

# NORMALIZED JENSEN FUNCTIONAL, SUPERQUADRACITY AND RELATED INEQUALITIES

S. ABRAMOVICH AND S. S. DRAGOMIR

ABSTRACT. In this paper we generalize the inequality

$$MJ_n(f, \mathbf{x}, \mathbf{q}) \geq J_n(f, \mathbf{x}, \mathbf{p}) \geq mJ_n(f, \mathbf{x}, \mathbf{q})$$

where

$$J_n(f, \mathbf{x}, \mathbf{p}) = \sum_{i=1}^n p_i f(x_i) - f\left(\sum_{i=1}^n p_i x_i\right),$$

obtained by S.S. Dragomir for convex functions. We provide cases where we can improve the bounds  $m$  and  $M$  for convex functions, and also, we show that for the class of superquadratic functions nonzero lower bounds of  $J_n(f, \mathbf{x}, \mathbf{p}) - mJ_n(f, \mathbf{x}, \mathbf{q})$  and nonzero upper bounds of  $J_n(f, \mathbf{x}, \mathbf{p}) - MJ_n(f, \mathbf{x}, \mathbf{q})$  can be pointed out. Finally, an inequality related to the Čebyšev functional and superquadracity is also given.

## 1. INTRODUCTION

In this paper we consider the normalized Jensen functional

$$(1.1) \quad J_n(f, \mathbf{x}, \mathbf{p}) = \sum_{i=1}^n p_i f(x_i) - f\left(\sum_{i=1}^n p_i x_i\right),$$

where  $\sum_{i=1}^n p_i = 1$ ,  $f: I \rightarrow \mathbb{R}$ , and  $I$  is an interval in  $\mathbb{R}$ .

This type of functionals were considered by S. S. Dragomir in [5], where the following theorem was proved:

**Theorem 1** ([5, Theorem 1]). *Consider the normalized Jensen functional (1.1) where  $f: C \rightarrow \mathbb{R}$  is a convex function on the convex set  $C$  in a real linear space, and  $\mathbf{x} = (x_1, \dots, x_n) \in C^n$ ,  $\mathbf{p} = (p_1, \dots, p_n)$ ,  $\mathbf{q} = (q_1, \dots, q_n)$  are nonnegative  $n$ -tuples satisfying  $\sum_{i=1}^n p_i = 1$ ,  $\sum_{i=1}^n q_i = 1$ ,  $q_i > 0$ ,  $i = 1, \dots, n$ . Then*

$$(1.2) \quad MJ_n(f, \mathbf{x}, \mathbf{q}) \geq J_n(f, \mathbf{x}, \mathbf{p}) \geq mJ_n(f, \mathbf{x}, \mathbf{q}),$$

provided

$$m := \min_{1 \leq i \leq n} \left(\frac{p_i}{q_i}\right), \quad M := \max_{1 \leq i \leq n} \left(\frac{p_i}{q_i}\right).$$

In the following section we show when (1.2) holds for  $m^*$  larger than  $\min_{1 \leq i \leq n} \left(\frac{p_i}{q_i}\right)$ , and  $M^*$  smaller than  $\max_{1 \leq i \leq n} \left(\frac{p_i}{q_i}\right)$ . Although  $\mathbf{x} = (x_1, \dots, x_n)$ ,  $x_i \in I$ ,  $i = 1, \dots, n$

---

*Date:* 26 June, 2007.

*1991 Mathematics Subject Classification.* 26D15.

*Key words and phrases.* Convex Functions, Superquadratic Functions, Jensen Inequality, Jensen Steffensen Inequality, Čebyšev Inequality.

is not necessarily a monotonic  $n$ -tuple, we use Jensen-Steffensen's inequality that states that if  $f : I \rightarrow \mathbb{R}$  is convex, where  $I$  is an interval in  $\mathbb{R}$ , then

$$(1.3) \quad \sum_{i=1}^n a_i f(x_i) \geq A_n f(\bar{x}),$$

where  $\bar{x} := \frac{\sum_{i=1}^n a_i x_i}{A_n}$ ,  $\mathbf{x} = (x_1, \dots, x_n)$  is any monotone  $n$ -tuple in  $I^n$ , and  $\mathbf{a} = (a_1, \dots, a_n)$  is a real  $n$ -tuple that satisfies the condition:

$$(1.4) \quad 0 \leq A_i \leq A_n, \quad i = 1, \dots, n, \quad \text{where } A_i = \sum_{j=1}^i a_j, \quad \text{and } A_n > 0$$

(see for instance [6, page 43]).

In Section 2 we also show that for a class of superquadratic functions defined below, nonzero lower bounds of  $J_n(f, \mathbf{x}, \mathbf{p}) - mJ_n(f, \mathbf{x}, \mathbf{q})$  and of  $J_n(f, \mathbf{x}, \mathbf{p}) - m^*J_n(f, \mathbf{x}, \mathbf{q})$  and nonzero upper bounds of  $J_n(f, \mathbf{x}, \mathbf{p}) - MJ_n(f, \mathbf{x}, \mathbf{q})$  and of  $J_n(f, \mathbf{x}, \mathbf{p}) - M^*J_n(f, \mathbf{x}, \mathbf{q})$  are obtained. In addition, we get in the last section an inequality related to the Čebyšev's type functional and superquadracity.

**Definition 1** ([2, Definition 1]). *A function  $f$  defined on an interval  $I = [0, a]$  or  $[0, \infty)$  is superquadratic, if for each  $x$  in  $I$  there exists a real number  $C(x)$  such that*

$$(1.5) \quad f(y) - f(x) \geq f(|y - x|) + C(x)(y - x)$$

for all  $y \in I$ .

For example, the functions  $x^p$ ,  $p \geq 2$  and the functions  $-x^p$ ,  $0 \leq p \leq 2$  are superquadratic functions as well as the function  $f(x) = x^2 \log x$ ,  $x > 0$ ,  $f(0) = 0$ .

In Section 2 we use also the following lemmas and theorem for superquadratic functions:

**Lemma 1** ([2, Lemma 2.1]). *Let  $f$  be a superquadratic function with  $C(x)$  as in (1.5).*

- (i) *Then  $f(0) \leq 0$*
- (ii) *If  $f(0) = f'(0) = 0$ , then  $C(x) = f'(x)$  wherever  $f$  is differentiable at  $x > 0$ .*
- (iii) *If  $f \geq 0$ , then  $f$  is convex and  $f(0) = f'(0) = 0$ .*

**Lemma 2** ([3, Lemma 2.3]). *Suppose that  $f$  is superquadratic. Let  $x_i \geq 0$ ,  $i = 1, \dots, n$  and let  $\bar{x} := \sum_{i=1}^n a_i x_i$ , where  $a_i \geq 0$ ,  $i = 1, \dots, n$  and  $\sum_{i=1}^n a_i = 1$ . Then*

$$(1.6) \quad \sum_{i=1}^n a_i f(x_i) - f(\bar{x}) \geq \sum_{i=1}^n a_i f(|x_i - \bar{x}|).$$

The following Theorem 2 was proved in [1, Theorem 1] for differentiable positive superquadratic functions  $f$ , but because of Lemma 1 (iii) it holds also when  $f$  is not always differentiable.

**Theorem 2.** *Let  $f : I \rightarrow \mathbb{R}$ , where  $I$  is  $[0, a]$  or  $[0, \infty)$ , be nonnegative superquadratic function. Let  $\mathbf{x}$  be a monotone nonnegative  $n$ -tuple in  $I^n$  and  $\mathbf{a}$  satisfies (1.4). Let*

$$(1.7) \quad \bar{x} := \frac{\sum_{i=1}^n a_i x_i}{A_n}$$

Then

$$(1.8) \quad \sum_{i=1}^n a_i f(x_i) - A_n f(\bar{x}) \geq (n-1) A_n f\left(\frac{\sum_{i=1}^n a_i |x_i - \bar{x}|}{(n-1) A_n}\right).$$

## 2. THE MAIN RESULTS

In this section we use the following notations:

Let  $\mathbf{x}_\uparrow = (x_{(1)}, \dots, x_{(n)})$  be the *increasing rearrangement* of  $\mathbf{x} = (x_1, \dots, x_n)$ . Let  $\pi$  be the permutation that transfers  $\mathbf{x}$  into  $\mathbf{x}_\uparrow$  and let  $(\bar{p}_1, \dots, \bar{p}_n)$  and  $(\bar{q}_1, \dots, \bar{q}_n)$  be the  $n$ -tuples obtained by the same permutation  $\pi$  on  $(p_1, \dots, p_n)$  and  $(q_1, \dots, q_n)$  respectively. Then for an  $n$ -tuple  $\mathbf{x} = (x_1, \dots, x_n)$ ,  $x_i \in I$ ,  $i = 1, \dots, n$  where  $I$  is an interval in  $\mathbb{R}$  we get the following results:

**Theorem 3.** Let  $\mathbf{p} = (p_1, \dots, p_n)$ , where  $0 \leq \sum_{j=1}^i \bar{p}_j \leq 1$ ,  $i = 1, \dots, n$ ,  $\sum_{i=1}^n p_i = 1$ , and  $\mathbf{q} = (q_1, \dots, q_n)$ ,  $0 < \sum_{j=1}^i \bar{q}_j < 1$ ,  $i = 1, \dots, n-1$ ,  $\sum_{i=1}^n q_i = 1$ , and  $\mathbf{p} \neq \mathbf{q}$ . Denote

$$(2.1) \quad m_i := \frac{\sum_{j=1}^i \bar{p}_j}{\sum_{j=1}^i \bar{q}_j}, \quad \bar{m}_i := \frac{\sum_{j=i}^n \bar{p}_j}{\sum_{j=i}^n \bar{q}_j}, \quad i = 1, \dots, n$$

where  $(\bar{p}_1, \dots, \bar{p}_n)$  and  $(\bar{q}_1, \dots, \bar{q}_n)$  are as denoted above, and

$$(2.2) \quad m^* := \min_{1 \leq i \leq n} \{m_i, \bar{m}_i\}, \quad M^* := \max_{1 \leq i \leq n} \{m_i, \bar{m}_i\}.$$

If  $\mathbf{x} = (x_1, \dots, x_n)$  is any  $n$ -tuple in  $I^n$ , where  $I$  is an interval in  $\mathbb{R}$ , then

$$(2.3) \quad M^* J_n(f, \mathbf{x}, \mathbf{q}) \geq J_n(f, \mathbf{x}, \mathbf{p}) \geq m^* J_n(f, \mathbf{x}, \mathbf{q}),$$

where  $f : I \rightarrow \mathbb{R}$  is a convex function on the interval  $I$ .

*Proof.* As  $\mathbf{p} \neq \mathbf{q}$  it is clear that  $m^* < 1$ ,  $m^* \geq \min_{1 \leq i \leq n} \left(\frac{p_i}{q_i}\right)$ , and  $M^* > 1$ ,

$$M^* \leq \max_{1 \leq i \leq n} \left(\frac{p_i}{q_i}\right).$$

As  $\sum_{i=1}^n q_i = 1$  and  $q_i > 0$  it is obvious that there is an integer  $k$ ,  $2 \leq k \leq n$  such that  $x_{(k-1)} \leq \sum_{i=1}^n q_i x_i \leq x_{(k)}$ .

We apply Jensen-Steffensen's inequality for the increasing  $(n+1)$ -tuple  $\mathbf{y} = (y_1, \dots, y_{n+1})$

$$(2.4) \quad y_i = \begin{cases} x_{(i)}, & i = 1, \dots, k-1 \\ \sum_{j=1}^n q_j x_j, & i = k \\ x_{(i-1)}, & i = k+1, \dots, n+1 \end{cases}$$

and to

$$(2.5) \quad a_i = \begin{cases} \bar{p}_i - m^* \bar{q}_i, & i = 1, \dots, k-1 \\ m^*, & i = k \\ \bar{p}_{i-1} - m^* \bar{q}_{i-1}, & i = k+1, \dots, n+1 \end{cases}$$

where  $m^*$  is defined in (2.2).

It is clear that  $\mathbf{a}$  satisfies (1.4). Therefore, (1.3) holds for the increasing  $(n+1)$ -tuple  $\mathbf{y}$  and for a convex function  $f$ .

Hence

$$\begin{aligned} \sum_{i=1}^{n+1} a_i f(y_i) &= m^* f\left(\sum_{i=1}^n q_i x_i\right) + \sum_{i=1}^n (p_i - m^* q_i) f(x_i) \\ &\geq f\left(m^* \sum_{i=1}^n q_i x_i + \sum_{i=1}^n (p_i - m^* q_i) x_i\right) = f\left(\sum_{i=1}^n p_i x_i\right). \end{aligned}$$

In other words

$$\sum_{i=1}^n p_i f(x_i) - f\left(\sum_{i=1}^n p_i x_i\right) \geq m^* \left(\sum_{i=1}^n a_i f(x_i) - f\left(\sum_{i=1}^n q_i x_i\right)\right).$$

This completes the proof of the right side inequality in (2.3).

The proof of the left side of (2.3) is similar:

We define an increasing  $(n+1)$ -tuple  $\mathbf{z}$

$$(2.6) \quad z_i = \begin{cases} x_{(i)}, & i = 1, \dots, s-1 \\ \sum_{j=1}^n p_j x_j, & i = s \\ x_{(i-1)}, & i = s+1, \dots, n+1 \end{cases}$$

and to

$$(2.7) \quad b_i = \begin{cases} \bar{q}_i - \frac{\bar{p}_i}{M^*}, & i = 1, \dots, s-1 \\ \frac{1}{M^*}, & i = s \\ \bar{q}_{i-1} - \frac{\bar{p}_{i-1}}{M^*}, & i = s+1, \dots, n+1, \end{cases}$$

where  $s$  satisfies  $x_{s-1} \leq \sum_{j=1}^n p_j x_j \leq x_s$ . As  $\mathbf{b}$  satisfies (1.4) and  $\sum_{i=1}^{n+1} b_i = 1$ , by using Jensen-Steffensen's inequality, we get the left side of (2.3).

This completes the proof.  $\blacksquare$

**Remark 1.** If  $\min_{1 \leq i \leq n} \left(\frac{p_i}{q_i}\right) = \frac{\bar{p}_k}{\bar{q}_k}$ ,  $k \neq 1, n$  and  $\max_{1 \leq i \leq n} \left(\frac{p_i}{q_i}\right) = \frac{\bar{p}_s}{\bar{q}_s}$ ,  $s \neq 1, n$  then it is clear that for  $p_i \geq 0$ , and  $q_i > 0$ , we get that  $m^* > m$  and  $M^* < M$  and in these cases (2.3) refines (1.2).

In Theorem 4 that deals with superquadratic functions we use the same techniques as used in [5] to prove Theorem 1 for convex functions.

**Theorem 4.** Under the same conditions and definitions on  $\mathbf{p}$ ,  $\mathbf{q}$ ,  $\mathbf{x}$ ,  $m$  and  $M$  as in Theorem 1, if  $I$  is  $[0, a)$  or  $[0, \infty)$  and  $f(x)$  is a superquadratic function on  $I$ , then

$$(2.8) \quad \begin{aligned} &J_n(f, \mathbf{x}, \mathbf{p}) - mJ_n(f, \mathbf{x}, \mathbf{q}) \\ &\geq mf\left(\left|\sum_{i=1}^n (q_i - p_i) x_i\right|\right) + \sum_{i=1}^n (p_i - m q_i) f\left(\left|x_i - \sum_{i=1}^n p_j x_j\right|\right) \end{aligned}$$

and

$$(2.9) \quad \begin{aligned} &J_n(f, \mathbf{x}, \mathbf{p}) - MJ_n(f, \mathbf{x}, \mathbf{q}) \\ &\leq -\left(\sum_{i=1}^n (M q_i - p_i) f\left(\left|x_i - \sum_{i=1}^n q_j x_j\right|\right) + f\left(\left|\sum_{i=1}^n (p_i - q_i) x_i\right|\right)\right). \end{aligned}$$

*Proof.* To prove (2.8) we define  $\mathbf{y}$  as

$$y_i = \begin{cases} x_i, & i = 1, \dots, n \\ \sum_{j=1}^n q_j x_j, & i = n+1 \end{cases},$$

and  $\mathbf{d}$  as

$$d_i = \begin{cases} p_i - mq_i, & i = 1, \dots, n \\ m, & i = n+1 \end{cases}.$$

Then (1.6) for  $\mathbf{y}$  and  $\mathbf{d}$  is

$$\begin{aligned} \sum_{i=1}^n (p_i - mq_i) f(x_i) + mf\left(\sum_{i=1}^n q_i x_i\right) &= \sum_{i=1}^{n+1} d_i f(y_i) - f\left(\sum_{i=1}^{n+1} d_i y_i\right) \\ &\geq \sum_{i=1}^{n+1} d_i f\left(\left|y_i - \sum_{j=1}^{n+1} a_j y_j\right|\right) \\ &= \sum_{i=1}^n (p_i - mq_i) f\left(\left|x_i - \sum_{j=1}^n p_j x_j\right|\right) + mf\left(\left|\sum_{i=1}^n (p_i - q_i) x_i\right|\right) \end{aligned}$$

which is (2.8).

To get (2.9), we choose  $\mathbf{z}$  and  $\mathbf{r}$  as

$$z_i = \begin{cases} x_i, & i = 1, \dots, n \\ \sum_{j=1}^n p_j x_j, & i = n+1 \end{cases},$$

and

$$r_i = \begin{cases} q_i - \frac{p_i}{M}, & i = 1, \dots, n \\ \frac{1}{M}, & i = n+1 \end{cases}$$

where  $s$  is any integer  $1 \leq s \leq n-1$ .

Then, as  $f$  is superquadratic and  $\sum_{i=1}^n r_i = 1$ ,  $r_i \geq 0$ , we get that

$$\begin{aligned} \sum_{i=1}^n \left(q_i - \frac{p_i}{M}\right) f(x_i) + \frac{1}{M} f\left(\sum_{i=1}^n p_i x_i\right) - f\left(\sum_{i=1}^n q_i x_i\right) \\ &= \sum_{i=1}^{n+1} r_i f(z_i) - f\left(\sum_{i=1}^{n+1} r_i z_i\right) \\ &\geq \sum_{i=1}^{n+1} r_i f\left(\left|z_i - \sum_{i=1}^{n+1} r_i z_i\right|\right) \\ &= \sum_{i=1}^n \left(q_i - \frac{p_i}{M}\right) f\left(\left|x_i - \sum_{j=1}^n q_j x_j\right|\right) + \frac{1}{M} f\left(\left|\sum_{i=1}^n (p_i - q_i) x_i\right|\right) \end{aligned}$$

which is equivalent to (2.9). ■

**Remark 2.** *If the superquadratic function is also positive and therefore according to Lemma 1 is convex, then (2.8) and (2.9) refine Theorem 1.*

The following result is proved for superquadratic functions using the same technique used in Theorem 3 for convex functions and by using Theorem 2, therefore, the proof is omitted.

**Theorem 5.** Let  $f(x)$  be a positive superquadratic function on  $[0, a]$ . Let  $\mathbf{x}$ ,  $\mathbf{p}$ ,  $\mathbf{q}$ ,  $m^*$ ,  $M^*$  be the same as in Theorem 3. Then

$$(2.10) \quad J_n(f, \mathbf{x}, \mathbf{p}) - m^* J_n(f, \mathbf{x}, \mathbf{q}) \\ \geq n f \left( \frac{\sum_{i=1}^n (p_i - m^* q_i) |x_i - \sum_{j=1}^n p_j x_j| + m^* |\sum_{i=1}^n (p_i - q_i) x_i|}{n} \right) \geq 0,$$

and

$$(2.11) \quad J_n(f, \mathbf{x}, \mathbf{p}) - M^* J_n(f, \mathbf{x}, \mathbf{q}) \\ \leq -n f \left( \frac{\sum_{i=1}^n (q_i - \frac{p_i}{M^*}) |x_i - \sum_{j=1}^n q_j x_j| + |\sum_{j=1}^n (q_j - p_j) x_j|}{n} \right) \leq 0.$$

In the following we state another generalisation of the Jensen inequality for superquadratic functions, and then we extend Theorems 4 and 5.

**Theorem 6.** Assume that  $\mathbf{x} = (x_1, \dots, x_n)$  with  $x_i \geq 0$  for  $i \in \{1, \dots, n\}$ ,  $\mathbf{p} = (p_1, \dots, p_n)$  is a probability sequence and  $\mathbf{q} = (q_1, \dots, q_k)$  is another probability sequence with  $n, k \geq 2$ . Then for any superquadratic function  $f : [0, \infty) \rightarrow \mathbb{R}$  we have the inequality

$$(2.12) \quad \sum_{i_1, \dots, i_k=1}^n p_{i_1} \dots p_{i_k} f \left( \sum_{j=1}^k q_j x_{i_j} \right) \\ \geq f \left( \sum_{i=1}^n p_i x_i \right) + \sum_{i_1, \dots, i_k=1}^n p_{i_1} \dots p_{i_k} f \left( \left| \sum_{j=1}^k q_j x_{i_j} - \sum_{i=1}^n p_i x_i \right| \right).$$

*Proof.* By the definition of superquadratic functions, we have

$$(2.13) \quad f \left( \sum_{j=1}^k q_j x_{i_j} \right) \\ \geq f \left( \sum_{i=1}^n p_i x_i \right) + C \left( \sum_{i=1}^n p_i x_i \right) \left( \sum_{j=1}^k q_j x_{i_j} - \sum_{i=1}^n p_i x_i \right) \\ + f \left( \left| \sum_{j=1}^k q_j x_{i_j} - \sum_{i=1}^n p_i x_i \right| \right)$$

for any  $x_{i_j} \geq 0, i_j \in \{1, \dots, n\}$ .

Now, if we multiply (2.13) with  $p_{i_1} \dots p_{i_k} \geq 0$ , sum over  $i_1, \dots, i_k$  from 1 to  $n$  and take into account that  $\sum_{i_1, \dots, i_k=1}^n p_{i_1} \dots p_{i_k} = 1$  we deduce

$$\begin{aligned}
 (2.14) \quad & \sum_{i_1, \dots, i_k=1}^n p_{i_1} \dots p_{i_k} f \left( \sum_{j=1}^k q_j x_{i_j} \right) \\
 & \geq f \left( \sum_{i=1}^n p_i x_i \right) + C \left( \sum_{i=1}^n p_i x_i \right) \sum_{i_1, \dots, i_k=1}^n p_{i_1} \dots p_{i_k} \left( \sum_{j=1}^k q_j x_{i_j} - \sum_{i=1}^n p_i x_i \right) \\
 & + \sum_{i_1, \dots, i_k=1}^n p_{i_1} \dots p_{i_k} f \left( \left| \sum_{j=1}^k q_j x_{i_j} - \sum_{i=1}^n p_i x_i \right| \right).
 \end{aligned}$$

However

$$\begin{aligned}
 I & : = \sum_{i_1, \dots, i_k=1}^n p_{i_1} \dots p_{i_k} \left( \sum_{j=1}^k q_j x_{i_j} - \sum_{i=1}^n p_i x_i \right) \\
 & = \sum_{i_1, \dots, i_k=1}^n p_{i_1} \dots p_{i_k} \left( \sum_{j=1}^k q_j x_{i_j} \right) - \sum_{i=1}^n p_i x_i
 \end{aligned}$$

and since

$$\begin{aligned}
 & \sum_{i_1, \dots, i_k=1}^n p_{i_1} \dots p_{i_k} \left( \sum_{j=1}^k q_j x_{i_j} \right) \\
 & = q_1 \sum_{i_1=1}^n p_{i_1} x_{i_1} \sum_{i_2, \dots, i_k=1}^n p_{i_2} \dots p_{i_k} + \dots + q_k \sum_{i_k=1}^n p_{i_k} x_{i_k} \sum_{i_1, \dots, i_{k-1}=1}^n p_{i_1} \dots p_{i_{k-1}} \\
 & = q_1 \sum_{i=1}^n p_i x_i + \dots + q_k \sum_{i=1}^n p_i x_i = \sum_{i=1}^n p_i x_i
 \end{aligned}$$

hence  $I = 0$  and by (2.14) we get the desired result (2.12). ■

**Theorem 7.** Assume that  $\mathbf{x} = (x_1, \dots, x_n)$  with  $x_i \in I$ ,  $i = 1, \dots, n$ ,  $I$  is an interval in  $\mathbb{R}$ ,  $\mathbf{p} = (p_1, \dots, p_n)$ ,  $\mathbf{r} = (r_1, \dots, r_n)$ ,  $r_i > 0$ ,  $i = 1, \dots, n$  are probability sequences, and  $\mathbf{q} = (q_1, \dots, q_k)$ , another probability sequence with  $n, k \geq 2$ . Then, for any convex function  $f$  on  $I$  we have the inequality

$$\begin{aligned}
 (2.15) \quad & M \left( \sum_{i_1, \dots, i_k=1}^n r_{i_1} \dots r_{i_k} f \left( \sum_{j=1}^k q_j x_{i_j} \right) - f \left( \sum_{i=1}^n r_i x_i \right) \right) \\
 & \geq \left( \sum_{i_1, \dots, i_k=1}^n p_{i_1} \dots p_{i_k} f \left( \sum_{j=1}^k q_j x_{i_j} \right) - f \left( \sum_{i=1}^n p_i x_i \right) \right) \\
 & \geq m \left( \sum_{i_1, \dots, i_k=1}^n r_{i_1} \dots r_{i_k} f \left( \sum_{j=1}^k q_j x_{i_j} \right) - f \left( \sum_{i=1}^n r_i x_i \right) \right)
 \end{aligned}$$

where  $m := \min_{1 \leq i_1, \dots, i_k \leq n} \left( \frac{p_{i_1} \dots p_{i_k}}{r_{i_1} \dots r_{i_k}} \right)$ ,  $M := \max_{1 \leq i_1, \dots, i_k \leq n} \left( \frac{p_{i_1} \dots p_{i_k}}{r_{i_1} \dots r_{i_k}} \right)$ .

*Proof.* The proof is similar to the proof of Theorem 1:

We will prove the right side of the inequality. The left side of the inequality is similar.

As

$$\begin{aligned} & m \sum_{i=1}^n r_i x_i + \sum_{i_1, i_2, \dots, i_k=1}^n (p_{i_1} \dots p_{i_k} - m r_{i_1} \dots r_{i_k}) \sum_{j=1}^k q_j x_{i_j} \\ &= \sum_{i_1 \dots i_k}^n p_{i_1} \dots p_{i_k} \sum_{j=1}^k q_j x_{i_j} = \sum_{i=1}^n p_i x_i, \end{aligned}$$

$0 \leq m \leq 1$ ,  $0 \leq p_{i_1} \dots p_{i_k} - m r_{i_1} \dots r_{i_k} \leq 1$  and  $m + \sum_{i_1 \dots i_k=1}^n (p_{i_1} \dots p_{i_k} - m r_{i_1} \dots r_{i_k}) = 1$  we get as a result of the convexity of  $f$  that

$$\begin{aligned} & m f \left( \sum_{i=1}^n r_i x_i \right) + \sum_{i_1 \dots i_k=1}^n (p_{i_1} \dots p_{i_k} - m r_{i_1} \dots r_{i_k}) f \left( \sum_{j=1}^k q_j x_{i_j} \right) \\ & \geq f \left( m \sum_{i=1}^n r_i x_i + \sum_{i_1 \dots i_k=1}^n (p_{i_1} \dots p_{i_k} - m r_{i_1} \dots r_{i_k}) f \left( \sum_{j=1}^k q_j x_{i_j} \right) \right) \\ & = f \left( \sum_{i=1}^n p_i x_i \right). \end{aligned}$$

This completes the proof of the right inequality of (2.15). ■

Below we state the analogue to Theorem 7 for superquadratic functions. The proof is similar to the proof of Theorem 4 and hence it is omitted.

**Theorem 8.** *Under the same conditions on  $\mathbf{p}$ ,  $\mathbf{q}$ ,  $\mathbf{r}$ ,  $m$  and  $M$  as in Theorem 7, if  $I$  is  $[0, a)$  or  $[0, \infty)$  and  $f(x)$  is a superquadratic function on  $I$ , then:*

$$\begin{aligned} & \sum_{i_1, \dots, i_k=1}^n p_{i_1} \dots p_{i_k} f \left( \sum_{j=1}^k q_j x_{i_j} \right) - f \left( \sum_{i=1}^n p_i x_i \right) \\ & - m \left( \sum_{i_1, \dots, i_k=1}^n r_{i_1} \dots r_{i_k} f \left( \sum_{j=1}^k q_j x_{i_j} \right) - f \left( \sum_{i=1}^n r_i x_i \right) \right) \\ & \geq m f \left( \left| \sum_{i=1}^n (r_i - p_i) x_i \right| \right) \\ & + \sum_{i_1, \dots, i_k=1}^n (p_{i_1} p_{i_2} \dots p_{i_k} - m r_{i_1} \dots r_{i_k}) f \left( \left| \sum_{j=1}^k q_j x_{i_j} - \sum_{s=1}^n p_s x_s \right| \right) \end{aligned}$$

and

$$\begin{aligned}
 & \sum_{i_1, \dots, i_k=1}^n p_{i_1} \dots p_{i_k} f \left( \sum_{j=1}^k q_j x_{i_j} \right) - f \left( \sum_{i=1}^n p_i x_i \right) \\
 & - M \left( \sum_{i_1, \dots, i_k=1}^n r_{i_1} \dots r_{i_k} f \left( \sum_{j=1}^k q_j x_{i_j} \right) - f \left( \sum_{i=1}^n r_i x_i \right) \right) \\
 & \leq -f \left( \left| \sum_{i=1}^n (r_i - p_i) x_i \right| \right) \\
 & - \sum_{i_1, \dots, i_k=1}^n (p_{i_1} p_{i_2} \dots p_{i_k} - M r_{i_1} \dots r_{i_k}) f \left( \left| \sum_{j=1}^k q_j x_{i_j} - \sum_{j=1}^n r_s x_s \right| \right)
 \end{aligned}$$

If  $f$  is also positive, then this inequality refines (2.15).

### 3. OTHER INEQUALITIES

The definition of superquadratic functions and their properties draw our attention to the possibility of using the *Čebyšev functional* and its properties to get new type of reverse Jensen Inequality.

For a function  $C : [0, \infty) \rightarrow \mathbb{R}$  we consider the Čebyšev type functional

$$T(C, \mathbf{x}, \mathbf{p}) := \sum_{i=1}^n p_i x_i C(x_i) - \sum_{i=1}^n p_i x_i \sum_{i=1}^n p_i C(x_i).$$

It is well known that, if  $C$  is monotonic nondecreasing function on  $[0, \infty)$  then the sequences  $\mathbf{x}$  and  $C(\mathbf{x}) := (C(x_1), \dots, C(x_n))$  are synchronous and for any probability sequence  $\mathbf{p}$  we have the *Čebyšev inequality*

$$T(C, \mathbf{x}, \mathbf{p}) \geq 0.$$

If certain bounds for the values of the function  $C(x_i)$  are known, namely

$$(3.1) \quad -\infty < m \leq C(x_i) \leq M < \infty \quad \text{for any } i \in \{1, \dots, n\}$$

then the following inequality due to Cerone & Dragomir [4] holds:

$$(3.2) \quad |T(C, \mathbf{x}, \mathbf{p})| \leq \frac{1}{2} (M - m) \sum_{i=1}^n p_i \left| x_i - \sum_{j=1}^n p_j x_j \right|.$$

The constant  $\frac{1}{2}$  is best possible in the sense that it cannot be replaced by a smaller quantity.

We can state now the following reverse of the Jensen inequality for superquadratic functions:

**Theorem 9.** *Assume that  $\mathbf{x} = (x_1, \dots, x_n)$  with  $x_i \geq 0$  for  $i \in \{1, \dots, n\}$ , and  $\mathbf{p} = (p_1, \dots, p_n)$  is a probability sequence with  $n \geq 2$ . Then for any superquadratic function  $f : [0, \infty) \rightarrow \mathbb{R}$  with  $C(x_i)$  satisfying (2.16), where  $C(x)$  is as in Definition*

1 we have the inequality,

$$(3.3) \quad \frac{1}{2} (M - m) \sum_{i=1}^n p_j \left| x_j - \sum_{i=1}^n p_i x_i \right| - \sum_{j=1}^n p_j f \left( \left| \sum_{i=1}^n p_i x_i - x_j \right| \right) \\ \geq \sum_{j=1}^n p_j f(x_j) - f \left( \sum_{i=1}^n p_i x_i \right) \left( \geq \sum_{j=1}^n p_j f \left( \left| \sum_{i=1}^n p_i x_i - x_j \right| \right) \right).$$

*Proof.* Utilising the definition of the superquadratic functions we have

$$(3.4) \quad f \left( \sum_{i=1}^n p_i x_i \right) \geq f(x_j) + C(x_j) \left( \sum_{i=1}^n p_i x_i - x_j \right) + f \left( \left| \sum_{i=1}^n p_i x_i - x_j \right| \right)$$

for any  $j \in \{1, \dots, n\}$ .

If we multiply (3.4) by  $p_j \geq 0, j \in \{1, \dots, n\}$ , sum over  $j$  from 1 to  $n$  and take into account that  $\sum_{j=1}^n p_j = 1$  we get

$$(3.5) \quad f \left( \sum_{i=1}^n p_i x_i \right) \\ \geq \sum_{j=1}^n p_j f(x_j) + \sum_{j=1}^n p_j C(x_j) \left( \sum_{i=1}^n p_i x_i - x_j \right) + \sum_{j=1}^n p_j f \left( \left| \sum_{i=1}^n p_i x_i - x_j \right| \right).$$

Since

$$\sum_{j=1}^n p_j C(x_j) \left( \sum_{i=1}^n p_i x_i - x_j \right) = -T(C, \mathbf{x}, \mathbf{p})$$

hence by (3.2) and (3.5) we deduce the desired result (3.3). ■

**Remark 3.** We observe that, as a "by-product" from (3.3) we get the following inequality

$$\frac{1}{2} T(C, \mathbf{x}, \mathbf{p}) \geq \sum_{j=1}^n p_j f \left( \left| \sum_{i=1}^n p_i x_i - x_j \right| \right)$$

while from (3.3) we get

$$\frac{1}{2} (M - m) \sum_{i=1}^n p_j \left| x_j - \sum_{i=1}^n p_i x_i \right| \geq \sum_{j=1}^n p_j f \left( \left| \sum_{i=1}^n p_i x_i - x_j \right| \right).$$

## REFERENCES

- [1] S. Abramovich, S. Banić, M. Matic, and J. Pečarić, Jensen- Steffensen's and related inequalities for superquadratic functions, *Math. Ineq. Appl.* **10** (2007), pp. .
- [2] S. Abramovich, G. Jameson and G. Sinnamon, Refining Jensen's inequality, *Bull.Math. Soc. Math. Roum.* **47**, (2004), pp. 3-14.
- [3] S. Abramovich, G. Jameson and G. Sinnamon, Inequalities for averages of convex and superquadratic functions, *J. Inequal. Pure & Appl. Math.* **5** (4), (2004), Article 91.
- [4] P. Cerone and S.S. Dragomir, A Refinement of the Grüss inequality and applications, *Tamkang J. Math.* **38**(2007), No. 1, pp. 37-49, Preprint *RGMA Res. Rep. Coll.*, **5**(2002), No. 2, Article 14, [ONLINE <http://rgmia.vu.edu.au/v5n2.html>].
- [5] S. S. Dragomir, Bounds for the normalised Jensen functional, *Bull. Austral. Math. Soc.* **74** (2006), pp. 471-478.

- [6] J. Pečarić, F. Proschan, Y. L. Tong, *Convex Functions, Partial Orderings, and Statistical Applications*, Academic Press, Inc. (1992).

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HAIFA, HAIFA 31905, ISRAEL

*E-mail address:* `abramos@math.haifa.ac.il`

SCHOOL OF COMPUTER SCIENCE & MATHEMATICS, VICTORIA UNIVERSITY, PO Box 14428,,  
MELBOURNE CITY, MC 8001, AUSTRALIA.

*E-mail address:* `Sever.Dragomir@vu.edu.au`