A DOUBLE INEQUALITY FOR DIVIDED DIFFERENCES AND SOME IDENTITIES OF PSI AND POLYGAMMA FUNCTIONS

FENG QI

ABSTRACT. In this short note, from the logarithmically completely monotonic property of the function $(x + c)^{b-a}\frac{\Gamma(x+a)}{\Gamma(x+b)}$, a double inequality for the divided differences and some identities of the psi and polygamma functions are presented.

1. INTRODUCTION

Recall [1, 8] that a positive function f is called logarithmically completely monotonic on an interval I if f has derivatives of all orders on I and its logarithm $\ln f$ satisfies $(-1)^k[\ln f(x)]^{(k)} \geq 0$ for all $k \in \mathbb{N}$ on I. For more detailed information, please refer to [1, 2, 3, 4, 7, 10, 11] and the related references therein.

It is well known that the classical Euler's gamma function $\Gamma(x)$ plays a central role in the theory of special functions and has much extensive applications in many branches, for example, statistics, physics, engineering, and other mathematical sciences. The logarithmic derivative of $\Gamma(x)$, denoted by $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$, is called psi or digamma function, and $\psi^{(i)}(x)$ for $i \in \mathbb{N}$ are known as the polygamma or multigamma functions.

For real numbers α and β with $\alpha \neq \beta$, $(\alpha, \beta) \neq (0, 1)$ and $(\alpha, \beta) \neq (1, 0)$ and for $t \in \mathbb{R}$, let

$$q_{\alpha,\beta}(t) = \begin{cases} \dfrac{e^{-\alpha t} - e^{-\beta t}}{1 - e^{-t}}, & t \neq 0, \\ \beta - \alpha, & t = 0. \end{cases} \tag{1}$$

From the necessary and sufficient conditions such that the function $q_{\alpha,\beta}(t)$ is monotonic, which were established in [5, 6], the following logarithmically complete monotonicity was obtained.

Lemma 1 ([9]). *Let a, b and c be real numbers and $\rho = \min\{a, b, c\}$. Then the function*

$$H_{a,b,c}(x) = (x + c)^{b-a}\frac{\Gamma(x + a)}{\Gamma(x + b)} \tag{2}$$

is logarithmically completely monotonic in $(-\rho, \infty)$ if and only if

$$(a, b, c) \in D_1(a, b, c) \triangleq \{(a, b, c) : (b - a)(1 - a - b + 2c) \geq 0\}$$
$$\cap \{(a, b, c) : (b - a)(|a - b| - a - b + 2c) \geq 0\}$$

$$\setminus \{(a,b,c) : a = c + 1 = b + 1\} \setminus \{(a,b,c) : b = c + 1 = a + 1\}, \quad (3)$$

so is $H_{b,a,c}(x)$ in $(-\rho, \infty)$ if and only if

$$(a,b,c) \in D_2(a,b,c) \triangleq \{(a,b,c) : (b-a)(1-a-b+2c) \leq 0\}$$
$$\cap \{(a,b,c) : (b-a)(|a-b|-a-b+2c) \leq 0\}$$
$$\setminus \{(a,b,c) : b = c + 1 = a + 1\} \setminus \{(a,b,c) : a = c + 1 = b + 1\}. \quad (4)$$

The first aim of this short note is to deduce a double inequality for the divided differences of the polygamma functions from Lemma 1 as follows.

Theorem 1. *Let $b > a \geq 0$ and $k \in \mathbb{N}$. Then the double inequality*

$$\frac{(k-1)!}{(x+\alpha)^k} \leq \frac{(-1)^{k-1}\big[\psi^{(k-1)}(x+b) - \psi^{(k-1)}(x+a)\big]}{b-a} \leq \frac{(k-1)!}{(x+\beta)^k} \quad (5)$$

for $x \in (0, \infty)$ holds if and only if $\alpha \geq \max\big\{a, \frac{a+b-1}{2}\big\}$ and $0 \leq \beta \leq \min\big\{a, \frac{a+b-1}{2}\big\}$.

The second aim of this short note is to show some identities of the psi and polygamma functions by using Theorem 1.

Theorem 2. *Let $k \in \mathbb{N}$. then the following identities of polygamma functions are valid:*

$$\psi^{(k-1)}\left(\left(\frac{\sqrt{5}+1}{2}\right)^2\right) - \psi^{(k-1)}\left(\frac{\sqrt{5}+1}{2}\right) = (-1)^{k-1}(k-1)!\left(\frac{\sqrt{5}-1}{2}\right)^k, \quad (6)$$

$$\psi^{(k-1)}\left(\left(\frac{\sqrt[3]{9-\sqrt{69}}+\sqrt[3]{9+\sqrt{69}}}{\sqrt[3]{18}}\right)^3\right) - \psi^{(k-1)}\left(\frac{\sqrt[3]{9-\sqrt{69}}+\sqrt[3]{9+\sqrt{69}}}{\sqrt[3]{18}}\right)$$
$$= (-1)^{k-1}(k-1)!\left(\frac{\sqrt[3]{18}}{\sqrt[3]{9-\sqrt{69}}+\sqrt[3]{9+\sqrt{69}}}\right)^k, \quad (7)$$

$$\psi^{(k-1)}\left(\frac{1}{8}\left(\sqrt{a-b+\frac{2}{\sqrt{b-a}}}+\sqrt{b-a}\right)^4\right)$$
$$- \psi^{(k-1)}\left(\frac{1}{2}\sqrt{a-b+\frac{2}{\sqrt{b-a}}}+\frac{\sqrt{b-a}}{2}\right)$$
$$= (-1)^{k-1}2^k(k-1)!\left(\sqrt{a-b+\frac{2}{\sqrt{b-a}}}+\sqrt{b-a}\right)^{-k}, \quad (8)$$

where $a = 4\sqrt[3]{\frac{2}{3(9+\sqrt{849})}}$ and $b = \sqrt[3]{\frac{9+\sqrt{849}}{18}}$.

For $v > 1$ and $\alpha > 1$, let $v_0 > 1$ denote the real root of equation $v^\alpha - v - 1 = 0$, then

$$v_0^k\big[\psi^{(k-1)}(v_0^\alpha) - \psi^{(k-1)}(v_0)\big] = (-1)^{k-1}(k-1)!. \quad (9)$$

For $0 < v < 1$ and $\alpha < 0$, let $v_0 < 1$ be the real root of equation $v^\alpha - v - 1 = 0$, then identity (9) is also valid.

2. Proofs of theorems

Proof of Theorem 1. From the logarithmically complete monotonicity of the function $H_{a,b,c}(x)$ in Lemma 1, it follows that

$$0 \leq (-1)^k [\ln H_{a,b,c}(x)]^{(k)}$$
$$= (-1)^k \left[\psi^{(k-1)}(x+a) - \psi^{(k-1)}(x+b) + \frac{(-1)^{k-1}(b-a)(k-1)!}{(x+c)^k} \right] \quad (10)$$

for $(a,b,c) \in D_1(a,b,c)$, then the left hand side inequality in (5) is deduced straightforwardly by standard arguments.

The right hand side inequality in (5) can be deduced from $(-1)^k [\ln H_{b,a,c}(x)]^{(k)} \geq 0$ for $(a,b,c) \in D_2(a,b,c)$. □

Proof of Theorem 2. Inequality (5) in Theorem 1 can be rearranged as

$$\frac{(k-1)!}{[\max\{v, (u+v-1)/2\}]^k} \leq \frac{(-1)^{k-1}[\psi^{(k-1)}(u) - \psi^{(k-1)}(v)]}{u-v}$$
$$= \frac{(-1)^{k-1}}{u-v} \int_v^u \psi^{(k)}(t)\,\mathrm{d}t \leq \frac{(k-1)!}{[\min\{v, (u+v-1)/2\}]^k} \quad (11)$$

for $u > v > 0$.

Substituting $u = v^2$ for $v > 1$ in (11) yields

$$\frac{(k-1)!(v^2-v)}{[\max\{v, (v^2+v-1)/2\}]^k} \leq (-1)^{k-1}[\psi^{(k-1)}(v^2) - \psi^{(k-1)}(v)]$$
$$\leq \frac{(k-1)!(v^2-v)}{[\min\{v, (v^2+v-1)/2\}]^k}. \quad (12)$$

Since equation $v^2 - v - 1 = 0$ has a unique root $\frac{\sqrt{5}+1}{2}$ greater than 1, then, if $1 < v \leq \frac{\sqrt{5}+1}{2}$,

$$(k-1)! \left(\frac{1}{v^{k-2}} - \frac{1}{v^{k-1}} \right) \leq (-1)^{k-1}[\psi^{(k-1)}(v^2) - \psi^{(k-1)}(v)]$$
$$\leq \frac{(k-1)!2^k v(v-1)}{(v^2+v-1)^k}; \quad (13)$$

if $v \geq \frac{\sqrt{5}+1}{2}$, above inequality reverses. Taking $v \to \frac{\sqrt{5}+1}{2}$ in (12) or (13) yields identity (6).

It is easy to see that equation $v^3 - v - 1 = 0$ has a unique real root

$$\sqrt[3]{\frac{1}{2} - \frac{1}{6}\sqrt{\frac{23}{3}}} + \sqrt[3]{\frac{1}{2} + \frac{1}{6}\sqrt{\frac{23}{3}}} = \frac{\sqrt[3]{9 - \sqrt{69}} + \sqrt[3]{9 + \sqrt{69}}}{\sqrt[3]{2}\,\sqrt[3]{9}} = 1.324\cdots. \quad (14)$$

Substituting $u = v^3$ for $v > 1$ in (11) yields

$$\frac{(k-1)!(v^3-v)}{[\max\{v, (v^3+v-1)/2\}]^k} \leq (-1)^{k-1}[\psi^{(k-1)}(v^3) - \psi^{(k-1)}(v)]$$
$$\leq \frac{(k-1)!(v^3-v)}{[\min\{v, (v^3+v-1)/2\}]^k}. \quad (15)$$

If $1 < v \le \dfrac{\sqrt[3]{9-\sqrt{69}}+\sqrt[3]{9+\sqrt{69}}}{\sqrt[3]{2}\,\sqrt[3]{9}}$,

$$(k-1)!\left(\frac{1}{v^{k-3}} - \frac{1}{v^{k-1}}\right) \le (-1)^{k-1}\big[\psi^{(k-1)}(v^3) - \psi^{(k-1)}(v)\big]$$

$$\le \frac{(k-1)!\,2^k v(v^2-1)}{(v^3+v-1)^k}; \quad (16)$$

if $v \ge \dfrac{\sqrt[3]{9-\sqrt{69}}+\sqrt[3]{9+\sqrt{69}}}{\sqrt[3]{2}\,\sqrt[3]{9}}$, above inequality reverses. Identity (7) follows from taking $v \to \dfrac{\sqrt[3]{9-\sqrt{69}}+\sqrt[3]{9+\sqrt{69}}}{\sqrt[3]{2}\,\sqrt[3]{9}}$ in (15) or (16).

It is not difficult to see that the quartic equation $v^4 - v - 1 = 0$ has a unique real root

$$\frac{1}{2}\sqrt{4\sqrt[3]{\frac{2}{3\left(9+\sqrt{849}\right)}} - \sqrt[3]{\frac{9+\sqrt{849}}{18}} + \frac{2}{\sqrt{\sqrt[3]{\frac{9+\sqrt{849}}{18}} - 4\sqrt[3]{\frac{2}{3\left(9+\sqrt{849}\right)}}}}}$$

$$+ \frac{1}{2}\sqrt{\sqrt[3]{\frac{9+\sqrt{849}}{18}} - 4\sqrt[3]{\frac{2}{3\left(9+\sqrt{849}\right)}}} = 1.220\cdots \quad (17)$$

Replacing u by v^4 for $v > 1$ in (11) gives

$$\frac{(k-1)!(v^4-v)}{[\max\{v,(v^4+v-1)/2\}]^k} \le (-1)^{k-1}\big[\psi^{(k-1)}(v^4) - \psi^{(k-1)}(v)\big]$$

$$\le \frac{(k-1)!(v^4-v)}{[\min\{v,(v^4+v-1)/2\}]^k}. \quad (18)$$

If $1 < v \le \frac{1}{2}\sqrt{a-b+\frac{2}{\sqrt{b-a}}} + \frac{1}{2}\sqrt{b-a}$, then

$$(k-1)!\left(\frac{1}{v^{k-4}} - \frac{1}{v^{k-1}}\right) \le (-1)^{k-1}\big[\psi^{(k-1)}(v^4) - \psi^{(k-1)}(v)\big]$$

$$\le \frac{(k-1)!\,2^k v(v^3-1)}{(v^4+v-1)^k}; \quad (19)$$

if $v \ge \frac{1}{2}\sqrt{a-b+\frac{2}{\sqrt{b-a}}} + \frac{1}{2}\sqrt{b-a}$, above inequality reverses. Identity (8) follows from taking $v \to \frac{1}{2}\sqrt{a-b+\frac{2}{\sqrt{b-a}}} + \frac{1}{2}\sqrt{b-a}$ in (18) or (19).

For $v > 1$ and $\alpha > 1$, since the function $f_\alpha(v) = v^\alpha - v - 1$ satisfying

$$\lim_{v\to 1^+} f_\alpha(v) = -1 \quad \text{and} \quad \lim_{v\to\infty} f_\alpha(v) = \infty, \quad (20)$$

the equation $v^\alpha - v - 1 = 0$ must have at least one root v_0 greater than 1. Letting $u = v^\alpha > v > 1$ and taking limit $v \to v_0$ in (11) leads to

$$\psi^{(k-1)}(v_0^4) - \psi^{(k-1)}(v_0) = \frac{(-1)^{k-1}(k-1)!}{v_0^k}. \quad (21)$$

Identity (9) is proved for $v > 1$ and $\alpha > 1$.

For $0 < v < 1$ and $\alpha < 0$, since the function $f_\alpha(v) = v^\alpha - v - 1$ satisfying

$$\lim_{v\to 1^+} f_\alpha(v) = -1 \quad \text{and} \quad \lim_{v\to 0^+} f_\alpha(v) = \infty, \quad (22)$$

the equation $v^\alpha - v - 1 = 0$ must have at least one root v_0 less than 1. Letting $u = v^\alpha > 1 > v$ and taking limit $v \to v_0$ in (11) leads to (21). Hence, identity (9) is proved for $0 < v < 1$ and $\alpha < 0$. $\hspace{1cm}\square$

References

[1] R. D. Atanassov and U. V. Tsoukrovski, *Some properties of a class of logarithmically completely monotonic functions*, C. R. Acad. Bulgare Sci. **41** (1988), no. 2, 21–23.

[2] C. Berg, *Integral representation of some functions related to the gamma function*, Mediterr. J. Math. **1** (2004), no. 4, 433–439.

[3] A. Z. Grinshpan and M. E. H. Ismail, *Completely monotonic functions involving the gamma and q-gamma functions*, Proc. Amer. Math. Soc. **134** (2006), 1153–1160.

[4] F. Qi, *Certain logarithmically N-alternating monotonic functions involving gamma and q-gamma functions*, Nonlinear Funct. Anal. Appl. (2007), in press. RGMIA Res. Rep. Coll. **8** (2005), no. 3, Art. 5, 413–422; Available online at `http://rgmia.vu.edu.au/v8n3.html`.

[5] F. Qi, *Monotonicity and logarithmic convexity for a class of elementary functions involving the exponential function*, RGMIA Res. Rep. Coll. **9** (2006), no. 3, Art. 3; Available online at `http://rgmia.vu.edu.au/v9n3.html`.

[6] F. Qi, *Three-log-convexity for a class of elementary functions involving exponential function*, J. Math. Anal. Approx. Theory **1** (2006), no. 2, 100–103.

[7] F. Qi and Ch.-P. Chen, *A complete monotonicity property of the gamma function*, J. Math. Anal. Appl. **296** (2004), no. 2, 603–607.

[8] F. Qi and B.-N. Guo, *Complete monotonicities of functions involving the gamma and digamma functions*, RGMIA Res. Rep. Coll. **7** (2004), no. 1, Art. 8, 63–72; Available online at `http://rgmia.vu.edu.au/v7n1.html`.

[9] F. Qi and B.-N. Guo, *Wendel-Gautschi-Kershaw's inequalities and sufficient and necessary conditions that a class of functions involving ratio of gamma functions are logarithmically completely monotonic*, RGMIA Res. Rep. Coll. **10** (2007), no. 1, Art. 2; Available online at `http://rgmia.vu.edu.au/v10n1.html`.

[10] F. Qi, B.-N. Guo, and Ch.-P. Chen, *Some completely monotonic functions involving the gamma and polygamma functions*, J. Austral. Math. Soc. **80** (2006), 81–88.

[11] F. Qi, B.-N. Guo, and Ch.-P. Chen, *Some completely monotonic functions involving the gamma and polygamma functions*, RGMIA Res. Rep. Coll. **7** (2004), no. 1, Art. 5, 31–36; Available online at `http://rgmia.vu.edu.au/v7n1.html`.

(F. Qi) Research Institute of Mathematical Inequality Theory, Henan Polytechnic University, Jiaozuo City, Henan Province, 454010, China

E-mail address: `qifeng618@gmail.com, qifeng618@hotmail.com, qifeng618@qq.com`

URL: `http://rgmia.vu.edu.au/qi.html`