A NEW REFINEMENT OF YOUNG'S INEQUALITY

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ABSTRACT. In this short note, the well-known Young's inequality is refined by a double inequality.

The original Young's inequality is as follows.

Theorem 1 ([7])**.** *Let $f(x)$ be a continuous and strictly increasing function on $[0, A]$ for $A > 0$. If $f(0) = 0$, $a \in [0, A]$ and $b \in [0, f(A)]$, then*

$$\int_0^a f(x)\,\mathrm{d}x + \int_0^b f^{-1}(x)\,\mathrm{d}x \geq ab, \tag{1}$$

where f^{-1} is the inverse function of f. Equality in (1) is valid if and only if $b = f(a)$.

The following theorem is a converse of Theorem 1 which was proved in [5].

Theorem 2 ([5])**.** *If the functions $f(x)$ and $g(x)$ for $x \geq 0$ are continuous and strictly increasing with $f(0) = g(0) = 0$, $g^{-1}(x) \geq f(x)$ and*

$$\int_0^a f(x)\,\mathrm{d}x + \int_0^b g(x)\,\mathrm{d}x \geq ab \tag{2}$$

for all $a > 0$ and $b > 0$, then $f = g^{-1}$.

The following reversed version of Young's inequality (1) was obtained in [6].

Theorem 3 ([6])**.** *Under the assumptions of Theorem 1, inequality*

$$\min\left\{1, \frac{b}{f(a)}\right\} \int_0^a f(t)\,\mathrm{d}t + \min\left\{1, \frac{a}{f^{-1}(b)}\right\} \int_0^b f^{-1}(t)\,\mathrm{d}t \leq ab, \tag{3}$$

holds. Equality in (3) is valid if and only if $b = f(a)$.

For more information on Young's inequality, please refer to [1, pp. 651–653], [2, pp. 48–50], [3, Chapter XIV, pp. 379–389] and the references therein.

In this short note, we would like to refine Young's inequality (1) by a double inequality below.

Theorem 4. *Let $f(x)$ be a continuous, differentiable and strictly increasing function on $[0, A]$ for $A > 0$. If $f(0) = 0$, $a \in [0, A]$, $b \in [0, f(A)]$ and $f'(x)$ is strictly monotonic on $[0, A]$, then*

$$\frac{m}{2}\big[a - f^{-1}(b)\big]^2 \leq \int_0^a f(x)\,\mathrm{d}x + \int_0^b f^{-1}(x)\,\mathrm{d}x - ab \leq \frac{M}{2}\big[a - f^{-1}(b)\big]^2, \tag{4}$$

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where $m = \min\{f'(a), f'(f^{-1}(b))\}$ *and* $M = \max\{f'(a), f'(f^{-1}(b))\}$. *Equalities in* (4) *are valid if and only if* $b = f(a)$.

Proof. Changing variable of integration by $x = f(y)$ and integrating by part of the second integral in (4) yields

$$
\begin{aligned}
\int_0^a f(x)\,\mathrm{d}x + \int_0^b f^{-1}(x)\,\mathrm{d}x &= \int_0^a f(x)\,\mathrm{d}x + \int_0^{f^{-1}(b)} y f'(y)\,\mathrm{d}y \\
&= \int_0^a f(x)\,\mathrm{d}x + b f^{-1}(b) - \int_0^{f^{-1}(b)} f(x)\,\mathrm{d}x \\
&= b f^{-1}(b) + \int_{f^{-1}(b)}^a f(x)\,\mathrm{d}x \\
&= ab + \int_{f^{-1}(b)}^a \left[f(x) - b\right]\,\mathrm{d}x.
\end{aligned}
\tag{5}
$$

From the fourth line in (5), it is easy to see that if $f^{-1}(b) = a$ then equalities in (4) hold.

If $f^{-1}(b) < a$, since $f(x)$ is strictly increasing, then $f(x) - b > 0$ for $x \in (f^{-1}(b), a)$. By mean value theorem, it is obtained that there exists $c = c(x)$ satisfying $f^{-1}(b) < c < x \le a$ such that $0 < f(x) - b = \left[x - f^{-1}(b)\right]f'(c)$. Further, by virtue of the monotonicity of $f'(x)$ on $[0, A]$, it is revealed that

$$
0 < m \triangleq \min\{f'(a), f'(f^{-1}(b))\} < f'(c) < \max\{f'(a), f'(f^{-1}(b))\} \triangleq M.
$$

Consequently,

$$
0 < m\left[x - f^{-1}(b)\right] < f(x) - b < M\left[x - f^{-1}(b)\right].
$$

As a result,

$$
m \int_{f^{-1}(b)}^a \left[x - f^{-1}(b)\right]\,\mathrm{d}x < \int_{f^{-1}(b)}^a \left[f(x) - b\right]\,\mathrm{d}x < M \int_{f^{-1}(b)}^a \left[x - f^{-1}(b)\right]\,\mathrm{d}x
$$

which is equivalent to

$$
\frac{m}{2}\left[a - f^{-1}(b)\right]^2 < \int_{f^{-1}(b)}^a \left[f(x) - b\right]\,\mathrm{d}x < \frac{M}{2}\left[a - f^{-1}(b)\right]^2.
\tag{6}
$$

If $f^{-1}(b) > a$, inequalities in (6) can be deduced by a similar argument as above. Substituting (6) into (5) leads to (4). The proof of Theorem 4 is complete. $\qquad\square$

Remark 1. Taking $f(x) = \sqrt[4]{x^4 + 1} - 1$, $a = 3$ and $b = 2$ in Theorem 4 and direct calculation gives

$$
\int_0^3 f(x)\,\mathrm{d}x = \int_0^3 \sqrt[4]{x^4 + 1}\,\mathrm{d}x - 3,
$$

$$
\int_0^2 f^{-1}(x)\,\mathrm{d}x = \int_0^2 \sqrt[4]{(x+1)^4 - 1}\,\mathrm{d}x = \int_1^3 \sqrt[4]{x^4 - 1}\,\mathrm{d}x,
$$

$$
f'(x) = \frac{x^3}{\sqrt[4]{(x^4+1)^3}}, \quad f'(3) = \frac{27}{\sqrt[4]{82^3}}, \quad f'(f^{-1}(2)) = f'(\sqrt[4]{80}) = \frac{8\sqrt[4]{5^3}}{27},
$$

$$
m = \frac{2\sqrt[4]{5}}{27}, \quad M = \frac{27}{\sqrt[4]{82^3}}
$$

and
$$9+\frac{\sqrt[4]{5}}{27}\left[3-2\sqrt[4]{5}\right]^2 < \int_0^3 \sqrt[4]{x^4+1}\,\mathrm{d}x + \int_1^3 \sqrt[4]{x^4-1}\,\mathrm{d}x < 9+\frac{27}{2\sqrt[4]{82^3}}\left[3-2\sqrt[4]{5}\right]^2$$

which can be computed numerically as
$$9.000004792\cdots < \int_0^3 \sqrt[4]{x^4+1}\,\mathrm{d}x + \int_1^3 \sqrt[4]{x^4-1}\,\mathrm{d}x < 9.000042871\cdots.$$

This refines the following double inequality
$$9 < \int_0^3 \sqrt[4]{x^4+1}\,\mathrm{d}x + \int_1^3 \sqrt[4]{x^4-1}\,\mathrm{d}x < 9.0001$$

in [4, Problem 3].

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