THE EDGE-TANGENT SPHERE OF A CIRCUMSCRIPTIBLE $n-$SIMPLEX

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Abstract. A $n-$simplex is circumscriptible if there is a sphere tangent to each of its $n(n+1)/2$ edges. We prove that the radius of the edge-tangent sphere is at least $\sqrt{\frac{n(n-1)}{2}}$ times the radius of its inscribed sphere. This settles affirmatively a part of a problem posed by the authors.

1. Introduction and Main Result

Every $n-$simplex has a circumscribed sphere passing through its $n+1$ vertices and an inscribed sphere tangent to each of its $n+1$ faces. A $n-$simplex is circumscriptible if there is a sphere tangent to each of its $n(n+1)/2$ edges. We call this the edge-tangent sphere of the $n-$simplex. However, it's not that every $n-$simplex $(n \geq 3)$ to have an edge-tangent sphere.

In 1995, Lin and Zhu [1] gave a sufficient and necessary condition for a simplex to have an edge-tangent sphere.

Theorem 1. *Supposing the edge lengths of an $n-$simplex $\Omega = P_0P_1P_2\cdots P_n$ are $P_iP_j = a_{ij}$ for $0 \leq i < j \leq n$. The $n-$simplex has an edge-tangent sphere if and only if there exist $x_i > 0$ with $0 \leq i \leq n$ satisfying $a_{ij} = x_i + x_j$ for $0 \leq i < j \leq n$.*

For the tetrahedron, Lin and Zhu ([2], see also [3, p. 252]) posed the following open problem that is settled affirmatively by Wu and Zhang [4].

Conjecture 1. *For any tetrahedron $\mathscr{P} = P_0P_1P_2P_3$ which has the edge-tangent sphere, denote ℓ be the radius of edge-tangent sphere of $\mathscr{P}$, r the radius of inscribed sphere, prove or disprove that*

$$\ell^2 \geq 3r^2.$$

As a generalization of this problem in tetrahedron, the authors concluded an analogous conjecture for the circumscriptible $n-$simplex in the end of [4].

Conjecture 2. *For an circumscriptible $n-$simplex with a circumscribed sphere of radius R, an inscribed sphere of radius r and an edge-tangent sphere of radius ℓ, prove or disprove that*

$$R \geq \sqrt{\frac{2n}{n-1}}\ell \geq nr. \tag{1}$$

The main purpose of this paper is affirmatively to give a proof of the right hand of inequality (1).

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Theorem 2. *For an circumscriptible $n-$simplex with an inscribed sphere of radius r and an edge-tangent sphere of radius ℓ,*

$$\ell^2 \geq \frac{n(n-1)}{2} r^2. \tag{2}$$

Let $\{A_0, A_1, \cdots, A_n\}$ denote the vertex set of an n-dimensional simplex Ω in the n-dimensional Euclidean space $\mathbb{E}^n$, r the radius of the inscribed sphere and ℓ the radius of the edge-tangent sphere of Ω, r_i the radius of the inscribed sphere of the $(n-1)$-dimensional face Ω_i spanned by the vertex set $\{A_0, \cdots, A_{i-1}, A_{i+1}, \cdots, A_n\}$ for $0 \leq i \leq n$, and ℓ_i the radius of the edge-tangent sphere of the $(n-1)$-dimensional face Ω_i.

2. Lemmas

In order to prove Theorem 2, we require several lemmas.

Lemma 1. ([5]) *Every face of a circumscriptible simplex is circumscriptible.*

Lemma 2. ([5]) *The radius of the edge-tangent sphere of Ω is given by*

$$\ell^2 = \frac{2(n-1)}{\left(\sum\limits_{i=0}^{n} \frac{1}{x_i}\right)^2 - (n-1) \sum\limits_{i=0}^{n} \frac{1}{x_i^2}}.$$

Lemma 3. *If $x_i > 0$ with $0 \leq i \leq n$, then*

$$\sum_{j=0}^{n} \left[\left(\sum_{i=0, i\neq j}^{n} \frac{1}{x_i} \right)^2 - (n-2) \sum_{i=0, i\neq j}^{n} \frac{1}{x_i}^2 \right] \geq n \left[\left(\sum_{i=0}^{n} \frac{1}{x_i} \right)^2 - (n-1) \sum_{i=0}^{n} \frac{1}{x_i^2} \right].$$

Equality holds if and only if $x_0 = x_1 = \cdots = x_n$.

Proof. For $x_i > 0$ with $0 \leq i \leq n$, we have

$$\begin{aligned}
&\sum_{j=0}^{n} \left[\left(\sum_{i=0, i\neq j}^{n} \frac{1}{x_i} \right)^2 - (n-2) \sum_{i=0, i\neq j}^{n} \frac{1}{x_i}^2 \right] \\
&- n \left[\left(\sum_{i=0}^{n} \frac{1}{x_i} \right)^2 - (n-1) \sum_{i=0}^{n} \frac{1}{x_i^2} \right] \\
&= n(3-n) \sum_{i=0}^{n} \frac{1}{x_i^2} + 2(n-1) \sum_{0 \leq i < j \leq n} \frac{1}{x_i x_j} \\
&- n(2-n) \sum_{i=0}^{n} \frac{1}{x_i^2} - 2n \sum_{0 \leq i < j \leq n} \frac{1}{x_i x_j} \\
&= \sum_{0 \leq i < j \leq n} \left(\frac{1}{x_i} - \frac{1}{x_j} \right)^2 \geq 0.
\end{aligned}$$

Thus, the proof of Lemma 3 is completed. □

According to Lemma 1, we consider the relation of ℓ and ℓ_i with $0 \le i \le n$, and give the following result.

Lemma 4. *For any $n \in \mathbb{N}$ with $n \ge 3$, we have*

$$(n-2)\sum_{i=0}^{n}\frac{1}{\ell_i^2} \ge n(n-1)\cdot\frac{1}{\ell^2}. \tag{3}$$

Equality is valid if and only if Ω is regular.

Proof. From Lemma 2 and 3, it is deduced easily that

$$\begin{aligned}&2\left[(n-2)\sum_{i=0}^{n}\frac{1}{\ell_i^2} - n(n-1)\cdot\frac{1}{\ell^2}\right]\\ &=\sum_{j=0}^{n}\left[\left(\sum_{i=0,i\neq j}^{n}\frac{1}{x_i}\right)^2 - (n-2)\sum_{i=0,i\neq j}^{n}\frac{1}{x_i}^2\right]\\ &-n\left[\left(\sum_{i=0}^{n}\frac{1}{x_i}\right)^2 - (n-1)\sum_{i=0}^{n}\frac{1}{x_i^2}\right] \ge 0.\end{aligned}$$

This is immediately to get inequality (3). Therefore, Lemma 4 is proved. ☐

Lemma 5. ([6])*If $n \in \mathbb{N}$ for $n \ge 3$, then*

$$\frac{n-1}{r^2} \ge \sum_{i=0}^{n}\frac{1}{r_i^2}.$$

Equality is valid if and only if Ω is regular.

3. The Proof of Theorem 2

We will verify Theorem 2 by mathematical induction.

Proof. In the case of $n = 3$, the required result is proved in [4].
Assume inequality (2) holds when $n = k$ with $k \ge 3$, then in circumscriptible $k-$simplex $\Omega_i = A_0 \cdots A_{i-1}A_{i+1}\cdots A_{k+1}$ for $0 \le i \le k+1$, we have

$$\ell_i^2 \ge \frac{k(k-1)}{2}\cdot r_i^2. \tag{4}$$

For circumscriptible $(k+1)-$simplex $\Omega = A_0A_1\cdots A_{k+1}$, from Lemma 4-5 and inequality (4), one reveals

$$\begin{aligned}\frac{2}{k-1}\cdot\frac{1}{r^2} = \frac{2}{k(k-1)}\cdot\frac{k}{r^2} &\ge \frac{2}{k(k-1)}\sum_{i=0}^{k+1}\frac{1}{r_i^2}\\ &\ge \sum_{i=0}^{k+1}\frac{1}{\ell_i^2} \ge \frac{k(k+1)}{k-1}\cdot\frac{1}{\ell^2}.\end{aligned}$$

That find immediately inequality (2) when $n = k+1$.

Hence, by induction, it is showed that inequality (2) holds for all $n \ge 3$. ☐

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