

# On a certain neutral type Volterra integrodifferential equation \*

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## Abstract

In this paper we study the existence, uniqueness and other properties of solutions of a certain nonlinear neutral type Volterra integrodifferential equation. The Banach fixed point theorem and a certain integral inequality with explicit estimate are used to establish the results.

## 1 Introduction

Let  $R^n$  denotes the real n-dimensional Euclidean space with appropriate norm denoted by  $|\cdot|$ . We denote by  $R_+ = [0, \infty)$  the given subset of  $R$ , the set of real numbers and  $E = R^n \times R^n$  and  $C(A, B)$  the class of continuous functions from the set  $A$  to the set  $B$ . In 1996 A. Constantin [3] studied the global existence of solution of the integrodifferential equation

$$x'(t) = F \left( t, x(t), \int_0^t K(t, s, x(s)) ds \right), x(0) = 0, \quad (1.1)$$

by using the topological transversality argument and a certain integral inequality with explicit estimate on the unknown function (see also [7,10]). In [5, p.185] C. Corduneanu dealt with the Volterra functional-differential equation of the form

$$x'(t) + (Lx')(t) + f(x(t)) = g(t), \quad (1.2)$$

in which the operator  $L$  is defined by

$$(Lx)(t) = \int_0^t k(t, s) x(s) ds, \quad (1.3)$$

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for  $t \in R_+$ . In the literature there are many papers dealing with the special and even more general versions of equations (1.1) and (1.2) by using different techniques (see [1,5,6,9]). Owing to the importance of equations of the forms (1.1) and (1.2) in applications, in the present paper we study the existence, uniqueness and other properties of solutions of the following neutral type Volterra integrodifferential equation

$$y'(t) = f \left( t, y(t), y'(t), \int_0^t g(t, \sigma, y(\sigma), y'(\sigma)) d\sigma \right), \quad (1.4)$$

with the given initial condition

$$y(0) = y_0, \quad (1.5)$$

for  $t \in R_+$ , where  $f \in C(R_+ \times R^n \times R^n \times R^n, R^n)$  and  $g \in C(R_+^2 \times R^n \times R^n, R^n)$ . The main tools employed in the analysis are based on the applications of the well known Banach fixed point theorem (see [5, p. 37]) coupled with Bielecki type norm [2] and a suitable integral inequality with explicit estimate (see [8, Theorems 1.4.1 and 3.5.5]).

## 2 Existence and uniqueness

For a function  $y(t)$  and its derivative  $y'(t)$  in  $C(R_+, R^n)$  we denote by  $|y(t)|_E = |y(t)| + |y'(t)|$ . Let  $S$  be the space of those functions  $(\phi(t), \phi'(t)) \in E$  which are continuous for  $t \in R_+$  and fulfil the condition

$$|\phi(t)|_E = o(\exp(\lambda t)). \quad (2.1)$$

for  $t \in R_+$ , where  $\lambda > 0$  is a constant. In the space  $S$  we define the norm

$$|\phi|_S = \sup_{t \in R_+} [|\phi(t)|_E \exp(-\lambda t)]. \quad (2.2)$$

It is easy to see that  $S$  with norm defined in (2.2) is a Banach space. We note that the condition (2.1) implies that there exists a constant  $N \geq 0$  such that  $|\phi(t)|_E \leq N \exp(\lambda t)$  for  $t \in R_+$ . Using this fact in (2.2) we observe that

$$|\phi|_S \leq N. \quad (2.3)$$

We need the following integral inequality similar to those of given in [8, Theorems 1.4.1 and 3.3.5]. We shall state it in the following lemma for completeness.

**Lemma.** Let  $u(t), f(t) \in C(R_+, R_+)$  and for  $0 \leq s \leq t < \infty$ ,  $e(t, s)$ ,  $\frac{\partial}{\partial t} e(t, s), k(t, s) \in C(R_+^2, R_+)$  and  $c \geq 0$  is a constant. If

$$u(t) \leq c + \int_0^t \left[ f(s) u(s) + e(t, s) u(s) + \int_0^s k(s, \sigma) u(\sigma) d\sigma \right] ds, \quad (2.4)$$

for  $t \in R_+$ , then

$$u(t) \leq c \exp \left( \int_0^t [f(s) + A(s)] ds \right), \quad (2.5)$$

for  $t \in R_+$ , where

$$A(t) = e(t, t) + \int_0^t \left\{ k(t, \sigma) + \frac{\partial}{\partial t} e(t, \sigma) \right\} d\sigma. \quad (2.6)$$

**Proof.** Define a function  $w(t)$  by the right hand side of (2.4). Then  $w(t) \geq 0$ ,  $w(0) = c$ ,  $u(t) \leq w(t)$ ,  $w(t)$  is nondecreasing in  $t$  and

$$\begin{aligned} w'(t) &= f(t)u(t) + e(t, t)u(t) + \int_0^t \frac{\partial}{\partial t} e(t, s)u(s) ds + \int_0^t k(t, \sigma)u(\sigma) d\sigma \\ &\leq f(t)w(t) + e(t, t)w(t) + \int_0^t \frac{\partial}{\partial t} e(t, s)w(s) ds + \int_0^t k(t, \sigma)w(\sigma) d\sigma \\ &\leq [f(t) + A(t)]w(t). \end{aligned} \quad (2.7)$$

The inequality (2.7) implies the estimate

$$w(t) \leq c \exp \left( \int_0^t [f(s) + A(s)] ds \right). \quad (2.8)$$

Using (2.8) in  $u(t) \leq w(t)$  we get the required inequality in (2.5).

Now we are in a position to formulate the main result of this section.

**Theorem 1.** Assume that

(i) the functions  $f$  and  $g$  in equation (1.4) satisfy the conditions

$$|f(t, y, z, u) - f(t, \bar{y}, \bar{z}, \bar{u})| \leq k[|y - \bar{y}| + |z - \bar{z}|] + |u - \bar{u}|, \quad (2.9)$$

$$|g(t, y, z) - g(t, \bar{y}, \bar{z})| \leq h(t, s)[|y - \bar{y}| + |z - \bar{z}|], \quad (2.10)$$

where  $k \geq 0$  is a constant and  $h \in C(R_+^2, R_+)$ ,

(ii) for  $\lambda$  as in (2.1)

(a) there exists a nonnegative constant  $\lambda$  such that  $\alpha < 1$  and

$$H(t) + \int_0^t H(s) ds \leq \alpha \exp(\lambda t), \quad (2.11)$$

for  $t \in R_+$ , where

$$H(t) = k \exp(\lambda t) + \int_0^t h(t, \sigma) \exp(\lambda \sigma) d\sigma, \quad (2.12)$$

(b) there exists a nonnegative constant  $\beta$  such that

$$\begin{aligned} & |y_0| + \left| f \left( t, 0, 0, \int_0^t g(t, \sigma, 0, 0) d\sigma \right) \right| \\ & + \int_0^t \left| f \left( s, 0, 0, \int_0^s g(s, \sigma, 0, 0) d\sigma \right) \right| ds \leq \beta \exp(\lambda t). \end{aligned} \quad (2.13)$$

Then the initial value problem (IVP for short) (1.4)-(1.5) has a unique solution on  $R_+$ .

**Proof.** Let  $y(t) \in S$  and define the operator  $T$  by

$$(Ty)(t) = y_0 + \int_0^t f \left( s, y(s), y'(s), \int_0^s g(s, \sigma, y(\sigma), y'(\sigma)) d\sigma \right) ds. \quad (2.14)$$

Differentiating both sides of (2.14) with respect to  $t$  we get

$$(Ty)'(t) = f \left( t, y(t), y'(t), \int_0^t g(t, \sigma, y(\sigma), y'(\sigma)) d\sigma \right). \quad (2.15)$$

First we shall show that  $Ty$  maps  $S$  into itself. Evidently  $Ty$  is continuous on  $R_+$  and  $Ty \in R^n$ . We verify that (2.1) is fulfilled. From (2.14), (2.15), using the hypotheses and (2.3) we have

$$\begin{aligned} & |(Ty)(t)| + |(Ty)'(t)| \\ & \leq |y_0| + \int_0^t \left| f \left( s, y(s), y'(s), \int_0^s g(s, \sigma, y(\sigma), y'(\sigma)) d\sigma \right) \right. \\ & \quad \left. - f \left( s, 0, 0, \int_0^s g(s, \sigma, 0, 0) d\sigma \right) \right| ds \\ & \quad + \int_0^t \left| f \left( s, 0, 0, \int_0^s g(s, \sigma, 0, 0) d\sigma \right) \right| ds \end{aligned}$$

$$\begin{aligned}
& + \left| f \left( t, y(t), y'(t), \int_0^t g(t, \sigma, y(\sigma), y'(\sigma)) d\sigma \right) \right. \\
& \quad \left. - f \left( t, 0, 0, \int_0^t g(t, \sigma, 0, 0) d\sigma \right) \right| \\
& \quad + \left| f \left( t, 0, 0, \int_0^t g(t, \sigma, 0, 0) d\sigma \right) \right| \\
& \leq \beta \exp(\lambda t) + \int_0^t \left\{ k[|y(s)| + |y'(s)|] + \int_0^s h(s, \sigma)[|y(\sigma)| + |y'(\sigma)|] d\sigma \right\} ds \\
& \quad + k[|y(t)| + |y'(t)|] + \int_0^t h(t, \sigma)[|y(\sigma)| + |y'(\sigma)|] d\sigma \\
& \leq \beta \exp(\lambda t) + |y|_S \left\{ \int_0^t H(s) ds + H(t) \right\} \\
& \leq [\beta + N\alpha] \exp(\lambda t). \tag{2.16}
\end{aligned}$$

From (2.16) it follows that  $Ty \in S$ . This proves that  $T$  maps  $S$  into itself.

Next, we verify that the operator  $T$  is a contraction map. Let  $y(t), z(t) \in S$ . From (2.14), (2.15) and using the hypotheses we have

$$\begin{aligned}
& |(Ty)(t) - (Tz)(t)| + |(Ty)'(t) - (Tz)'(t)| \\
& \leq \int_0^t \left| f \left( s, y(s), y'(s), \int_0^s g(s, \sigma, y(\sigma), y'(\sigma)) d\sigma \right) \right. \\
& \quad \left. - f \left( s, z(s), z'(s), \int_0^s g(s, \sigma, z(\sigma), z'(\sigma)) d\sigma \right) \right| ds \\
& \quad + \left| f \left( t, y(t), y'(t), \int_0^t g(t, \sigma, y(\sigma), y'(\sigma)) d\sigma \right) \right. \\
& \quad \left. - f \left( t, z(t), z'(t), \int_0^t g(t, \sigma, z(\sigma), z'(\sigma)) d\sigma \right) \right|
\end{aligned}$$

$$\begin{aligned}
& \leq \int_0^t \left\{ k [|y(s) - z(s)| + |y'(s) - z'(s)|] \right. \\
& + \int_0^s h(s, \sigma) [|y(\sigma) - z(\sigma)| + |y'(\sigma) - z'(\sigma)|] d\sigma \left. \right\} ds \\
& \quad + k [|y(t) - z(t)| + |y'(t) - z'(t)|] \\
& \quad + \int_0^t h(t, \sigma) [|y(\sigma) - z(\sigma)| + |y'(\sigma) - z'(\sigma)|] d\sigma \\
& \leq |y - z|_S \left\{ \int_0^t H(s) ds + H(t) \right\} \\
& \leq |y - z|_S \alpha \exp(\lambda t). \tag{2.17}
\end{aligned}$$

From (2.17) we observe that

$$|Ty - Tz|_S \leq \alpha |y - z|_S.$$

Since  $\alpha < 1$ , it follows from Banach fixed point theorem (see [5, p. 37]) that  $T$  has a unique fixed point in  $S$ . The fixed point of  $T$  is however a solution of IVP (1.4)-(1.5). The proof is complete.

**Remark 1.** The norm  $|\cdot|_S$  defined in (2.2) was first used by Bielecki [2] for proving global existence and uniqueness of solutions of ordinary differential equations. For a detailed discussion related to this topic, see [4] and the references cited therein.

The following theorem deals with the uniqueness of solutions of IVP (1.4)-(1.5) in the whole space  $R^n$ , without existence part.

**Theorem 2.** Assume that the functions  $f$  and  $g$  in equation (1.4) satisfy the conditions

$$|f(t, y, z, u) - f(t, \bar{y}, \bar{z}, \bar{u})| \leq d[|y - \bar{y}| + |z - \bar{z}|] + |u - \bar{u}|, \tag{2.18}$$

$$|g(t, y, z) - g(t, \bar{y}, \bar{z})| \leq r(t, s)[|y - \bar{y}| + |z - \bar{z}|], \tag{2.19}$$

where  $d$  is a nonnegative constant such that  $d < 1$  and for  $0 \leq s \leq t < \infty$ ,  $r(t, s), \frac{\partial}{\partial t} r(t, s) \in C(R_+^2, R_+)$ . Then the IVP (1.4)-(1.5) has at most one solution on  $R_+$ .

**Proof.** Let  $y_1(t)$  and  $y_2(t)$  be two solutions of IVP (1.4)-(1.5) and  $u(t) = |y_1(t) - y_2(t)| + |y_1'(t) - y_2'(t)|$ . Then by hypotheses we have

$$\begin{aligned}
u(t) &\leq \int_0^t \left| f \left( s, y_1(s), y_1'(s), \int_0^s g(s, \sigma, y_1(\sigma), y_1'(\sigma)) d\sigma \right) \right. \\
&\quad \left. - f \left( s, y_2(s), y_2'(s), \int_0^s g(s, \sigma, y_2(\sigma), y_2'(\sigma)) d\sigma \right) \right| ds \\
&\quad + \left| f \left( t, y_1(t), y_1'(t), \int_0^t g(t, \sigma, y_1(\sigma), y_1'(\sigma)) d\sigma \right) \right. \\
&\quad \left. - f \left( t, y_2(t), y_2'(t), \int_0^t g(t, \sigma, y_2(\sigma), y_2'(\sigma)) d\sigma \right) \right| \\
&\leq \int_0^t \left\{ d[|y_1(s) - y_2(s)| + |y_1'(s) - y_2'(s)|] \right. \\
&\quad \left. + \int_0^s r(s, \sigma) [|y_1(\sigma) - y_2(\sigma)| + |y_1'(\sigma) - y_2'(\sigma)|] d\sigma \right\} ds \\
&\quad + d[|y_1(t) - y_2(t)| + |y_1'(t) - y_2'(t)|] \\
&\quad + \int_0^t r(t, \sigma) [|y_1(\sigma) - y_2(\sigma)| + |y_1'(\sigma) - y_2'(\sigma)|] d\sigma. \tag{2.20}
\end{aligned}$$

From (2.20) we observe that

$$u(t) \leq \frac{1}{1-d} \int_0^t \left\{ du(s) + r(t, s)u(s) + \int_0^s r(s, \sigma)u(\sigma) d\sigma \right\} ds. \tag{2.21}$$

Now a suitable application of Lemma (with  $e(t, s) = k(t, s) = r(t, s)$  and  $c = 0$ ) to (2.21) yields

$$|y_1(t) - y_2(t)| + |y_1'(t) - y_2'(t)| \leq 0,$$

which implies  $y_1(t) = y_2(t)$  for  $t \in R_+$ . Thus there is at most one solution to the IVP (1.4)-(1.5) on  $R_+$ .

### 3 Boundedness and continuous dependence

In this section we shall study the boundedness of solutions of IVP (1.4)-(1.5) and the continuous dependence of solutions of equation (1.4) on the given initial data and the functions involved therein.

The following theorem contains the estimate on the solution of IVP (1.4)-(1.5).

**Theorem 3.** Assume that

$$|f(t, y, z, u)| \leq \gamma [|y| + |z|] + |u|, \quad (3.1)$$

$$|g(t, s, y, z)| \leq q(t, s) [|y| + |z|], \quad (3.2)$$

where  $\gamma$  is a nonnegative constant such that  $\gamma < 1$  and for  $0 \leq s \leq t < \infty$ ,  $q(t, s), \frac{\partial}{\partial t}q(t, s) \in C(R_+^2, R_+)$ . If  $y(t), t \in R_+$  is any solution of IVP (1.4)-(1.5), then

$$|y(t)| + |y'(t)| \leq \frac{|y_0|}{1-\gamma} \exp\left(\int_0^t \left[\frac{\gamma}{1-\gamma} + \bar{A}(s)\right] ds\right), \quad (3.3)$$

for  $t \in R_+$ , where

$$\bar{A}(t) = \frac{1}{1-\gamma} \left[ q(t, t) + \int_0^t \left\{ q(t, \sigma) + \frac{\partial}{\partial t}q(t, \sigma) \right\} d\sigma \right]. \quad (3.4)$$

**Proof.** Using the fact that  $y(t)$  is a solution of IVP (1.4)-(1.5) and the hypotheses we have

$$\begin{aligned} |y(t)| + |y'(t)| &\leq |y_0| + \int_0^t \left| f\left(s, y(s), y'(s), \int_0^s g(s, \sigma, y(\sigma), y'(\sigma)) d\sigma\right) \right| ds \\ &\quad + \left| f\left(t, y(t), y'(t), \int_0^t g(t, \sigma, y(\sigma), y'(\sigma)) d\sigma\right) \right| \\ &\leq |y_0| + \int_0^t \left\{ \gamma [|y(s)| + |y'(s)|] + \int_0^s q(s, \sigma) [|y(\sigma)| + |y'(\sigma)|] d\sigma \right\} ds \\ &\quad + \gamma [|y(t)| + |y'(t)|] + \int_0^t q(t, \sigma) [|y(\sigma)| + |y'(\sigma)|] d\sigma. \end{aligned} \quad (3.5)$$

From (3.5) we observe that

$$\begin{aligned} |y(t)| + |y'(t)| &\leq \frac{|y_0|}{1-\gamma} + \frac{1}{1-\gamma} \int_0^t \left\{ \gamma [|y(s)| + |y'(s)|] + q(t, s) [|y(s)| + |y'(s)|] \right. \\ &\quad \left. + \int_0^s q(s, \sigma) [|y(\sigma)| + |y'(\sigma)|] d\sigma \right\} ds. \end{aligned} \quad (3.6)$$

Now a suitable application of Lemma to (3.6) yields (3.3).

**Remark 2.** We note that the estimate obtained in (3.3) yields not only the bound on the solution  $y(t)$  of IVP (1.4)-(1.5) but also the bound on  $y'(t)$  for  $t \in R_+$ . If the estimate on the right hand side in (3.3) is bounded, then the solution  $y(t)$  of IVP (1.4)-(1.5) and also  $y'(t)$  are bounded on  $R_+$ .

The next theorem deals with the continuous dependence of solutions of equation (1.4) on given initial values.

**Theorem 4.** Assume that the functions  $f, g$  in equation (1.4) satisfy the conditions (2.18), (2.19). Let  $y_1(t)$  and  $y_2(t)$  be the solutions of equation (1.4) with the given initial conditions

$$y_1(0) = c_1, \tag{3.7}$$

and

$$y_2(0) = c_2, \tag{3.8}$$

respectively, where  $c_1, c_2$  are constants. Then

$$|y_1(t) - y_2(t)| + |y_1'(t) - y_2'(t)| \leq \frac{|c_1 - c_2|}{1-d} \exp \left( \int_0^t \left[ \frac{d}{1-d} + B(s) \right] ds \right), \tag{3.9}$$

for  $t \in R_+$ , where

$$B(t) = \frac{1}{1-d} \left[ r(t, t) + \int_0^t \left\{ r(t, \sigma) + \frac{\partial}{\partial t} r(t, \sigma) \right\} d\sigma \right]. \tag{3.10}$$

**Proof.** Let  $u(t) = |y_1(t) - y_2(t)| + |y_1'(t) - y_2'(t)|$  for  $t \in R_+$ . From the hypotheses we have

$$\begin{aligned} u(t) &\leq |c_1 - c_2| + \int_0^t \left| f \left( s, y_1(s), y_1'(s), \int_0^s g(s, \sigma, y_1(\sigma), y_1'(\sigma)) \right) \right. \\ &\quad \left. - f \left( s, y_2(s), y_2'(s), \int_0^s g(s, \sigma, y_2(\sigma), y_2'(\sigma)) \right) \right| ds \\ &\quad + \left| f \left( t, y_1(t), y_1'(t), \int_0^t g(t, \sigma, y_1(\sigma), y_1'(\sigma)) \right) \right. \\ &\quad \left. - f \left( t, y_2(t), y_2'(t), \int_0^t g(t, \sigma, y_2(\sigma), y_2'(\sigma)) \right) \right| \end{aligned}$$

$$\begin{aligned}
&\leq |c_1 - c_2| + \int_0^t \left\{ d[|y_1(s) - y_2(s)| + |y_1'(s) - y_2'(s)|] \right. \\
&\quad \left. + \int_0^s r(s, \sigma)[|y_1(\sigma) - y_2(\sigma)| + |y_1'(\sigma) - y_2'(\sigma)|] d\sigma \right\} ds \\
&\quad + d[|y_1(t) - y_2(t)| + |y_1'(t) - y_2'(t)|] \\
&\quad + \int_0^t r(t, \sigma)[|y_1(\sigma) - y_2(\sigma)| + |y_1'(\sigma) - y_2'(\sigma)|] d\sigma. \quad (3.11)
\end{aligned}$$

From (3.11) we observe that

$$u(t) \leq \frac{|c_1 - c_2|}{1-d} + \frac{1}{1-d} \int_0^t \left\{ du(s) + r(t, s)u(s) + \int_0^s r(s, \sigma)u(\sigma) d\sigma \right\} ds. \quad (3.12)$$

Now an application of Lemma to (3.12) yields the bound in (3.9), which shows the dependency of solutions of equation (1.4) on given initial values.

Next, we consider the IVP (1.4)-(1.5) and the corresponding IVP

$$z'(t) = F \left( t, z(t), z'(t), \int_0^t G(t, \sigma, z(\sigma), z'(\sigma)) d\sigma \right), z(0) = z_0, \quad (3.13)$$

for  $t \in R_+$ , where  $F \in C(R_+ \times R^n \times R^n \times R^n, R^n)$  and  $G \in C(R_+^2 \times R^n \times R^n, R^n)$ .

The following theorem deals with the continuous dependence of solutions of IVP (1.4)-(1.5) on the functions involved therein.

**Theorem 5.** Assume that the functions  $f, g$  in equation (1.4) satisfy the conditions (2.18), (2.19) and

$$\begin{aligned}
&|y_0 - z_0| + \int_0^t \left| f \left( s, z(s), z'(s), \int_0^s g(s, \sigma, z(\sigma), z'(\sigma)) d\sigma \right) \right. \\
&\quad \left. - F \left( s, z(s), z'(s), \int_0^s G(s, \sigma, z(\sigma), z'(\sigma)) d\sigma \right) \right| ds \\
&\quad + \left| f \left( t, z(t), z'(t), \int_0^t g(t, \sigma, z(\sigma), z'(\sigma)) d\sigma \right) \right.
\end{aligned}$$

$$-F \left( t, z(t), z'(t), \int_0^t G(t, \sigma, z(\sigma), z'(\sigma)) d\sigma \right) \Big| \leq \varepsilon, \quad (3.14)$$

where  $y_0, f, g$  and  $z_0, F, G$  are as in IVP (1.4)-(1.5) and IVP (3.13),  $\varepsilon \geq 0$  is a constant and  $z(t)$  is a solution of IVP (3.13). Then the solution  $y(t)$  of IVP (1.4)-(1.5) depends continuously on the functions involved therein.

**Proof.** Let  $u(t) = |y(t) - z(t)| + |y'(t) - z'(t)|$  for  $t \in R_+$ . From the hypotheses we have

$$\begin{aligned} u(t) &\leq |y_0 - z_0| + \int_0^t \left| f \left( s, y(s), y'(s), \int_0^s g(s, \sigma, y(\sigma), y'(\sigma)) d\sigma \right) \right. \\ &\quad \left. - f \left( s, z(s), z'(s), \int_0^s g(s, \sigma, z(\sigma), z'(\sigma)) d\sigma \right) \right| ds \\ &\quad + \int_0^t \left| f \left( s, z(s), z'(s), \int_0^s g(s, \sigma, z(\sigma), z'(\sigma)) d\sigma \right) \right. \\ &\quad \left. - F \left( s, z(s), z'(s), \int_0^s G(s, \sigma, z(\sigma), z'(\sigma)) d\sigma \right) \right| ds \\ &\quad + \left| f \left( t, y(t), y'(t), \int_0^t g(t, \sigma, y(\sigma), y'(\sigma)) d\sigma \right) \right. \\ &\quad \left. - f \left( t, z(t), z'(t), \int_0^t g(t, \sigma, z(\sigma), z'(\sigma)) d\sigma \right) \right| \\ &\quad + \left| f \left( t, z(t), z'(t), \int_0^t g(t, \sigma, z(\sigma), z'(\sigma)) d\sigma \right) \right. \\ &\quad \left. - F \left( t, z(t), z'(t), \int_0^t G(t, \sigma, z(\sigma), z'(\sigma)) d\sigma \right) \right| \\ &\leq \varepsilon + \int_0^t \left\{ d[|y(s) - z(s)| + |y'(s) - z'(s)|] \right. \\ &\quad \left. + \int_0^s r(s, \sigma) [|y(\sigma) - z(\sigma)| + |y'(\sigma) - z'(\sigma)|] d\sigma \right\} ds \end{aligned}$$

$$\begin{aligned}
& +d[|y(t) - z(t)| + |y'(t) - z'(t)|] \\
& + \int_0^t r(t, \sigma) [|y(\sigma) - z(\sigma)| + |y'(\sigma) - z'(\sigma)|] d\sigma. \quad (3.15)
\end{aligned}$$

From (3.15) we observe that

$$u(t) \leq \frac{\varepsilon}{1-d} + \frac{1}{1-d} \int_0^t \left\{ du(s) + r(t, s)u(s) + \int_0^s r(s, \sigma)u(\sigma) d\sigma \right\} ds. \quad (3.16)$$

Now a suitable application of Lemma to (3.16) yields

$$u(t) \leq \frac{\varepsilon}{1-d} \exp \left( \int_0^t \left[ \frac{d}{1-d} + B(s) \right] ds \right), \quad (3.17)$$

for  $t \in R_+$ , where  $B(t)$  is given by (3.10). From (3.17) it follows that the solutions of IVP (1.4)-(1.5) depends continuously on the functions involved therein.

**Remark 3.** We note that our approach to the study of IVP (1.4)-(1.5) is different from those in [3,5] and we believe that the results given here are of independent interest. We also note that the idea employed here can be extended to the study of higher order integrodifferential equation of the form

$$y^{(n)}(t) = f \left( t, y(t), \dots, y^{(n-1)}(t), y^{(n)}(t), Qy(t) \right), \quad (3.18)$$

with the given initial conditions

$$y^{(k)}(0) = c_k, k = 0, 1, \dots, n-1, \quad (3.19)$$

where

$$Qy(t) = \int_0^t g \left( t, \sigma, y(\sigma), \dots, y^{(n-1)}(\sigma), y^{(n)}(\sigma) \right) d\sigma. \quad (3.20)$$

Naturally, these considerations will make the analysis more complicated. However, the detailed discussion of such results is left to another place.

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