APPROXIMATION OF THE RIEMANN-STIELTJES INTEGRAL WITH (l, L)-LIPSCHITZIAN INTEGRATORS

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ABSTRACT. Sharp error estimates in approximating the Riemann-Stieltjes integral with (l, L)-Lipschitzian integrators and applications for the Čebyšev functional are given. Some inequalities that complement the classical results due to Čebyšev, Grüss, Ostrowski and Lupaş are also provided.

1. INTRODUCTION

In order to accurately approximate the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ with the simpler quantity

$$[u(a) - u(b)] \cdot \frac{1}{b-a} \int_a^b f(t)\,dt,$$

S.S. Dragomir and I. Fedotov introduced in [9] the following *error functional*

$$D(f; u) := \int_a^b f(t)\,du(t) - [u(a) - u(b)] \cdot \frac{1}{b-a} \int_a^b f(t)\,dt$$

provided the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ and the Riemann integral $\int_a^b f(t)\,dt$ exist. In the same paper, the authors have shown that

$$|D(f; u)| \leq \frac{1}{2} \cdot L(M - m)(b - a), \tag{1.1}$$

provided that u is $L-$Lipschitzian, i.e., $|u(t) - u(s)| \leq L(t - x)$ for any $t, s \in [a, b]$ and f is Riemann integrable and bounded below by m and above by M. The constant $\frac{1}{2}$ is best possible in the sense that it cannot be replaced by a smaller quantity.

In the follow-up paper [10], the same authors established a different result, namely

$$|D(f; u)| \leq \frac{1}{2} K(b - a) \bigvee_a^b (u), \tag{1.2}$$

provided that u is of bounded variation and f is $K-$Lipschitzian with a constant $K > 0$. Here $\frac{1}{2}$ is also best possible.

In [7], by the use of the following representation

$$D(f; u) = \int_a^b \Phi_u(t)\,df(t), \tag{1.3}$$

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where

$$\Phi_u(t) := \frac{1}{b-a}\left[(t-a)u(b)+(b-t)u(a)\right]-u(t), \quad t\in[a,b], \tag{1.4}$$

the author has established the following inequality as well

$$|D(f;u)| \tag{1.5}$$
$$\leq \begin{cases} \sup_{t\in[a,b]} |\Phi_u(t)| \cdot \bigvee_a^b (f) & \text{if } u \text{ is continuous and } f \text{ is of bounded variation;} \\ \\ L\int_a^b |\Phi_u(t)|\,dt & \text{if } u \text{ is Riemann integrable and } f \text{ is } L\text{-Lipschitzian;} \\ \\ \int_a^b |\Phi_u(t)|\,dt & \text{if } u \text{ is continuous and } f \text{ is monotonic nondecreasing.} \end{cases}$$

If u is monotonic nondecreasing, then

$$|D(f;u)| \leq \frac{1}{2}L(b-a)\left[u(b)-u(a)-K(u)\right] \leq \frac{1}{2}L(b-a)\left[u(b)-u(a)\right], \tag{1.6}$$

where

$$K(u) := \frac{4}{(b-a)^2}\int_a^b \left(t-\frac{a+b}{2}\right)u(t)\,dt\,(\geq 0),$$

and f is $L-$Lipschitzian, and

$$|D(f;u)| \leq \left[u(b)-u(a)-Q(u)\right]\cdot\bigvee_a^b(f) \leq \left[u(b)-u(a)\right]\cdot\bigvee_a^b(f), \tag{1.7}$$

where

$$Q(u) := \frac{1}{b-a}\int_a^b u(t)\operatorname{sgn}\left(t-\frac{a+b}{2}\right)dt\,(\geq 0),$$

and f is of bounded variation, The constant $\frac{1}{2}$ in (1.6) and the first inequality in (1.7) are sharp.

The main aim of the present paper is to provide other bounds for $D(f;u)$ in the case where the integrator u is $(l,L)-$Lipschitzian (see Definition 1). Natural applications for the Čebyšev functional that complement the classical results due to Čebyšev, Grüss, Ostrowski and Lupaş are also given.

2. (l,L)-Lipschitzian Functions

We say that a function $v:[a,b]\to\mathbb{R}$ is $K-$Lipschitzian with $K>0$ if $|v(t)-v(s)|\leq K|t-s|$ for any $t,s\in[a,b]$. The following lemma may be stated:

Lemma 1. *Let $u:[a,b]\to\mathbb{R}$ and $l,L\in\mathbb{R}$ with $L>l$. The following statements are equivalent:*

(i) *The function $u-\frac{l+L}{2}\cdot e$, where $e(t)=t$, $t\in[a,b]$ is $\frac{1}{2}(L-l)-$Lipschitzian;*
(ii) *We have the inequalities*

$$l \leq \frac{u(t)-u(s)}{t-s} \leq L \quad \text{for each } t,s\in[a,b] \quad \text{with } t\neq s; \tag{2.1}$$

(iii) *We have the inequalities*

$$l(t-s) \leq u(t)-u(s) \leq L(t-s) \quad \text{for each } t,s\in[a,b] \quad \text{with } t>s. \tag{2.2}$$

Following [13], we can introduce the definition of (l,L)-Lipschitzian functions:

Definition 1. *The function $u:[a,b]\to\mathbb{R}$ which satisfies one of the equivalent conditions (i) – (iii) from Lemma 1 is said to be (l,L)-Lipschitzian on $[a,b]$.*

If $L>0$ and $l=-L$, then $(-L,L)-$Lipschitzian means L-Lipschitzian in the classical sense.

Utilising *Lagrange's mean value theorem*, we can state the following result that provides examples of (l,L)-Lipschitzian functions.

Proposition 1. *Let $u:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). If $-\infty<l=\inf_{t\in[a,b]}u'(t)$ and $\sup_{t\in[a,b]}u'(t)=L<\infty$, then u is (l,L)-Lipschitzian on $[a,b]$.*

The following result holds.

Theorem 1. *If $u:[a,b]\to\mathbb{R}$ is (l,L)-Lipschitzian on $[a,b]$, then*

$$|\Phi_u(t)|\le\frac{(L-l)(b-t)(t-a)}{b-a}\le\frac{1}{4}(L-l)(b-a) \tag{2.3}$$

for each $t\in[a,b]$.

The inequalities are sharp and the constant $\frac{1}{4}$ is best possible.

Proof. First of all, let us observe that

$$\Phi_u(t)=\Phi_{u-\frac{l+L}{2}e}(t)\quad\text{for each}\quad t\in[a,b]. \tag{2.4}$$

Now, if $v:[a,b]\to\mathbb{R}$ is $K-$Lipschitzian, then by the definition of Φ_v we have

$$\begin{aligned}|\Phi_v(t)|&=\left|\frac{(t-b)(v(t)-v(a))+(t-a)(v(b)-v(t))}{b-a}\right| \\ &\le\frac{(b-t)|v(t)-v(a)|+(t-a)|v(b)-v(t)|}{b-a} \\ &\le\frac{2K(b-t)(t-a)}{b-a},\end{aligned} \tag{2.5}$$

for any $t\in[a,b]$.

Now, applying (2.5) for $v=u-\frac{l+L}{2}e$ which is $\frac{1}{2}(L-l)-$Lipschitzian, we deduce

$$\left|\Phi_{u-\frac{l+L}{2}e}(t)\right|\le\frac{(L-l)(b-t)(t-a)}{b-a},\quad t\in[a,b]$$

which together with (2.4) produces the first inequality in (2.3).

The second inequality in (2.3) is obvious.

For $t=\frac{a+b}{2}$, we deduce from (2.3) the following inequality:

$$\left|\frac{u(a)+u(b)}{2}-u\left(\frac{a+b}{2}\right)\right|\le\frac{1}{4}(L-l)(b-a). \tag{2.6}$$

We will show that $\frac{1}{4}$ is best possible in (2.6).

Consider the function $u:[a,b]\to\mathbb{R}$, $u(t)=\left|t-\frac{a+b}{2}\right|$. Then u is $(-1,1)-$Lipschitzian, $u(a)=u(b)=\frac{b-a}{2}$, $u\left(\frac{a+b}{2}\right)=0$ and introducing these values in (2.6) we obtain an equality with both terms $\frac{1}{2}(b-a)$. ∎

Corollary 1. *With the assumptions of Theorem 1, we have the inequality:*

$$\left|\frac{u(a)+u(b)}{2}-u\left(\frac{a+b}{2}\right)\right|\le\frac{1}{4}(L-l)(b-a). \tag{2.7}$$

The constant $\frac{1}{4}$ is best possible.

3. Sharp Bounds for (l, L)-Lipschitzian Integrators

The following result can be stated:

Theorem 2. *Let $f, u : [a,b] \to \mathbb{R}$ be such that u is (l,L)-Lipschitzian and f is of bounded variation, then*

$$|D(f;u)| \leq \frac{1}{4}(L-l)(b-a)\bigvee_a^b (f). \tag{3.1}$$

The constant $\frac{1}{4}$ is best possible in (3.1).

Proof. We use the following representation of the Grüss type functional $D(f;u)$ that has been obtained in [7] (see also [5]):

$$D(f;u) = \int_a^b \Phi_u(t)\, df(t). \tag{3.2}$$

It is well known that if $p : [\alpha,\beta] \to \mathbb{R}$ is continuous and $v : [\alpha,\beta] \to \mathbb{R}$ is of bounded variation, then the Riemann-Stieljtes integral $\int_\alpha^\beta p(t)\, dv(t)$ exists and $\left|\int_\alpha^\beta p(t)\, dv(t)\right| \leq \sup_{t\in[\alpha,\beta]} |p(t)| \bigvee_\alpha^\beta (v)$.

Applying this property we then have

$$|D(f;u)| = \left|\int_a^b \Phi_u(t)\, df(t)\right| \leq \sup_{t\in[a,b]} |\Phi_u(t)| \bigvee_a^b (f) \leq \frac{1}{4}(L-l)(b-a)\bigvee_a^b (f)$$

and the inequality (3.1) is obtained.

To prove the sharpness of the constant $\frac{1}{4}$, assume that there is a constant $A > 0$ so that

$$|D(f;u)| \leq A(L-l)(b-a)\bigvee_a^b (f), \tag{3.3}$$

where u and f are as in the statement of the theorem.

Consider $u, f : [a,b] \to \mathbb{R}$, $u(t) = \left|t - \frac{a+b}{2}\right|$ and $f(t) = \operatorname{sgn}\left(t - \frac{a+b}{2}\right)$. Then u is $(-1,1)$-Lipschitzian, f is of bounded variation with $\bigvee_a^b (f) = 2$ and

$$D(f;u) = \int_a^{\frac{a+b}{2}} (-1)\cdot d\left(\frac{a+b}{2} - t\right) + \int_{\frac{a+b}{2}}^b (+1)\cdot d\left(t - \frac{a+b}{2}\right) = b - a$$

and the inequality (3.3) becomes $b - a \leq 4A(b-a)$, which implies that $A \geq \frac{1}{4}$. ∎

The following result also holds:

Theorem 3. *Let $f, u : [a,b] \to \mathbb{R}$ be such that u is (l,L)-Lipschitzian and f is K-Lipschitzian on $[a,b]$, then*

$$|D(f;u)| \leq \frac{1}{6}K(L-l)(b-a)^2. \tag{3.4}$$

Proof. It is known that, if $p : [\alpha,\beta] \to \mathbb{R}$ is Riemann integrable and $v : [a,b] \to \mathbb{R}$ is L-Lipschitzian, then the Riemann-Stieltjes integral $\int_\alpha^\beta p(t)\, dv(t)$ exists and $\left|\int_\alpha^\beta p(t)\, dv(t)\right| \leq L\int_a^b |v(t)|\, dt$.

If we apply this property to the integral $\int_a^b \Phi_u(t)\, df(t)$ and use the identity (3.2), we then have

$$|D(f;u)| = \left| \int_a^b \Phi_u(t)\, df(t) \right| \le K \int_a^b |\Phi_u(t)|\, dt$$
$$\le \frac{K(L-l)}{b-a} \int_a^b (b-t)(t-a)\, dt = \frac{1}{6} K (L-l)(b-a)^2$$

and the inequality (3.4) is proved. ∎

Remark 1. *It is an open problem whether or not the constant* $\frac{1}{6}$ *is the best possible constant in (3.4).*

The following result can be stated as well:

Theorem 4. *Let* $f, u : [a,b] \to \mathbb{R}$ *be such that* u *is* $(l,L)-$*Lipschitzian and* f *is monotonic nondecreasing, then*

$$|D(f;u)| \le 2 \cdot \frac{L-l}{b-a} \int_a^b \left(t - \frac{a+b}{2} \right) f(t)\, dt \tag{3.5}$$
$$\le \begin{cases} \frac{1}{2}(L-l) \max\{|f(a)|, |f(b)|\}(b-a); \\ \frac{1}{(q+1)^{\frac{1}{q}}}(L-l)\|f\|_p (b-a)^{\frac{1}{q}} & \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ (L-l)\|f\|_1, \end{cases}$$

where $\|f\|_p := \left(\int_a^b |f(t)|^p\, dt \right)^{\frac{1}{p}}$, $p \ge 1$ *are the Lebesgue norms.*

The constants 2 *and* $\frac{1}{2}$ *are best possible in (3.5).*

Proof. It is well known that if $p : [\alpha,\beta] \to \mathbb{R}$ is continuous and $v : [\alpha,\beta] \to \mathbb{R}$ is monotonic nondecreasing, then the Riemann-Stieltjes integral $\int_\alpha^\beta p(t)\, dv(t)$ exists and $\left| \int_\alpha^\beta p(t)\, dv(t) \right| \le \int_\alpha^\beta |p(t)|\, dv(t)$. Then, on applying this property for the integral $\int_a^b \Phi_u(t)\, df(t)$, we have

$$|D(f;u)| = \left| \int_a^b \Phi_u(t)\, df(t) \right| \le \int_a^b |\Phi_u(t)|\, df(t) \tag{3.6}$$
$$\le \frac{L-l}{b-a} \int_a^b (b-t)(t-a)\, df(t),$$

where, for the last inequality, we have used the inequality (2.3).

Integrating by parts in the Riemann-Stieltjes integral, we have

$$\int_a^b (b-t)(t-a)\, df(t) = f(t)(b-t)(t-a)\Big|_a^b - \int_a^b [-2t+(a+b)] f(t)\, dt$$
$$= 2\int_a^b \left(t - \frac{a+b}{2} \right) f(t)\, dt,$$

which together with (3.6) produces the first inequality in (3.5).

The last part follows on utilising the Hölder inequality, namely

$$\int_a^b \left(t - \frac{a+b}{2}\right) f(t)\, dt$$
$$\leq \begin{cases} \sup_{t\in[a,b]} |f(t)| \int_a^b \left|t - \frac{a+b}{2}\right| dt \\ \\ \left(\int_a^b |f(t)|^p dt\right)^{\frac{1}{p}} \left(\int_a^b \left|t - \frac{a+b}{2}\right|^q dt\right)^{\frac{1}{q}} & \text{if } p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \\ \sup_{t\in[a,b]} \left|t - \frac{a+b}{2}\right| \int_a^b |f(t)|\, dt \end{cases}$$
$$\leq \begin{cases} \frac{1}{4} \max\{|f(a)|, |f(b)|\} (b-a)^2; \\ \\ \frac{1}{2} \cdot \frac{1}{(q+1)^{\frac{1}{q}}} \|f\|_p (b-a)^{1+\frac{1}{q}} & \text{if } p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \\ \frac{1}{2} \|f\|_1 (b-a). \end{cases}$$

For the best possible constant, assume that there exists a $B > 0$ such that

$$|D(f;u)| \leq B \cdot \frac{L-l}{b-a} \int_a^b \left(t - \frac{a+b}{2}\right) f(t)\, dt. \tag{3.7}$$

Consider $u(t) := \left|t - \frac{a+b}{2}\right|$ and $f(t) = \operatorname{sgn}\left(t - \frac{a+b}{2}\right)$. Then u is $(-1,1)$ −Lipschitzian, f is monotonic nondecreasing, $D(f;u) = b - a$, $\int_a^b \left(t - \frac{a+b}{2}\right) f(t)\, dt = \frac{(b-a)^2}{4}$ and by (3.7) we get $b - a \leq \frac{B(b-a)}{2}$, which implies that $B \geq 2$.

The fact that $\frac{1}{2}$ is also the best constant follows likewise and we omit the details. ∎

4. Applications for the Čebyšev Functional

For two Lebesgue integrable functions, $f, g : [a,b] \to \mathbb{R}$ with fg an integrable function, consider the *Čebyšev functional* $C(\cdot,\cdot)$ defined by

$$C(f,g) := \frac{1}{b-a} \int_a^b f(t) g(t)\, dt - \frac{1}{b-a} \int_a^b f(t)\, dt \cdot \frac{1}{b-a} \int_a^b g(t)\, dt. \tag{4.1}$$

In 1934, Grüss [12] showed that

$$|C(f,g)| \leq \frac{1}{4} (M-m)(N-n), \tag{4.2}$$

provided m, M, n, N are real numbers with the property

$$-\infty < m \leq f \leq M < \infty, \quad -\infty < n \leq g \leq N < \infty \quad \text{a.e. on} \quad [a,b]. \tag{4.3}$$

The constant $\frac{1}{4}$ is best possible in (4.1) in the sense that it cannot be replaced by a smaller quantity.

Another lesser known inequality, even though it was derived in 1882 by Čebyšev [1], under the assumption that f', g' exist and are continuous in $[a,b]$ and is given by

$$|C(f,g)| \leq \frac{1}{12} \|f'\|_\infty \|g'\|_\infty (b-a)^2, \tag{4.4}$$

where $\|f'\|_\infty := \sup_{t\in[a,b]} |f'(t)|$. The constant $\frac{1}{12}$ cannot be improved in the general case. We notice that the Čebyšev inequality (4.4) also holds if $f, g : [a,b] \to \mathbb{R}$ are absolutely continuous on $[a,b]$ and $f', g' \in L_\infty[a,b]$.

In 1970, Ostrowski [15] proved, amongst others, the following result that is in a sense a combination of the Čebyšev and Grüss results, namely

$$|C(f,g)| \leq \frac{1}{8}(b-a)(M-m)\|g'\|_\infty, \tag{4.5}$$

provided that f satisfies (4.3) while g is absolutely continuous and $f', g' \in L_\infty[a,b]$. The constant $\frac{1}{8}$ is best possible in (4.5).

Finally, let us recall that in 1973, Lupaş [14], proved the following inequality in terms of the Euclidean norm:

$$|C(f,g)| \leq \frac{1}{\pi^2}(b-a)\|f'\|_2\|g'\|_2, \tag{4.6}$$

provided that f, g are absolutely continuous and $f', g' \in L_2[a,b]$. The constant $\frac{1}{\pi^2}$ is best possible.

In the recent paper [2], Cerone and Dragomir proved amongst others the following result:

$$|C(f,g)| \leq \inf_{\gamma\in\mathbb{R}}\|g-\gamma\|_\infty \cdot \frac{1}{b-a}\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| dt, \tag{4.7}$$

provided $f \in L[a,b]$ and $g \in C[a,b]$.

As particular cases of (4.7), one can deduce the results

$$|C(f,g)| \leq \|g\|_\infty \frac{1}{b-a}\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| dt, \tag{4.8}$$

where $g \in C[a,b]$ and $f \in L[a,b]$, and

$$|C(f,g)| \leq \frac{1}{2}(M-m)\frac{1}{b-a}\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| dt, \tag{4.9}$$

where $m \leq g(x) \leq M$, $x \in [a,b]$.

The multiplicative constants 1 in (4.8) and $\frac{1}{2}$ in (4.9) are best possible. The inequality (4.9) has been obtained before in a different way in [4].

For generalisations in abstract Lebesgue spaces, best constants and discrete versions, see [3].

For other results on the Čebyšev functional, see [6], [8] and [11].

Now, assume that $g : [a,b] \to \mathbb{R}$ is Lebesgue integrable on $[a,b]$ and $-\infty < m \leq g(t) \leq M < \infty$ for a.e. $t \in [a,b]$. Then the function $u(t) := \int_a^t g(s)\,ds$ is (m,M)-Lipschitzian on $[a,b]$ and

$$\tilde{\Phi}_g(t) := \Phi_u(t) = \int_a^t g(s)\,ds - \frac{t-a}{b-a}\int_a^b g(s)\,ds, \quad t \in [a,b]. \tag{4.10}$$

On utilising the Theorem 1 we can state the following result that provides a sharp bound for $\tilde{\Phi}_g(t)$ in (4.10).

Proposition 2. *If $g : [a,b] \to \mathbb{R}$ is Lebesgue integrable on $[a,b]$ and $-\infty < m \leq g(s) \leq M < \infty$ for a.e. $s \in [a,b]$, then*

$$\left|\tilde{\Phi}_g(t)\right| \leq \frac{(M-m)(b-t)(t-a)}{b-a} \leq \frac{1}{4}(M-m)(b-a), \tag{4.11}$$

for a.e. $t \in [a,b]$. *The first inequality is sharp. The constant* $\frac{1}{4}$ *is best possible.*

The inequality is obvious by (3.1). The sharpness follows on choosing $t = \frac{a+b}{2}$ and $g(t) = \operatorname{sgn}\left(t - \frac{a+b}{2}\right)$ in (4.11). The details are omitted.

The following result for the Čebyšev functional can be stated:

Proposition 3. *If* $f : [a,b] \to \mathbb{R}$ *is of bounded variation on* $[a,b]$ *and* $g : [a,b] \to \mathbb{R}$ *is Lebesgue integrable and satisfies the bounds*

$$-\infty < m \le g \le M < \infty \quad \text{a.e. on } [a,b], \tag{4.12}$$

then

$$|C(f,g)| \le \frac{1}{4}(M-m)\bigvee_a^b (f). \tag{4.13}$$

The constant $\frac{1}{4}$ *is best possible.*

Proof. We observe that, for $u(t) = \int_a^b g(s)\,ds$,

$$D(f;u) = (b-a)\,C(f,g), \tag{4.14}$$

which by (3.1) produces the desired inequality (4.13).

Now, assume that (4.13) holds with a constant $D > 0$, i.e.,

$$|C(f,g)| \le D(M-m)\bigvee_a^b (f). \tag{4.15}$$

If we choose $f(t) = g(t) = \operatorname{sgn}\left(t - \frac{a+b}{2}\right)$, then $M = 1$, $m = -1$, $\bigvee_a^b (f) = 2$, $C(f,g) = 1$ and by (4.15) we get $1 \le 4D$ which implies $D \ge \frac{1}{4}$. ∎

The following result can be stated as well.

Proposition 4. *Assume that* $g : [a,b] \to \mathbb{R}$ *is as in Proposition 3. If* $f : [a,b] \to \mathbb{R}$ *is monotonic nondecreasing on* $[a,b]$, *then*

$$|C(f,g)| \le 2 \cdot \frac{(M-m)}{b-a}\int_a^b \left(t - \frac{a+b}{2}\right) f(t)\,dt \tag{4.16}$$

$$\le \begin{cases} \frac{1}{2}(M-m)\max\{|f(a)|,|f(b)|\}; \\ \\ \frac{1}{(q+1)^{\frac{1}{q}}}(M-m)\|f\|_p (b-a)^{-\frac{1}{p}} & \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ (M-m)\frac{1}{b-a}\|f\|_1. \end{cases}$$

The constants 2 *and* $\frac{1}{2}$ *are best possible.*

The proof of the inequalities in (4.16) are obvious from (3.5). The sharpness of the constants follows on choosing $f(t) = g(t) = \operatorname{sgn}\left(t - \frac{a+b}{2}\right)$, $t \in [a,b]$.

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