

MAJORIZED PROOF AND EXTENSIONS OF BERNOULLI'S INEQUALITY

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ABSTRACT. By using methods on the theory of majorization, three known extensions of Bernoulli's inequality are proved and a new extension of this inequality is established by the Schur-concavity of the elementary symmetric function and simple majorizations.

1. INTRODUCTION

Let $x > -1$ and n is a positive integer. Then

$$(1.1) \quad (1+x)^n \geq 1+nx.$$

(1.1) is known as the Bernoulli's inequality which play an important role in analysis and its applications. So, during the past few years, many researchers obtained various generalizations, extensions of inequality (1.1). For example, the following extensions and variants of (1.1) were recorded in [2, pp.127-128]:

Theorem A. *Let $x > -1$. If $\alpha > 1$ or $\alpha < 0$, then*

$$(1.2) \quad (1+x)^\alpha \geq 1+\alpha x,$$

if $0 < \alpha < 1$, then

$$(1.3) \quad (1+x)^\alpha \leq 1+\alpha x.$$

In (1.2) and (1.3), equalities holding if and only if $x = 0$.

Theorem B. *Let $a_i \geq 0$, $x_i > -1$, $i = 1, \dots, n$, and $\sum_{i=1}^n a_i \leq 1$. Then*

$$(1.4) \quad \prod_{i=1}^n (1+x_i)^{a_i} \leq 1 + \sum_{i=1}^n a_i x_i,$$

if $a_i \geq 1$ and $x_i > 0$, or $a_i \leq 0$ and $x_i < 0$, $i = 1, \dots, n$, then

$$(1.5) \quad \prod_{i=1}^n (1+x_i)^{a_i} \geq 1 + \sum_{i=1}^n a_i x_i.$$

Theorem C. *Let $x > 0, x \neq 1$. If $0 < \alpha < 1$, then*

$$(1.6) \quad \alpha x^{\alpha-1}(x-1) < x^\alpha - 1 < \alpha(x-1),$$

if $\alpha > 1$ or $\alpha < 0$, then the inequalities in (1.6) are all reversed.

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Throughout the paper we assume that the set of n -dimensional row vector on real number field by \mathbb{R}^n .

$$\begin{aligned}\mathbb{R}_+^n &= \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_i \geq 0, i = 1, \dots, n\}, \\ \mathbb{R}_{++}^n &= \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_i > 0, i = 1, \dots, n\}.\end{aligned}$$

In particular, \mathbb{R}^1 , \mathbb{R}_+^1 and \mathbb{R}_{++}^1 denoted by \mathbb{R} , \mathbb{R}_+ and \mathbb{R}_{++} respectively.

Let $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$. Its elementary symmetric functions are

$$E_k(\mathbf{x}) = E_k(x_1, \dots, x_n) = \sum_{1 \leq i_1 < \dots < i_k \leq n} \prod_{j=1}^k x_{i_j}, \quad k = 1, \dots, n.$$

In particular, $E_n(\mathbf{x}) = \prod_{i=1}^n x_i$, $E_1(\mathbf{x}) = \sum_{i=1}^n x_i$, and defined $E_0(\mathbf{x}) = 1$ and $E_k(\mathbf{x}) = 0$ for $k < 0$ or $k > n$.

Seven proofs for Theorem A are recorded in [1-3], in this paper, by using methods on the theory of majorization, three known extensions of Bernoulli's inequality are proved and a new extension of this inequality is established by the Schur-concatity of the elementary symmetric function and simple majoricotions.

We need the following definitions and lemmas.

Definition 1.1 ([4]). Let $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$.

- (i) \mathbf{x} is said to be majorized by \mathbf{y} (in symbols $\mathbf{x} \prec \mathbf{y}$) if $\sum_{i=1}^k x_{[i]} \leq \sum_{i=1}^k y_{[i]}$ for $k = 1, 2, \dots, n-1$ and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$, where $x_{[1]} \geq \dots \geq x_{[n]}$ and $y_{[1]} \geq \dots \geq y_{[n]}$ are rearrangements of \mathbf{x} and \mathbf{y} in a descending order, and \mathbf{x} is said to strictly majorized by \mathbf{y} (in symbols $\mathbf{x} \prec \prec \mathbf{y}$) if \mathbf{x} is not permutation of \mathbf{y} .
- (ii) $\mathbf{x} \geq \mathbf{y}$ means $x_i \geq y_i$ for all $i = 1, 2, \dots, n$. let $\Omega \subset \mathbb{R}^n$, $\varphi: \Omega \rightarrow \mathbb{R}$ is said to be increasing if $\mathbf{x} \geq \mathbf{y}$ implies $\varphi(\mathbf{x}) \geq \varphi(\mathbf{y})$. φ is said to be decreasing if and only if $-\varphi$ is increasing.
- (iii) $\Omega \subset \mathbb{R}^n$ is called a convex set if $(\alpha x_1 + \beta y_1, \dots, \alpha x_n + \beta y_n) \in \Omega$ for any \mathbf{x} and $\mathbf{y} \in \Omega$, where α and $\beta \in [0, 1]$ with $\alpha + \beta = 1$.
- (iv) let $\Omega \subset \mathbb{R}^n$, $\varphi: \Omega \rightarrow \mathbb{R}$ is said to be a Schur-convex function on Ω if $\mathbf{x} \prec \mathbf{y}$ on Ω implies $\varphi(\mathbf{x}) \leq \varphi(\mathbf{y})$. φ is said to be a Schur-concave function on Ω if and only if $-\varphi$ is Schur-convex function. φ is said to be a strictly Schur-convex function on Ω if $\mathbf{x} \prec \prec \mathbf{y}$ on Ω implies $\varphi(\mathbf{x}) < \varphi(\mathbf{y})$, φ is said to be a strictly Schur-concave function on Ω if and only $-\varphi$ is strictly Schur-convex on Ω .

Lemma 1.1. [4, p. 5] Let $\mathbf{x} \in \mathbb{R}^n$ and $\bar{\mathbf{x}} = \frac{1}{n} \sum_{i=1}^n x_i$. Then $(\bar{\mathbf{x}}, \dots, \bar{\mathbf{x}}) \prec \mathbf{x}$.

Lemma 1.2. [5] Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $x_1 \geq x_2 \geq \dots \geq x_n$ and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$. If for some $k, 1 \leq k \leq n$, $x_i \leq y_i, i = 1, \dots, k$, $x_i \geq y_i, i = k+1, \dots, n$, then $\mathbf{x} \prec \mathbf{y}$.

Lemma 1.3 ([4, pp. 48–49]). Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $I \subset \mathbb{R}$ be an interval and $g: I \rightarrow \mathbb{R}$.

- (1) $\mathbf{x} \prec \mathbf{y}$ if and only if $\sum_{i=1}^n g(x_i) \leq \sum_{i=1}^n g(y_i)$ for all convex functions g ,
- (2) $\mathbf{x} \prec \mathbf{y}$ if and only if $\sum_{i=1}^n g(x_i) \geq \sum_{i=1}^n g(y_i)$ for all concave functions g ,
- (3) $\mathbf{x} \prec \prec \mathbf{y}$ if and only if $\sum_{i=1}^n g(x_i) < \sum_{i=1}^n g(y_i)$ for all strictly convex functions g ,
- (4) $\mathbf{x} \prec \prec \mathbf{y}$ if and only if $\sum_{i=1}^n g(x_i) > \sum_{i=1}^n g(y_i)$ for all strictly concave functions g .

2. MAIN RESULTS AND THEIR PROOFS

Theorem I. Let m, n is a positive integer,

- (i) if $m \geq n$ and $x > -1$, then

$$(2.1) \quad C_m^k \left(1 + \frac{n}{m}x\right)^k \geq \sum_{i=0}^k C_n^i C_{m-n}^{k-i} (1+x)^i, \quad \text{for } k = 1, \dots, n$$

(ii) if $m < n$ and $x > -\frac{m}{n}$, then

$$(2.2) \quad C_m^k (1+x)^k \geq \sum_{i=0}^k C_m^i C_{n-m}^{k-i} \left(1 + \frac{m}{n}x\right)^i, \quad \text{for } k = 1, \dots, n$$

where $C_n^k = \frac{n!}{k!(n-k)!}$ is the number of combinations of n elements taken k at a time, defined $C_n^0 = 1$ and $C_n^k = 0$ for $k > n$. In (2.1) and (2.2), equalities holding if and only if $x = 0$.

Proof. From Lemma 1, we have

$$\mathbf{p} := \left(\underbrace{1 + \frac{n}{m}x, \dots, 1 + \frac{n}{m}x}_m \right) \prec \left(\underbrace{1+x, \dots, 1+x}_n, \underbrace{1, \dots, 1}_{m-n} \right) := \mathbf{q}$$

and $\mathbf{p} \prec \prec \mathbf{q}$ for $x \neq 0$. If $m \geq n$, from $x > -1$, we have $x+1 > 0$ and $1 + \frac{n}{m}x > 1 - \frac{n}{m} > 0$, i.e. $\mathbf{p}, \mathbf{q} \in \mathbb{R}_{++}^n$. Since $E_k(\mathbf{x})$ be increasing and Schur-concave on \mathbb{R}_+^n and it be increasing and strictly Schur-concave on \mathbb{R}_{++}^n for $k > 1$ (see Proposition 6.7 in [4]), we have $E_k(\mathbf{p}) \geq E_k(\mathbf{q})$, i.e. (2.1) holds, and equality holding if and only if $x = 0$.

The proof of Theorem 1 is completed. \square

Majorized Proof of Theorem A.

Proof. Now we consider three cases:

Case 1. $0 < \alpha < 1$. Notice that $C_n^k = 0$ for $k > n$, taking $k = m$ in (2.1), we get

$$\left(1 + \frac{n}{m}x\right)^m \geq (1+x)^n,$$

i.e.

$$(2.3) \quad (1+x)^{n/m} \leq 1 + \frac{n}{m}x$$

and equality holding if and only if $x = 0$. Taking $m = 1$, (2.3) is deduces to (1.1), so (2.1) is an extension of (1.1). And (2.3) is show that (1.3) is holds for the rational number α with $0 < \alpha < 1$. If α is a irrational number, then there exists sequence of rational numbers $\{r_k\}$ with $0 < \alpha < 1$, $k = 1, 2, \dots$, such that $r_k \rightarrow \alpha, (k \rightarrow \infty)$. From (2.3), we have $(1+x)^{r_k} \leq 1 + r_k x$, and let $k \rightarrow \infty$, it deduce to (1.3).

Case 2. $\alpha > 1$. If $\alpha x \leq -1$, i.e. $\alpha x + 1 \leq 0$, then it is clear that (1.2) is holds, if $\alpha x > -1$, since $0 < \frac{1}{\alpha} < 1$, from (1.3), we have $(1+\alpha x)^{1/\alpha} \leq 1 + \frac{1}{\alpha}\alpha x$, i.e. (1.2) is holds.

Case 3. $\alpha < 0$. Since $1 - \alpha > 1$ and $-\frac{x}{1+x} > -1$, from (1.2), we have

$$\left(\frac{1}{1+x}\right)^{1-\alpha} = \left(1 - \frac{x}{1+x}\right)^{1-\alpha} \geq 1 - \frac{(1-\alpha)x}{1+x},$$

i.e. $(1+x)^\alpha \geq 1 + \alpha x$. And then, the proof of Theorem A is completed, by Theorem I. \square

Theorem II. Let $a_i \geq 1$ and $x_i > 0$, or $a_i \leq 0$ and $0 \geq x_i \geq -1$. Then

$$(2.4) \quad C_n^k \left[\frac{1}{n} \sum_{i=1}^n (1+x_i)^{a_i} \right]^k \geq \sum_{1 \leq i_1 < \dots < i_k \leq n} \prod_{j=1}^k (1+x_{i_j})^{a_{i_j}} \\ \geq \sum_{1 \leq i_1 < \dots < i_k \leq n} \prod_{j=1}^k (1+a_{i_j} x_{i_j}) > C_{n-1}^k + C_{n-1}^{k-1} \left(1 + \sum_{i=1}^n a_i x_i \right)$$

Proof. Set $y = \frac{1}{n} \sum_{i=1}^n (1+x_i)^{a_i}$, from (1.2), we have $(1+x_i)^{a_i} \geq 1+a_ix_i$, $i = 1, \dots, n$, and by lemma 1, it follows that

$$\begin{aligned} \underbrace{(y, \dots, y)}_n &< ((1+x_1)^{a_1}, \dots, (1+x_n)^{a_n}) \\ &\geq (1+a_1x_1, \dots, 1+a_nx_n) << \left(1 + \sum_{i=1}^n a_ix_i, \underbrace{1, \dots, 1}_{n-1} \right). \end{aligned}$$

And then since $E_k(\mathbf{x})$ be increasing and Schur-concave on \mathbb{R}_+^n and it be increasing and strictly Schur-concave on \mathbb{R}_{++}^n for $k > 1$, we have

$$\begin{aligned} E_k(\underbrace{y, \dots, y}_n) &\geq E_k((1+x_1)^{a_1}, \dots, (1+x_n)^{a_n}) \\ &\geq E_k(1+a_1x_1, \dots, 1+a_nx_n) > E_k\left(1 + \sum_{i=1}^n a_ix_i, \underbrace{1, \dots, 1}_{n-1}\right), \end{aligned}$$

i.e. (2.4) is holds. The proof of Theorem 2 is completed. \square

Remark 2.1. When $k = n$, (2.4) is deduce to (1.5).

Majorized Proof of Theorem C.

Proof. Since the right inequality in (1.6) equivalent to (1.3), it is sufficient prove that the left inequality in (1.6) holds. Now we consider three cases:

Case 1. $\alpha > 1$. Let $m < n$, now proof

$$(2.5) \quad \frac{n}{m} x^{\frac{n}{m}-1} (x-1) > x^{\frac{n}{m}} - 1.$$

For $x > 1$, let

$$\mathbf{u} := \left(\underbrace{\frac{n}{m} \ln x, \dots, \frac{n}{m} \ln x}_m, \underbrace{\left(\frac{n}{m}-1\right) \ln x, \dots, \left(\frac{n}{m}-1\right) \ln x}_n \right)$$

and

$$\mathbf{v} := \left(\underbrace{\frac{n}{m} \ln x, \dots, \frac{n}{m} \ln x}_n, \underbrace{0, \dots, 0}_m \right).$$

Since $m < n$, we have $0 < \left(\frac{n}{m}-1\right) \ln x < \frac{n}{m} \ln x$, this is $u_i \leq v_i$, $i = 1, \dots, n$, and $u_i \geq v_i$, $i = n+1, \dots, m+n$, and $\sum_{i=1}^{m+n} u_i = \sum_{i=1}^{m+n} v_i = \frac{n^2}{m} \ln x$, from Lemma 2, it follows $u < v$ and $u << v$ for $x \neq 1$, and then by the strictly convexity of e^x on \mathbb{R} , from Lemma 3, we have $\sum_{i=1}^{m+n} e^{u_i} < \sum_{i=1}^{m+n} e^{v_i}$, i.e.

$$mx^{\frac{n}{m}} + nx^{\frac{n}{m}-1} < nx^{\frac{n}{m}} + m,$$

it equivalent to (2.5).

For $0 < x < 1$, let

$$\mathbf{u}' := \left(\underbrace{\left(\frac{n}{m}-1\right) \ln x, \dots, \left(\frac{n}{m}-1\right) \ln x}_n, \underbrace{\frac{n}{m} \ln x, \dots, \frac{n}{m} \ln x}_m \right)$$

and

$$v' := \left(\underbrace{0, \dots, 0}_m, \underbrace{\frac{n}{m} \ln x, \dots, \frac{n}{m} \ln x}_n \right).$$

Since $m < n$, we have $0 > \left(\frac{n}{m} - 1\right) \ln x > \frac{n}{m} \ln x$, this is $u'_i \leq v'_i, i = 1, \dots, m$, and $u'_i \geq v'_i, i = m + 1, \dots, m + n$, and $\sum_{i=1}^{m+n} u'_i = \sum_{i=1}^{m+n} v'_i = \frac{n^2}{m} \ln x$, from Lemma 2, it follows $u' \prec v'$ and $u' \prec\prec v'$ for $x \neq 1$, and then by the strictly convexity of e^x on \mathbb{R} , from Lemma 3, we have $\sum_{i=1}^{m+n} e^{u'_i} < \sum_{i=1}^{m+n} e^{v'_i}$, i.e.

$$mx^{\frac{n}{m}} + nx^{\frac{n}{m}-1} < nx^{\frac{n}{m}} + m,$$

it equivalent to (2.5).

To sum up, for $x > 0, x \neq 1, m < n$, (2.5) is holds, it is show that the inequality $\alpha x^{\alpha-1}(x-1) > x^\alpha - 1$ is holds for the rational number α with $\alpha < 1$. And then, By the rational numbers approximation, we can know that the inequality $\alpha x^{\alpha-1}(x-1) > x^\alpha - 1$ is holds for all real number α with $\alpha > 1$.

Case 2. $\alpha < 0$. Here $1 - \alpha > 0$, from $x > 0, x \neq 1$, we have $x^{-1} > 0, x^{-1} \neq 1$, by Case 1, it follows

$$(1 - \alpha) (x^{-1})^{(1-\alpha)-1} (x^{-1} - 1) > (x^{-1})^{1-\alpha} - 1,$$

i.e. $\alpha x^{\alpha-1}(x-1) > x^\alpha - 1$.

Case 3. $0 < \alpha < 1$. Here $\alpha^{-1} > 1$, from $x > 0, x \neq 1$, we have $x^\alpha > 0, x^\alpha \neq 1$, by Case 1, it follows

$$\alpha^{-1} (x^\alpha)^{(\alpha^{-1}-1)} \left((x^\alpha)^{-1} - 1 \right) > (x^\alpha)^{\alpha^{-1}} - 1,$$

i.e. $\alpha x^{\alpha-1}(x-1) < x^\alpha - 1$. The proof of Theorem C is completed. □

REFERENCES

- [1] Xiang-dong Wang, Hua-ming Su and Fang-uan Wang, *Theory and method of inequalities*(Bu deng shi.li lun.fang fa), Henan education Press China, 1994.(Chinese)
- [2] Ji-chang Kuang, *Applied Inequalities* (Chang yong bu deng shi) 3nd ed, Shandong Press of science and technology, Jinan, China, 2004. (Chinese).
- [3] Alfred Witkowski, *A new of the monotonicity of power menas*, J. Ineq. Pure. Appl. Math., 5 (1) (2004), Article 6. [http //jipam.vu.edu.au/](http://jipam.vu.edu.au/)
- [4] Bo-ying. Wang, *Foundations of Majorization Inequalities* (Kong zhi bu deng shi ji chu), Beijing Normal Univ. Press, Beijing, China, 1990. (Chinese)
- [5] Song-gui Wang, Mi-xia Wu and Zong-zhen Jia, *Inequalities in Matrix Theory*(Ju zhen bu deng shi)2nd ed. Science Press, Beijing, China, 2006. (Chinese)
- [6] Mitrinovic D S, Pecaric J E and Flnk A M., *Classical and New Inequalitis in Analysis*, Kluwer Academic Publishers, 1993, 65-82
- [7] Jia-jin Wen and Zhao Luo, *Optimal Generalization of Bernoullis Inequality and Its Applications*, Journal of Chengdu University (Natural science) 20 (4) (2001),1-8
- [8] Jian-ya Ge, *Some extensions on Bernoulli's inequality*, Journal of Zhejiang University (Natural science) 26 (2) (2003), 119-122.
- [9] Chang-xing Xue, *Separations and extensions on Bernoulli's inequality*, Journal of Gansu Education College(Natural Scienc Edition), 1999, 13 (3): 5-7

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