}^1_+$ Then for $k = 1, \ldots, n$, we have*

$$E_k^*\left(f\left(x_1\right), \ldots, f\left(x_n\right)\right) = \prod_{1 \le i_1 < \ldots < i_k \le n} \sum\nolimits_{j=1}^{k} f\left(x_{i_j}\right) \le \left[k f(s/n)\right]^{C_n^k}. \tag{3}$$

In particular,

$$E_k^*(x_1, \ldots, x_n) = \prod_{1 \le i_1 < \ldots < i_k \le n} \sum\nolimits_{j=1}^{k} x_{i_j} \le \left[k(s/n)\right]^{C_n^k}, \tag{4}$$

with equality holding if and only if $x_1 = \cdots = x_n$, for $\boldsymbol{x} \in \mathbb{R}^n_{++}$.

Proof. Since $E_k^*(\boldsymbol{x})$ is increasing and Schur-concave on $\mathbb{R}^n_+$, and $f(x)$ is a nonnegative concave function, from proposition 6.16 (b) in [1], it follows that $E_k^*(f(x_1), \ldots, f(x_n))$ is also Schur-concave on $\mathbb{R}^n_+$. And then from

$$(s/n, \ldots, s/n) \prec (x_1, \ldots, x_n),$$

we have

$$E_k^*\left(f\left(x_1\right), \ldots, f\left(x_n\right)\right) \le E_k^*\left(f(s/n), \ldots, f(s/n)\right),$$

i.e. inequality(3) is hold. Since $E_k^*(x_1, \ldots, x_n)$ is strictly increasing and Schur-concave on $\mathbb{R}^n_{++}$, it follows inequality(4) is hold, and equality holding if and only if $x_1 = \cdots = x_n$, for $\boldsymbol{x} \in \mathbb{R}^n_{++}$. □

Corollary 2. *Let $\boldsymbol{x} \in \mathbb{R}^n_{++}, n \ge 2$, and let $\prod_{i=1}^n x_i = p > 0$. Then for $k = 1, \ldots, n, \alpha \ge 1$, we have*

$$E_k^*(x_1^\alpha, \ldots, x_n^\alpha) = \prod_{1 \le i_1 < \ldots < i_k \le n} \sum\nolimits_{j=1}^{k} x_{i_j}^\alpha \le \left(p^{\alpha/n}\right)^{C_n^k}, \tag{5}$$

with equality holding if and only if $x_1 = \cdots = x_n$, for $\alpha = 1$ and $\boldsymbol{x} \in \mathbb{R}^n_{++}$.

Proof. Since $E_k^*(\boldsymbol{x})$ is increasing and Schur-geometrically concave on $\mathbb{R}_{++}^n$, and x^α is convex on $\mathbb{R}_{++}^n$, for $\alpha \geq 1$, from proposition 6.16 (a) on [1], it follows that $E_k^*(x_1^\alpha, \ldots, x_n^\alpha)$ is also Schur-geometrically convex on $\mathbb{R}_{++}^n$. And then from

$$(\ln \sqrt[n]{p}, \ldots, \ln \sqrt[n]{p}) \prec (\ln x_1, \ldots, \ln x_n)\,,$$

we have

$$E_k^*\left(x_1^\alpha, \ldots, x_n^\alpha\right) \leq E_k^*\left((\sqrt[n]{p})^\alpha, \ldots, (\sqrt[n]{p})^\alpha\right),$$

i.e. inequality (5) is hold. When $\alpha = 1$, since $E_k^*(x_1^\alpha, \ldots, x_n^\alpha)$ is strictly increasing and Schur-concave on $\mathbb{R}_{++}^n$, it follows equality holding if and only if $x_1 = \cdots = x_n$, for $\boldsymbol{x} \in \mathbb{R}_{++}^n$. □

Corollary 3. *Let $\boldsymbol{x} \in \mathbb{R}_+^n, n \geq 2$, and let $\sum_{i=1}^n x_i = s > 0, c \geq s$. Then for $k = 1, \ldots, n, 0 \leq \alpha \leq 1$, we have*

$$\frac{E_k^*((c-x_1)^\alpha, \ldots, (c-x_n)^\alpha)}{E_k^*(x_1^\alpha, \ldots, x_n^\alpha)} \geq \left(\frac{nc}{s} - 1\right)^{\alpha C_n^k}, \tag{6}$$

with equality holding if and only if $x_1 = \cdots = x_n$, for $\alpha = 1$.

Proof. In [4], it is proved that

$$\left(\frac{(c-x_1)s}{nc-s}, \frac{(c-x_2)s}{nc-s}, \ldots, \frac{(c-x_n)s}{nc-s}\right) \prec (x_1, x_2, \ldots, x_n).$$

Combining the Schur-concavity of $E_k^*(x_1^\alpha, \ldots, x_n^\alpha)$ on $\mathbb{R}_{++}^n$, it follows that inequality (6) is hold. □

Corollary 4. *Let $\boldsymbol{x} \in \mathbb{R}_+^n, n \geq 2$, and let $\sum_{i=1}^n x_i = s > 0, c \geq 0$. Then for $k = 1, \ldots, n, 0 \leq \alpha \leq 1$, we have*

$$\frac{E_k^*((c+x_1)^\alpha, \ldots, (c+x_n)^\alpha)}{E_k^*(x_1^\alpha, \ldots, x_n^\alpha)} \geq \left(\frac{nc}{s} + 1\right)^{\alpha C_n^k}, \tag{7}$$

with equality holding if and only if $x_1 = \cdots = x_n$, for $\alpha = 1$.

Proof. In [4], it is proved that

$$\left(\frac{(c+x_1)s}{nc+s}, \frac{(c+x_2)s}{nc+s}, \ldots, \frac{(c-x_n)s}{nc-s}\right) \prec (x_1, x_2, \ldots, x_n).$$

Combining the Schur-concavity of $E_k^*(x_1^\alpha, \ldots, x_n^\alpha)$ on $\mathbb{R}_{++}^n$, it follows that inequality (7) is hold. □

Corollary 5. *Let $\boldsymbol{x} \in \mathbb{R}_+^n, n \geq 2$ and $0 < r \leq s$. Then*

$$\frac{E_k^*(x_1^r, \ldots, x_n^r)}{E_k^*(x_1^s, \ldots, x_n^s)} \geq \left(\frac{\sum_{i=1}^n x_i^r}{\sum_{i=1}^n x_i^s}\right)^{C_n^k}. \tag{8}$$

Proof. In [2, p.130], it is proved that

$$\left(\frac{x_1}{\sum_{i=1}^n x_i^r}, \ldots, \frac{x_2}{\sum_{i=1}^n x_i^r}\right) \prec \left(\frac{x_1}{\sum_{i=1}^n x_i^s}, \ldots, \frac{x_2}{\sum_{i=1}^n x_i^s}\right).$$

Combining the Schur-concavity of $E_k^*(x_1^\alpha, \ldots, x_n^\alpha)$ on $\mathbb{R}_{++}^n$, it follows that inequality (8) is hold. □

3. Applications

Theorem 2. *Let A be an n-dimensional simplex in n-dimensional Euclidean space $\mathbb{E}^n(n \geq 3\ )$ and $\{A_1, A_2, ..., A_{n+1}\}$is the set of vertices. Let P be an arbitrary point in the interior of A. If B_i is the intersection point of the extension line of A_iP and the $(n-1)$-dimensional hyperplane opposite to the point A_i. Then we have*

$$E_k^*\left(\left(\frac{PB_1}{A_1B_1}\right)^\alpha, \dots, \left(\frac{PB_{n+1}}{A_{n+1}B_{n+1}}\right)^\alpha\right) \leq \left[k\left(\frac{1}{n+1}\right)^\alpha\right]^{C_{n+1}^k}, \tag{9}$$

$$E_k^*\left(\left(\frac{A_1P}{A_1B_1}\right)^\alpha, \dots, \left(\frac{A_{n+1}P}{A_{n+1}B_{n+1}}\right)^\alpha\right) \leq \left[k\left(\frac{n}{n+1}\right)^\alpha\right]^{C_{n+1}^k} \tag{10}$$

and

$$E_k^*\left(\left(\frac{A_1P}{A_1B_1}\right)^\alpha, \dots, \left(\frac{A_{n+1}P}{A_{n+1}B_{n+1}}\right)^\alpha\right) \geq n^{C_{n+1}^k} E_k^*\left(\left(\frac{PB_1}{A_1B_1}\right)^\alpha, \dots, \left(\frac{PB_{n+1}}{A_{n+1}B_{n+1}}\right)^\alpha\right) \tag{11}$$

Proof. It is easy to known that

$$\sum_{i=1}^{n+1}\frac{PB_i}{A_iB_i} = 1, \frac{A_iP}{A_iB_i} = 1 - \frac{PB_i}{A_iB_i}, k = 1, 2, \dots, n+1, \sum_{i=1}^{n+1}\frac{A_iP}{A_iB_i} = n.$$

Taking $s = 1$ and $s = n$ in (3), it follows that (9) and (10) holds respectively, and taking $s = c = 1$ in (4), it follows that (11) holds. The proof of Theorem 2 is be completed. □

References

[1] B.-Y. Wang. *Foundations of majorization inequalities*, Beijing Normal Univ. Press, Beijing, China, 1990. (Chinese)

[2] A. M. Marshall and I. Olkin. *Inequalities:theory of majorization and its application.* New York: Academies Press, 1979.

[3] X. -M. Zhang. *Geometrically convex functions.* Hefei: An'hui University Press, 2004.(Chinese)

[4] H.-N.Shi *Majorized proof of a kind of inequalities for symmetric function* Chinese J. Math. for Techology1999, 15 (3): 140-142.

[5] D. S.Mitrinovic, J. E. Pecaric, V.Volenec. *Recent advances in geometric inequalities.* Kluwer Academic Publishers, 1989, 463-473.

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