On a Certain Second Order Integrodifferential Equation *

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Abstract

In this paper we study the existence, uniqueness and other properties of solutions of a certain second order integrodifferential equation. The well known Banach fixed point theorem coupled with Bilecki type norm and the new integral inequality with explicit estimate are used to establish the results.

1 Introduction

Consider the initial value problem (IVP for short) for second order integrodifferential equation of the form

$$x''(t) = F(t, x(t), x'(t), (Ax)(t), (Bx)(t)), \tag{1.1}$$

for $t \in I = [0, b], b > 0$, with the given initial conditions

$$x(0) = x_0, \ \ x'(0) = x_1, \tag{1.2}$$

where

$$(Ax)(t) = \int_0^t k_1(t,\tau) h_1(\tau, x(\tau), x'(\tau)) \, d\tau, \tag{1.3}$$

$$(Bx)(t) = \int_0^b k_2(t,\tau) h_2(\tau, x(\tau), x'(\tau)) \, d\tau. \tag{1.4}$$

In (1.1)-(1.4), $F \in C\left(I \times R^{4n}, R^n\right)$; for i=1,2 and $0 \leq \tau \leq t$, $k_i \in C\left(I^2, R\right)$, $h_i \in C\left(I \times R^{2n}, R^n\right)$ are given functions and R^n denotes the n-dimensional Euclidean space with norm $\|.\|$. In the literature, there are many papers dealing with the problems of existence, uniqueness and other properties of solutions of various special versions of IVP (1.1)-(1.2) by using different techniques. The monographs [3, 5-7] present a comprehensive accounts on different types of integral and integrodifferential equations.

Many nonlinear systems are too complicated to be analized. It is well known that, the fixed point theorems and the integral inequalities with explicit estimates are forceful tools in the analysis of various types of nonlinear systems. The main purpose of this paper is to study the existence, uniqueness and other properties of solutions of IVP (1.1)-(1.2). The analysis used in the proofs is based on the well known Banach fixed point theorem (see [5, p.37]) coupled with Bilecki type norm (see [2]) and the new integral inequality with explicit estimate, recently established by the present author in [14].

2 Existence and Uniqueness

For every continuous function $x(t) : I \to R^n$ together with the continuous first derivative $x'(t) : I \to R^n$, we denote by $\|x(t)\|_1 = \|x(t)\| + \|x'(t)\|$. Let G be a space of those functions $x(t) : I \to R^n$ which are continuous on I together with the continuous first derivative $x'(t) : I \to R^n$ and fulfil the condition

$$\|x(t)\|_1 = o(\exp(\lambda t)), \tag{2.1}$$

where λ is a positive constant. In the space G we define the norm (see [2])

$$\|x\|_G = \sup_{t \in I} \{\|x(t)\|_1 \exp(-\lambda t)\}. \tag{2.2}$$

It is easy to see that G with the norm defined in (2.2) is a Banach space. We note that the condition (2.1) implies that there is a constant $N_0 \geq 0$ such that

$$\|x(t)\|_1 \leq N_0 \exp(\lambda t).$$

Using this fact in (2.2), we observe that

$$\|x\|_G \leq N_0. \tag{2.3}$$

It is easy to observe that the IVP (1.1)-(1.2) is equivalent to the integral equation

$$x(t) = x_0 + t\,x_1 + \int_0^t (t-s)\,F(s, x(s), x'(s), (Ax)(s), (Bx)(s))ds. \tag{1.4}$$

Differentiating both sides of (2.4) with respect to t , it is easy to observe that

$$x'(t) = x_1 + \int_0^t F(s, x(s), x'(s), (Ax)(s), (Bx)(s))ds. \tag{2.5}$$

Our existence and uniqueness result is given in the following theorem.

Theorem 1 . Assume that

(i) the functions F, h_i $(i = 1, 2)$ satisfy the conditions

$$\|F(t, x_1, y_1, u_1, v_1) - F(t, x_2, y_2, u_2, v_2)\|$$

$$\leq p(t)\,[\|x_1 - x_2\| + \|y_1 - y_2\| + \|u_1 - u_2\| + \|v_1 - v_2\|], \tag{2.6}$$

$$\|h_i(t, x_1, y_1) - h_i(t, x_2, y_2)\| \leq q_i(t)\,[\|x_1 - x_2\| + \|y_1 - y_2\|], \tag{2.7}$$

where $p, q_i \in C(I, R_+)\,; R_+ = [0, \infty)$,

(ii) there exist nonnegative constants α_1, α_2 such that $0 \le \alpha_1 + \alpha_2 < 1$ and

$$\int_0^t bp(s)\left[\exp(\lambda s) + h_1^*(s) + h_2^*(s)\right] ds \le \alpha_1 \exp(\lambda t), \tag{2.8}$$

$$\int_0^t p(s)\left[\exp(\lambda s) + h_1^*(s) + h_2^*(s)\right] ds \le \alpha_2 \exp(\lambda t), \tag{2.9}$$

for $t \in I$, where

$$h_1^*(t) = \int_0^t |k_1(t,\tau)| q_1(\tau) \exp(\lambda\tau)\, d\tau, \tag{2.10}$$

$$h_2^*(t) = \int_0^b |k_2(t,\tau)| q_2(\tau) \exp(\lambda\tau)\, d\tau, \tag{2.11}$$

where λ is as given in (2.1),

(iii) there exist nonnegative constants P_1, P_2 such that

$$\left\| x_0 + t\,x_1 + \int_0^t (t-s)\, F(s, 0, 0, (A0)(s), (B0)(s))\, ds \right\| \le P_1 \exp(\lambda t), \tag{2.12}$$

$$\left\| x_1 + \int_0^t F(s, 0, 0, (A0)(s), (B0)(s))\, ds \right\| \le P_2 \exp(\lambda t), \tag{2.13}$$

where λ is as given in (2.1).

Then the IVP (1.1)-(1.2) has a unique solution $x(t)$ in G.

Proof . Let $x \in G$ and define the operator

$$(Tx)(t) = x_0 + t\,x_1 + \int_0^t (t-s)\, F(s, x(s), x'(s), (Ax)(s), (Bx)(s))\, ds. \tag{2.14}$$

Differentiating both sides of (2.14) with respect to t , we get

$$(Tx)'(t) = x_1 + \int_0^t F(s, x(s), x'(s), (Ax)(s), (Bx)(s))\, ds. \tag{2.15}$$

Now, we shall show that Tx maps G into itself. Evidently, $(Tx),(Tx)'$ are continuous on I and $(Tx),(Tx)' \in R^n$. We verify that (2.1) is fulfilled. From (2.14) and (2.12), (2.6) we have

$$\|(Tx)(t)\| \leq \left\| x_0 + t\,x_1 + \int\limits_0^t (t-s)\,F\left(0,0,0,(A0)(s),(B0)(s)\right)ds \right\|$$

$$+\int\limits_0^t (t-s)\,\|F\left(s,x(s),x'(s),(Ax)(s),(Bx)(s)\right)$$

$$-F\left(0,0,0,(A0)(s),(B0)(s)\right)\|\,ds$$

$$\leq P_1\exp(\lambda t) + \int\limits_0^t (t-s)\,p(s)\left[\|x(s)\| + \|x'(s)\|\right.$$

$$\left.+\|(Ax)(s)-(A0)(s)\| + \|(Bx)(s)-(B0)(s)\|\right]ds$$

$$\leq P_1\exp(\lambda t) + \int\limits_0^t bp(s)\left[\exp(\lambda s)\,\|x(s)\|_1\exp(-\lambda s)\right.$$

$$\left.+\|(Ax)(s)-(A0)(s)\| + \|(Bx)(s)-(B0)(s)\|\right]ds$$

$$\leq P_1\exp(\lambda t) + \int\limits_0^t bp(s)\left[\exp(\lambda s)\,\|x\|_G\right.$$

$$\left.+\|(Ax)(s)-(A0)(s)\| + \|(Bx)(s)-(B0)(s)\|\right]ds. \tag{2.16}$$

From (1.3), (2.7), (2.10), we observe that

$$\|(Ax)(s)-(A0)(s)\| \leq \int\limits_0^s |k_1(s,\tau)|\,\|h_1(\tau,x(\tau),x'(\tau)) - h_1(\tau,0,0)\|d\tau$$

$$\leq \int\limits_0^s |k_1(s,\tau)|\,q_1(\tau)\left[\|x(\tau)\| + \|x'(\tau)\|\right]d\tau$$

$$= \int\limits_0^s |k_1(s,\tau)|\,q_1(\tau)\exp(\lambda\tau)\,\|x(\tau)\|_1\exp(-\lambda\tau)d\tau$$

$$\leq \|x\|_G\,h_1^*(s). \tag{2.17}$$

Similarly, from (1.4), (2.7), (2.11) we observe that

$$\|(Bx)(s)-(B0)(s)\| \leq \|x\|_G\,h_2^*(s). \tag{2.18}$$

From (2.16), (2.17),(2.18), (2.3), (2.8) we get

$$\|(Tx)(t)\| \leq P_1 \exp(\lambda t) + \|x\|_G \int_0^t bp(s)\left[\exp(\lambda s) + h_1^*(s) + h_2^*(s)\right] ds$$

$$\leq P_1 \exp(\lambda t) + N_0 \alpha_1 \exp(\lambda t). \tag{2.19}$$

From (2.15) and (2.13),(2.6), (2.17), (2.18), (2.3), (2.9) and closely looking at the proof of (2.16) we have

$$\left\|(Tx)'(t)\right\| \leq P_2 \exp(\lambda t) + \|x\|_G \int_0^t p(s)\left[\exp(\lambda s) + h_1^*(s) + h_2^*(s)\right] ds$$

$$\leq P_2 \exp(\lambda t) + N_0 \alpha_2 \exp(\lambda t). \tag{2.20}$$

From (2.19) and (2.20) we observe that

$$\|(Tx)(t)\|_1 = \left\|(Tx)(t) + (Tx)'(t)\right\|$$

$$\leq \left[P_1 + P_2 + N_0(\alpha_1 + \alpha_2)\right] \exp(\lambda t). \tag{2.21}$$

From (2.21) it follows that $Tx \in G$. This proves that T maps G into itself.

Now, we verify that the operator T is a contraction map. Let $x, y \in G$. From (2.14) and (2.6) we have

$$\|(Tx)(t) - (Ty)(t)\| \leq \int_0^t (t-s)\left\|F\left(s, x(s), x'(s), (Ax)(s), (Bx)(s)\right)\right.$$

$$\left. - F\left(s, y(s), y'(s), (Ay)(s), (By)(s)\right)\right\| ds$$

$$\leq \int_0^t (t-s) p(s)\left[\|x(s) - y(s)\| + \|x'(s) - y'(s)\|\right.$$

$$\left. + \|(Ax)(s) - (Ay)(s)\| + \|(Bx)(s) - (By)(s)\|\right] ds$$

$$\leq \int_0^t bp(s)\left[\exp(\lambda s)\|x(s) - y(s)\|_1 \exp(-\lambda s)\right.$$

$$\left. + \|(Ax)(s) - (Ay)(s)\| + \|(Bx)(s) - (By)(s)\|\right] ds$$

$$\leq \int_0^t bp(s)\left[\exp(\lambda s)\|x - y\|_G\right.$$

$$\left. + \|(Ax)(s) - (Ay)(s)\| + \|(Bx)(s) - (By)(s)\|\right] ds. \tag{2.22}$$

Following the proofs of (2.17) and (2.18) we obtain

$$\|(Ax)(s)-(Ay)(s)\| \le \|x-y\|_G\, h_1^*(s), \tag{2.23}$$

and

$$\|(Bx)(s)-(By)(s)\| \le \|x-y\|_G\, h_2^*(s). \tag{2.24}$$

Using (2.23), (2.24) in (2.22) and the condition (2.8) we get

$$\|(Tx)(t)-(Ty)(t)\| \le \|x-y\|_G \int_0^t bp(s)\left[\exp(\lambda s) + h_1^*(s) + h_2^*(s)\right] ds$$

$$\le \|x-y\|_G\, \alpha_1 \exp(\lambda t). \tag{2.25}$$

Similarly, from (2.15), (2.6), (2.23), (2.24) and (2.9) we get

$$\left\|(Tx)'(t)-(Ty)'(t)\right\| \le \|x-y\|_G\, \alpha_2 \exp(\lambda t). \tag{2.26}$$

From (2.25) and (2.26) we have

$$\|(Tx)(t)-(Ty)(t)\|_1 = \|(Tx)(t)-(Ty)(t)\| + \left\|(Tx)'(t)-(Ty)'(t)\right\|$$

$$\le \|x-y\|_G\,(\alpha_1+\alpha_2)\exp(\lambda t).$$

Consequently, we have

$$\|Tx-Ty\|_G \le (\alpha_1+\alpha_2)\,\|x-y\|_G.$$

Since $\alpha_1+\alpha_2<1$, it follows from the Banach fixed point theorem (see [5, p.37]) that T has a unique fixed point in G . The fixed point of T is however a solution of IVP (1.1)-(1.2). The proof is complete.

Remark 1. We note that, there exists a vast literature on existence of solutions of various types of integral and integrodifferential equations, see [4-12] and the references given therein. The norm $\|.\|_G$ defined by (2.2), was first used by Bielecki [2] (see [4] for developments related to this topic), and has the role of improving the length of the interval on which the existence is assured.

3 Properties of solutions

In the study of nonlinear systems, in many situations, having the knowledge of the existence of solutions, we are interested in knowing the qualitative properties of solutions without the explicit knowledge of the solution process. It frequently happens that a method which works very efficiently to establish existence does not yield other properties of the solutions in any ready fashion. The integral inequalities with explicit estimates serves an important tool in the study of qualitative properties of solutions of various nonlinear systems.

We need the following new integral inequality, recently established in [14] to study the various properties of solutions of IVP (1.1)-(1.2).We shall state and prove it in the following lemma for completeness. For a detailed account on such inequalities, see [1,13].

Lemma (Pachpatte [14]). Let $u(t), f(t), g(t), h(t) \in C(I, R_+)$ and suppose

$$u(t) \le c + \int_0^t f(s) \left[u(s) + \int_0^s g(\sigma) u(\sigma) d\sigma + \int_0^b h(\sigma) u(\sigma) d\sigma \right] ds, \tag{3.1}$$

for $t \in I = [0, b]$, $b > 0$ where $c \ge 0$ is a constant. If

$$d = \int_0^b h(\sigma) \exp\left(\int_0^\sigma [f(\tau) + g(\tau)] d\tau \right) d\sigma < 1, \tag{3.2}$$

then

$$u(t) \le \frac{c}{1-d} \exp\left(\int_0^t [f(s) + g(s)] ds \right), \tag{3.3}$$

for $t \in I$.

Proof. Define a function $z(t)$ by the right side of (3.1). Then $z(0) = c, u(t) \le z(t)$ and

$$z'(t) = f(t) \left[u(t) + \int_0^t g(\sigma) u(\sigma) d\sigma + \int_0^b h(\sigma) u(\sigma) d\sigma \right]$$

$$\le f(t) \left[z(t) + \int_0^t g(\sigma) z(\sigma) d\sigma + \int_0^b h(\sigma) z(\sigma) d\sigma \right],$$

for $t \in I$. Define a function $v(t)$ by

$$v(t) = z(t) + \int_0^t g(\sigma) z(\sigma) d\sigma + \int_0^b h(\sigma) z(\sigma) d\sigma,$$

then we observe that $z(t) \le v(t), z'(t) \le f(t) v(t)$,

$$v(0) = c + \int_0^b h(\sigma) z(\sigma) d\sigma, \tag{3.4}$$

and

$$\begin{aligned}
v'(t) &= z'(t) + g(t)\, z(t) \\
&\leq f(t)\, v(t) + g(t)\, z(t) \\
&\leq [f(t) + g(t)]\, v(t),
\end{aligned}$$

which implies

$$v(t) \leq v(0) \exp\left(\int_0^t [f(s) + g(s)]\, ds\right), \tag{3.5}$$

for $t \in I$. Using the fact that $z(t) \leq v(t)$ and (3.5) on the right side of (3.4) and in view of (3.2), it is easy to observe that

$$v(0) \leq \frac{c}{1-d}. \tag{3.6}$$

Using (3.6) in (3.5) and the fact that $u(t) \leq z(t)$ we get the desired inequality in (3.3).

Remark 2. In the special cases, when (i) $g(t) = 0$ (ii) $h(t) = 0$, we get new inequalities which can be used conveniently in certain situations.

The following theorem deals with the estimate on the solution of IVP (1.1)-(1.2).

Theorem 2. Assume that the functions F, k_i, h_i $(i = 1, 2)$ satisfy the conditions

$$\|F(t, x, y, u, v)\| \leq p(t)\, [\|x\| + \|y\| + \|u\| + \|v\|], \tag{3.7}$$

$$|k_i(t, s)| \leq M_i, \tag{3.8}$$

$$\|h_i(t, x, y)\| \leq q_i(t)\, [\|x\| + \|y\|], \tag{3.9}$$

where $M_i \geq 0$ are constants and $p, q_i \in C(I, R_+)$. Let

$$d' = \int_0^b M_2 q_2(\sigma) \exp\left(\int_0^\sigma [(b+1)\, p(s) + M_1 q_1(s)]\, ds\right) d\sigma < 1. \tag{3.10}$$

If $x(t)$ is any solution of IVP (1.1)-(1.2), then

$$\|x(t)\| + \|x'(t)\| \leq \frac{\|x_0\| + (b+1)\, \|x_1\|}{1-d'} \exp\left(\int_0^t [(b+1)\, p(s) + M_1 q_1(s)]\, ds\right), \tag{3.11}$$

for $t \in I$.

Proof. The solution $x(t)$ of IVP (1.1)-(1.2) and its derivative $x'(t)$ satisfies the integral equations (2.4) and (2.5) respectively. Let $u(t) = \|x(t)\| + \|x'(t)\|$. Then from (2.4), (2.5) and the hypotheses (3.7)-(3.9) we observe that

$$u(t) \leq \|x_0\| + (b+1)\|x_1\| + \int_0^t (b+1)\, p(s) \Bigg[u(s) + \int_0^s M_1 q_1(\tau) u(\tau)\, d\tau + \int_0^b M_2 q_2(\tau) u(\tau)\, d\tau \Bigg] ds. \tag{3.12}$$

Now in view of the hypotheses (3.10), an application of Lemma to (3.12) yields (3.11). The proof is complete.

Remark 3. We note that the inequality (3.11) gives the bound in terms of the known functions which majorizes the solution $x(t)$ of IVP (1.1)-(1.2) as well as its derivative $x'(t)$ for $t \in I$. If the right side in (3.11) is finite, then the solution $x(t)$ (as well as its derivative $x'(t)$) is bounded on I .

Next, by applying Lemma we establish the uniqueness of solutions of IVP (1.1)-(1.2).

Theorem 3. Assume that the functions F, h_i $(i = 1, 2)$ satisfy the conditions (2.6), (2.7) and the functions k_i $(i = 1, 2)$ satisfy (3.8). Let d' be as in (3.10). Then the IVP (1.1)-(1.2) has at most one solution on I .

Proof. Let $x(t)$ and $y(t)$ be two solutions of IVP (1.1)-(1.2) on I and $v(t) = \|x(t) - y(t)\| + \|x'(t) - y'(t)\|$. Then it is easy to observe that

$$v(t) \leq \int_0^t (t-s)\, p(s) \left[v(s) + \|(Ax)(s) - (Ay)(s)\| + \|(Bx)(s) - (By)(s)\| \right] ds + \int_0^t p(s) \left[v(s) + \|(Ax)(s) - (Ay)(s)\| + \|(Bx)(s) - (By)(s)\| \right] ds. \tag{3.13}$$

Furthermore, we observe that

$$\|(Ax)(s) - (Ay)(s)\| \leq \int_0^s |k_1(s,\tau)| \, \|h_1(\tau, x(\tau), x'(\tau)) - h_1(\tau, y(\tau), y'(\tau))\| \, d\tau \leq \int_0^s M_1 q_1(\tau)\, v(\tau)\, d\tau, \tag{3.14}$$

and

$$\|(Bx)(s)-(By)(s)\| \leq \int_0^b M_2 q_2(\tau)\, v(\tau)\, d\tau. \tag{3.15}$$

Using (3.14), (3.15) in (3.13) we get

$$v(t) \leq \int_0^t (b+1)\, p(s) \left[v(s) + \int_0^s M_1 q_1(\tau)\, v(\tau)\, d\tau + \int_0^b M_2 q_2(\tau)\, v(\tau)\, d\tau \right] ds. \tag{3.16}$$

A suitable application of Lemma (with $c=0$) to (3.16) yields $v(t) \leq 0$ and consequently $x(t)=y(t)$, i.e. there is at most one solution of IVP (1.1)-(1.2). on I .

The following theorem shows the continuous dependence of solutions of IVP (1.1)-(1.2) on given initial data.

Theorem 4. Let $x(t)$ and $y(t)$ be solutions of (1.1) with initial data

$$x(0)=x_0, x'(0)=x_1, \tag{3.17}$$

and

$$y(0)=y_0, y'(0)=y_1, \tag{3.18}$$

respectively. Suppose that the functions $F, k_i, h_i\, (i=1,2)$ be as in Theorem 3. Let d' be as in (3.10). Then

$$\|x(t)-y(t)\| + \|x'(t)-y'(t)\|$$

$$\leq \frac{\|x_0-y_0\| + (b+1)\|x_1-y_1\|}{1-d'} \exp\left(\int_0^t [(b+1)\, p(s) + M_1 q_1(s)]\, ds \right), \tag{3.19}$$

for $t \in I$.

Proof. Let $w(t) = \|x(t)-y(t)\| + \|x'(t)-y'(t)\|$. Using the facts that $x(t)$ and $y(t)$ ate the solutions of IVP (1.1)-(3.17) and IVP (1.1)-(3.18) respectively, and using the hypotheses (2.6), (2.7), (3.8) and the estimates (3.14) and (3.15) we have

$$w(t) \leq \|x_0-y_0\| + (b+1)\|x_1-y_1\|$$

$$+ \int_0^t (b+1)\, p(s) \left[w(s) + \int_0^s M_1 q_1(\tau)\, w(\tau)\, d\tau + \int_0^b M_2 q_2(\tau)\, w(\tau)\, d\tau \right]. \tag{3.20}$$

Now an application of Lemma to (3.20) yields (3.19), which shows the dependency of solutions of IVP (1.1)-(3.17) and IVP (1.1)-(3.18) on given initial data.

We next consider the following initial value problems

$$x''(t) = H\left(t, x(t), x'(t), (Ax)(t), (Bx)(t), \mu\right), \tag{3.21}$$

$$x(0) = x_0, x'(0) = x_1, \tag{3.22}$$

and

$$x''(t) = H\left(t, x(t), x'(t), (Ax)(t), (Bx)(t), \mu_0\right),$$

$$x(0) = x_0, x'(0) = x_1, \tag{3.24}$$

where $H \in C\left(I \times R^{4n} \times R, R^n\right)$, $(Ax)(t)$, $(Bx)(t)$ are as given in (1.3), (1.4) and μ, μ_0 are real parameters.

The following theorem shows the dependency of solutions of IVP (3.21)-(3.22) and IVP (3.23)-(3.24) on parameters.

Theorem 5. Assume that the functions k_i, h_i $(i = 1, 2)$ satisfy the conditions (3.8), (2.7) and the function H in (3.21) (or (3.23) satisfies the conditions

$$\|H(t, x_1, y_1, u_1, v_1, \mu) - H(t, x_2, y_2, u_2, v_2, \mu)\|$$

$$\leq p(t)\left[\|x_1 - x_2\| + \|y_1 - y_2\| + \|u_1 - u_2\| + \|v_1 - v_2\|\right], \tag{3.25}$$

$$\|H(t, x, y, u, v, \mu) - H(t, x, y, u, v, \mu_0)\| \leq r(t)|\mu - \mu_0|, \tag{3.26}$$

where $p, r \in C(I, R_+)$ and

$$\int_0^t br(s)\,ds \leq k_1, \int_0^t r(s)\,ds \leq k_2, \tag{3.27}$$

where k_1, k_2 are nonnegative constants. Let d' be as in (3.10). Let $y_1(t)$ and $y_2(t)$ be the solutions of IVP (3.21)-(3.22) and IVP (3.23)-(3.24). Then

$$\|y_1(t) - y_2(t)\| + \|y_1'(t) - y_2'(t)\|$$

$$\leq \frac{(k_1 + k_2)|\mu - \mu_0|}{1 - d'} \exp\left(\int_0^t \left[(b+1)p(s) + M_1 q_1(s)\right] ds\right), \tag{3.28}$$

for $t \in I$.

Proof. Let $z(t) = \|y_1(t) - y_2(t)\| + \|y_1'(t) - y_2'(t)\|$, for $t \in I$. From the hypotheses , it is easy to observe that

$$\|y_1(t) - y_2(t)\|$$

$$= \left\| \int_0^t (t-s)\left\{H\left(s, y_1(s), y_1'(s), (Ay_1)(s), (By_1)(s), \mu\right)\right.\right.$$

$$
\begin{aligned}
&-H\left(s, y_2(s), y_2'(s), (Ay_2)(s), (By_2)(s), \mu\right) \\
&+H\left(s, y_2(s), y_2'(s), (Ay_2)(s), (By_2)(s), \mu\right) \\
&-H\left(s, y_2(s), y_2'(s), (Ay_2)(s), (By_2)(s), \mu_0\right)\} \, ds \quad \Big\| \\
&\le \int_0^t (t-s) \left\| H\left(s, y_1(s), y_1'(s), (Ay_1)(s), (By_1)(s), \mu\right) \right. \\
&\left. -H\left(s, y_2(s), y_2'(s), (Ay_2)(s), (By_2)(s), \mu\right) \right\| ds \\
&+\int_0^t (t-s) \left\| H\left(s, y_2(s), y_2'(s), (Ay_2)(s), (By_2)(s), \mu\right) \right. \\
&\left. -H\left(s, y_2(s), y_2'(s), (Ay_2)(s), (By_2)(s), \mu_0\right) \right\| ds \\
&\le \int_0^t (t-s)\, p(s) \left[\| y_1(s) - y_2(s) \| + \| y_1'(s) - y_2'(s) \| \right. \\
&\left. + \| (Ay_1(s)) - (Ay_2(s)) \| + \| (By_1(s)) - (By_2(s)) \| \right] ds \\
&+\int_0^t (t-s)\, r(s) \left| \mu - \mu_0 \right| ds \\
&\le \int_0^t b p(s) \left[z(s) \right. \\
&\left. + \| (Ay_1(s)) - (Ay_2(s)) \| + \| (By_1(s)) - (By_2(s)) \| \right] ds \\
&+\int_0^t b r(s) \left| \mu - \mu_0 \right| ds. \qquad (2.29)
\end{aligned}
$$

Similarly we get

$$
\begin{aligned}
\| y_1'(t) - y_2'(t) \| &\le \int_0^t p(s) \left[z(s) \right. \\
&\left. + \| (Ay_1(s)) - (Ay_2(s)) \| + \| (By_1(s)) - (By_2(s)) \| \right] ds \\
&+\int_0^t r(s) \left| \mu - \mu_0 \right| ds. \qquad (3.30)
\end{aligned}
$$

Furthermore, we observe that

$$
\| (Ay_1(s)) - (Ay_2(s)) \| \le \int_0^s \left| k_1(s,\tau) \right| \left\| h_1\left(\tau, y_1(\tau), y_1'(\tau)\right) - h_1\left(\tau, y_2(\tau), y_2'(\tau)\right) \right\| d\tau
$$

$$\leq \int_0^s M_1 q_1(\tau) z(\tau) d\tau, \tag{3.31}$$

and similarly,

$$\|(By_1(s)) - (By_2(s))\| \leq \int_0^b M_2 q_2(\tau) z(\tau) d\tau. \tag{3.32}$$

From (3.29)-(3.32) and (3.27) we get

$$z(t) \leq (k_1 + k_2)|\mu - \mu_0| + \int_0^t (b+1) p(s) \Big[z(s)$$

$$\left. + \int_0^s M_1 q_1(\tau) z(\tau) d\tau + \int_0^b M_2 q_2(\tau) z(\tau) d\tau \right] ds. \tag{3.33}$$

Now an application of Lemma to (3.33) yields (3.28), which shows the dependency of solutions of IVP (3.21)-(3.22) and IVP(3.23)-(3.24) on parameters μ, μ_0.

Remark 4. We note that the ideas of this paper can be extended to study the IVP for the second order semilinear integrodifferential equation of the form

$$x''(t) = Ax(t) + F(t, x(t), x'(t), (Mx)(t), (Nx)(t)), \tag{3.34}$$

with the given initial conditions

$$x(0) = x_0, x'(0) = x_1, \tag{3.35}$$

for $t \in I = [0, b]$, where A is a linear infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in R\}$ in a Banach space X and

$$(Mx)(t) = \int_0^t k_1(t,\tau) h_1(\tau, x(\tau), x'(\tau)) d\tau, \tag{3.36}$$

$$(Nx)(t) = \int_0^b k_2(t,\tau) h_2(\tau, x(\tau), x'(\tau)) d\tau, \tag{3.37}$$

under some suitable conditions on the functions involved in (3.34)-(3.37) in a Banach space X. The details of the formulation of such results are very close to those given above in view of the results given in [10, 15]. Here we do not discuss the details.

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