# ON AN INEQUALITY OF DIANANDA, IV

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Abstract. In this paper, we extend the results in part I-III on certain inequalities involving the weighted power means as well as the symmetric means. These inequalities can be largely viewed as concerning the bounds for ratios of differences of means and can be traced back to the work of Diananda.

## 1. Introduction

Let $M_{n,r}(\mathbf{x};\mathbf{q})$ be the weighted power means: $M_{n,r}(\mathbf{x};\mathbf{q}) = (\sum_{i=1}^n q_i x_i^r)^{\frac{1}{r}}$, where $M_{n,0}(\mathbf{x};\mathbf{q})$ denotes the limit of $M_{n,r}(\mathbf{x};\mathbf{q})$ as $r \to 0^+$, $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{q} = (q_1, \ldots, q_n)$ and $q_i > 0$ $(1 \le i \le n)$ are positive real numbers with $\sum_{i=1}^n q_i = 1$. In this paper, we let $q = \min q_i$ and unless otherwise specified, we assume $0 \le x_1 < x_2 < \cdots < x_n$.

Let $k \in \{0, 1, \ldots, n\}$, the $k$-th symmetric function $E_{n,k}$ of $\mathbf{x}$ and its mean $P_{n,k}$ are defined by

$$E_{n,r}(\mathbf{x}) = \sum_{1 \le i_1 < \cdots < i_r \le n} \prod_{j=1}^r x_{i_j}, P^r_{n,r}(\mathbf{x}) = \frac{E_{n,r}(\mathbf{x})}{\binom{n}{r}}, 1 \le r \le n; \quad E_{n,0} = P_{n,0} = 1.$$

We define $A_n(\mathbf{x};\mathbf{q}) = M_{n,1}(\mathbf{x};\mathbf{q}), G_n(\mathbf{x}) = M_{n,0}(\mathbf{x};\mathbf{q}), H_n(\mathbf{x};\mathbf{q}) = M_{n,-1}(\mathbf{x};\mathbf{q})$ and we shall write $M_{n,r}$ for $M_{n,r}(\mathbf{x};\mathbf{q})$ and similarly for other means when there is no risk of confusion.

For a real number $\alpha$ and mutually distinct numbers $r, s, t$, we define

$$\Delta_{r,s,t,\alpha} = \Big|\frac{M^\alpha_{n,r} - M^\alpha_{n,t}}{M^\alpha_{n,r} - M^\alpha_{n,s}}\Big|,$$

where we interpret $M^0_{n,r} - M^0_{n,s}$ as $\ln M_{n,r} - \ln M_{n,s}$. We also define $\Delta_{r,s,t}$ to be $\Delta_{r,s,t,1}$.

For $r > s > t \ge 0, \alpha > 0$, the author studied in [5]-[7] inequalities of the following two types:

$$C_{r,s,t}\big((1-q)^\alpha\big) \ge \Delta_{r,s,t,\alpha}, \tag{1.1}$$

and

$$\Delta_{r,s,t,\alpha} \ge C_{r,s,t}(q^\alpha), \tag{1.2}$$

where

$$C_{r,s,t}(x) = \frac{1 - x^{1/t - 1/r}}{1 - x^{1/s - 1/r}}, \quad t > 0; \qquad C_{r,s,0}(x) = \frac{1}{1 - x^{1/s - 1/r}}.$$

For any set $\{a, b, c\}$ with $a, b, c$ mutually distinct and non-negative, we let $r = \max\{a, b, c\}, t = \min\{a, b, c\}, s = \{a, b, c\} \backslash \{r, t\}$. By saying (1.1) (resp. (1.2)) holds for the set $\{a, b, c\}, \alpha > 0$ we mean (1.1) (resp. (1.2)) holds for $r > s > t \ge 0, \alpha > 0$.

A result of P. Diananda ([3], [4]) (see also [2], [12]) shows that (1.1) and (1.2) hold for $\{1, 1/2, 0\}$, $\alpha = 1$. When $\alpha = 1$, the sets $\{a, b, 1\}$'s for which (1.1) or (1.2) holds have been completely determined by the author in [5]-[7]. Moreover, it is shown in [7] that (1.2) doesn't hold in general unless $0 \in \{a, b, c\}$.

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For the consideration of more general $\alpha$'s, we first restrict our attention to the case $\{a, b, 0\}$. This is partially because of the following result in [6] (note there is a typo in the original statement though) :

**Theorem 1.1** ([6, Theorem 3.2]). *Let $r > s > 0$. If (1.1) holds for $\{r, s, 0\}$, $\alpha > 0$, then it also holds for $\{r, s, 0\}$, $k\alpha$ with $0 < k < 1$. If (1.2) holds for $\{r, s, 0\}$, $\alpha > 0$, then it also holds for $\{r, s, 0\}$, $k\alpha$ with $k > 1$.*

Moreover, for the unweighted case $q_1 = q_2 = \cdots = q_n = 1/n$, the author [7, Theorem 3.5] has shown that (1.1) holds for $\{1, 1/r, 0\}$ with $\alpha = n/r$ when $r \geq 2$ and (1.2) holds for $\{1, 1/r, 0\}$ with $\alpha = n/\big((n-1)r\big)$ when $1 < r \leq 2$. The values of $\alpha$'s are best possible here, namely, no larger $\alpha$'s can make (1.1) hold for $\{1, 1/r, 0\}$ and similarly no smaller $\alpha$'s can make (1.2) hold for $\{1, 1/r, 0\}$.

More generally, for arbitrary weights $\{q_i\}$'s, by using similar arguments as in [7], one sees that the largest $\alpha$ that can make (1.1) hold for $\{1, 1/r, 0\}$ is $1/(qr)$ and the smallest $\alpha$ that can make (1.2) hold for $\{1, 1/r, 0\}$ is $1/\big((1-q)r\big)$.

In Section 2, we will extend Theorem 3.5 of [7] to the case of arbitrary weights. Namely, we will prove

**Theorem 1.2.** *For $r \geq 2, 0 < p \leq 1/q$,*

$$A_n^p \leq (1-q)^{\frac{(r-1)p}{r}} M_{n,r}^p + \Big(1 - (1-q)^{\frac{(r-1)p}{r}}\Big) G_n^p. \tag{1.3}$$

*For $1 < r \leq 2, p \geq 1/(1-q)$,*

$$A_n^p \geq q^{\frac{(r-1)p}{r}} M_{n,r}^p + \Big(1 - q^{\frac{(r-1)p}{r}}\Big) G_n^p. \tag{1.4}$$

We note here by a change of variables: $x_i \to x_i^{1/r}$, one can easily rewrite (1.3) (resp. (1.4)) in the form of (1.1) (resp. (1.2)).

After studying (1.1) and (1.2) for the set $\{a, b, 0\}$, we move on in Section 3 to the case $\{a, b, c\}$ with $\min(a, b, c) > 0$. Our remark earlier allows us to focus on (1.1) only. In this case, we can recast (1.1), via a change of variables, as the following

$$A_n^\alpha \leq \lambda_{r,s}\big((1-q)^\alpha\big) M_{n,r}^\alpha + \Big(1 - \lambda_{r,s}\big((1-q)^\alpha\big)\Big) M_{n,s}^\alpha \tag{1.5}$$

where $\alpha > 0, r > 1 > s > 0$ and

$$\lambda_{r,s}(x) = 1 - \frac{1}{C_{r,1,s}(x)}.$$

We will show that (1.5) holds for all $n$ if and only if it holds for the case $n = 2$. Based on this, we will then be able to prove (1.5) for certain $r, s, \alpha$'s satisfying a natural condition.

One certainly expects analogues of (1.1) and (1.2) hold with weighted power means replaced by the symmetric means, one such example is given by the following result in [10]:

**Theorem 1.3.** *Let $q_i = 1/n$, then for any integer $2 \leq k \leq n$,*

$$\Big(\sum_{i=1}^n x_i\Big)^k \leq \Big(n^k - \tilde{\lambda}_{2,k}(n)\binom{n}{k}\Big) M_{n,2}^k + \tilde{\lambda}_{2,k}(n) E_{n,k},$$

*where for $2 \leq r \leq k \leq n$ (with $\binom{n-1}{n} = 0$ here),*

$$\tilde{\lambda}_{r,k}(n) = \frac{n^k(1-\frac{1}{n})^{k/r} - (n-1)^k}{\binom{n}{k}(1-\frac{1}{n})^{k/r} - \binom{n-1}{k}}.$$

As was pointed out in [7], the proof given in [10] for the above theorem is not quite correct. In Section 4, we will study inequalities involving the symmetric means and our results include a proof of Theorem 1.3.

## 2. Proof of Theorem 1.2

In view of Theorem 1.1, one only needs to prove (1.3) for $p = 1/q$ and similarly (1.4) for $p = 1/(1-q)$. In this proof we assume that $0 < x_1 \leq \cdots \leq x_n$. The case $x_1 = 0$ will follow by taking the limit. We first prove (1.3) and we define

$$f(\mathbf{x}) = \frac{A_n^{1/q} - (1-q)^{\frac{r-1}{qr}} M_{n,r}^{1/q}}{G_n^{1/q}}.$$

If $x_1 = \cdots = x_n$ then $f = 0$ otherwise we may assume $n \geq 2$ and $0 < x = x_1 = \cdots = x_k < x_{k+1}$ for some $1 \leq k < n$, then

$$\frac{\partial f}{\partial x} = \sum_{i=1}^{k} \frac{\partial f}{\partial x_i}.$$

We want to show that the right-hand side above is non-negative. It suffices to show each single term in the sum is non-negative. Without loss of generality, we now show that

$$\frac{\partial f}{\partial x_1} \geq 0.$$

We have

$$\frac{q x_1 G_n^{1/q}}{q_1} \frac{\partial f}{\partial x_1} = A_n^{1/q-1}\Big(x_1 - A_n\Big) - (1-q)^{\frac{r-1}{qr}} M_{n,r}^{1/q-r}\Big(x_1^r - M_{n,r}^r\Big).$$

Now we set

$$y(r) = \Big(\frac{(q_1 - q)x_1^r + \sum_{i=2}^n q_i x_i^r}{1-q}\Big)^{1/r},$$

so that

$$\begin{aligned} A_n^{1/q-1}\Big(x_1 - A_n\Big) &= (1-q)\Big(qx_1 + (1-q)y(1)\Big)^{1/q-1}\Big(x_1 - y(1)\Big) \\ &\geq (1-q)\Big(qx_1 + (1-q)y(r)\Big)^{1/q-1}\Big(x_1 - y(r)\Big). \end{aligned}$$

Hence

$$\frac{qG_n^{1/q}}{q_1(1-q)^{1/q}x_1^{1/q-1}} \frac{\partial f}{\partial x_1} \geq \Big(\frac{q}{1-q} + \frac{y(r)}{x_1}\Big)^{1/q-1}\Big(1 - \frac{y(r)}{x_1}\Big) - \Big(\frac{q}{1-q} + \frac{y^r(r)}{x_1^r}\Big)^{\frac{1}{qr}-1}\Big(1 - \frac{y^r(r)}{x_1^r}\Big).$$

We want to show the right-hand side expression above is non-negative and by setting $z = y(r)/x_1$, this is equivalent to showing that

$$g(z,q) = \frac{(\frac{q}{1-q} + z)^{1/q-1}(z-1)}{(\frac{q}{1-q} + z^r)^{\frac{1}{qr}-1}(z^r - 1)} \leq 1,$$

for $z \geq 1, 0 \leq q \leq 1/2$ and calculation yields that

$$\begin{aligned} &\frac{\Big((\frac{q}{1-q} + z^r)^{\frac{1}{qr}-1}(z^r - 1)\Big)^2}{(\frac{q}{1-q} + z)^{1/q-2}(\frac{q}{1-q} + z^r)^{\frac{1}{qr}-2}} \frac{\partial g}{\partial z} \\ = &\frac{1}{q}\Big((z - 1 + \frac{q}{1-q})(z^r - 1)(z^r + \frac{q}{1-q}) - (z + \frac{q}{1-q})(z^r - z^{r-1})(z^r + \frac{rq}{1-q} - 1)\Big). \end{aligned}$$

We now set $s = q/(1-q)$ with $0 \leq s \leq 1$ and we consider

$$a(z,s) = (z - 1 + s)(z^r - 1)(z^r + s) - (z + s)(z^r - z^{r-1})(z^r + rs - 1).$$

By Cauchy's mean value theorem,

$$\frac{\partial^2 a}{\partial s^2}=2\Big(z^r-1-rz^{r-1}(z-1)\Big)\leq 0.$$

It follows that for $0\leq s\leq 1$,

$$a(z,s)\geq \min\{a(z,0),a(z,1)\}=\min\{0,a(z,1)\}.$$

It follows from the discussion in [7] (see the function $a(y,1)$ defined in the proof of Theorem 3.5 there) that $a(z,1)\geq 0$. This implies that $a(z,s)\geq 0$ so that $g(z,q)$ is an increasing function of $z$ and we then deduce that

$$g(z,q)\leq \lim_{z\to+\infty} g(z,q)=1.$$

This shows that $\frac{\partial f}{\partial x_1}\geq 0$ and hence $\frac{\partial f}{\partial x}\geq 0$ and by letting $x\to x_{k+1}$ and repeating the above argument, we conclude that $f(\mathbf{x})\leq f(x_n,x_n,\ldots,x_n)=1-(1-q)^{\frac{r-1}{qr}}$ which completes the proof for (1.3).

Now, to prove (1.4), we consider

$$h(\mathbf{x})=\frac{A_n^{1/(1-q)}-q^{\frac{r-1}{(1-q)r}}M_{n,r}^{1/(1-q)}}{G_n^{1/(1-q)}}.$$

Similar to our discussion above, it suffices to show $\partial h/\partial x_n\geq 0$. Now

$$\frac{x_nG_n^{1/(1-q)}(1-q)}{q_n}\frac{\partial h}{\partial x_n}=A_n^{1/(1-q)-1}\Big(x_n-A_n\Big)-q^{\frac{r-1}{(1-q)r}}M_{n,r}^{1/(1-q)-r}\Big(x_n^r-M_{n,r}^r\Big).$$

Now we set

$$w(r)=\Big(\frac{(q_n-q)x_n^r+\sum_{i=1}^{n-1}q_ix_i^r}{1-q}\Big)^{1/r},$$

so that

$$\begin{aligned}A_n^{1/(1-q)-1}\Big(x_n-A_n\Big) &= (1-q)\Big(qx_n+(1-q)w(1)\Big)^{1/(1-q)-1}\Big(x_n-w(1)\Big)\\ &\geq (1-q)\Big(qx_n+(1-q)w(r)\Big)^{1/(1-q)-1}\Big(x_n-w(r)\Big).\end{aligned}$$

where the inequality follows from the observation that the function

$$z\mapsto\Big(qx_n+(1-q)z\Big)^{1/(1-q)-1}(x_n-z)$$

is decreasing for $0<z<x_n$.

We then deduce that

$$\begin{aligned}&\frac{x_nG_n^{1/(1-q)}}{q_nq^{1/(1-q)-1}w^{1/(1-q)}(r)}\frac{\partial h}{\partial x_n}\\ \geq\ &\Big(\frac{x_n}{w(r)}+\frac{1-q}{q}\Big)^{1/(1-q)-1}\Big(\frac{x_n}{w(r)}-1\Big)-\Big(\frac{x_n^r}{w^r(r)}+\frac{1-q}{q}\Big)^{\frac{1}{(1-q)r}-1}\Big(\frac{x_n^r}{w^r(r)}-1\Big).\end{aligned}$$

By proceeding similarly as in the proof of (1.3) above, one is then able to establish (1.4) and we shall omit all the details here.

## 3. A General Discussion on (1.5)

**Theorem 3.1.** *For fixed $\alpha > 0, r > 1 > s > 0$, (1.5) holds for all $n$ if and only if it holds for $n = 2$.*

*Proof.* We consider the function

$$f_n(\mathbf{x}; \mathbf{q}, q) := \lambda_{r,s}\big((1-q)^{\alpha}\big)M^{\alpha}_{n,r} + \Big(1 - \lambda_{r,s}\big((1-q)^{\alpha}\big)\Big)M^{\alpha}_{n,s} - A^{\alpha}_n.$$

The theorem asserts that in order to show $f_n \geq 0$, it suffices to check the case $n = 2$. To see this, we may assume by homogeneity that $0 \leq x_1 \leq x_2 \leq \cdots \leq x_{n-1} \leq x_n = 1$ and we let $\mathbf{a} = (a_1, \ldots, a_n) \in [0,1]^{n-1}$ be the point in which the absolute minimum of $f_n$ is reached. We may assume that $0 \leq a_1 < a_2 < \cdots < a_{n-1} < a_n = 1$ for otherwise if $a_i = a_{i+1}$ for some $1 \leq i \leq n-1$, by combining $a_i$ with $a_{i+1}$ and $q_i$ with $q_{i+1}$, and noticing that $\lambda_{r,s}(x)$ is an increasing function of $x$ by Lemma 2.1 of [6], we have

$$f_n(\mathbf{a}; \mathbf{q}, q) \geq f_{n-1}(\mathbf{a}'; \mathbf{q}', q'),$$

where $\mathbf{a}' = (a_1, \ldots, a_{i-1}, a_i + a_{i+1}, a_{i+2}, \ldots, a_n), \mathbf{q}' = (q_1, \ldots, q_{i-1}, q_i + q_{i+1}, q_{i+2}, \ldots, q_n), q' = \min(q_1, \ldots, q_{i-1}, q_i + q_{i+1}, q_{i+2}, \ldots, q_n)$. We can then reduce the determination of the absolute minimum of $f_n$ to that of $f_{n-1}$ with different weights.

If $\mathbf{a}$ is a boundary point of $[0,1]^{n-1}$, then $a_1 = 0$, and in this case we show that $f_n(\mathbf{a}; \mathbf{q}_n, q) \geq 0$ follows from $f_{n-1}(\mathbf{a}''; \mathbf{q}'', q) \geq 0$, where $\mathbf{a}'' = (a_2, \ldots, a_n), \mathbf{q}'' = (q_2, \ldots, q_n)/(1-q_1)$. On writing $f_{n-1}(\mathbf{a}''; \mathbf{q}'', q) \geq 0$ explicitly, we get

$$\Big(\sum_{i=2}^{n} q_i a_i\Big)^{\alpha} \leq \lambda_{r,s}\big((1-q)^{\alpha}\big)(1-q_1)^{\alpha}M^{\alpha}_{n-1,r}(\mathbf{a}''; \mathbf{q}'') + \Big(1 - \lambda_{r,s}\big((1-q)^{\alpha}\big)\Big)(1-q_1)^{\alpha}M^{\alpha}_{n-1,s}(\mathbf{a}''; \mathbf{q}'').$$

Meanwhile, $f_n(\mathbf{a}; \mathbf{q}_n, q) \geq 0$ is equivalent to

$$\Big(\sum_{i=2}^{n} q_i a_i\Big)^{\alpha} \leq \lambda_{r,s}\big((1-q)^{\alpha}\big)(1-q_1)^{\alpha/r}M^{\alpha}_{n-1,r}(\mathbf{a}''; \mathbf{q}'') + \Big(1 - \lambda_{r,s}\big((1-q)^{\alpha}\big)\Big)(1-q_1)^{\alpha/s}M^{\alpha}_{n-1,s}(\mathbf{a}''; \mathbf{q}'').$$

Thus, it amounts to show that

$$\begin{aligned} & \lambda_{r,s}\big((1-q)^{\alpha}\big)(1-q_1)^{\alpha}M^{\alpha}_{n-1,r}(\mathbf{a}''; \mathbf{q}'') + \Big(1 - \lambda_{r,s}\big((1-q)^{\alpha}\big)\Big)(1-q_1)^{\alpha}M^{\alpha}_{n-1,s}(\mathbf{a}''; \mathbf{q}'') \\ \leq\ & \lambda_{r,s}\big((1-q)^{\alpha}\big)(1-q_1)^{\alpha/r}M^{\alpha}_{n-1,r}(\mathbf{a}''; \mathbf{q}'') + \Big(1 - \lambda_{r,s}\big((1-q)^{\alpha}\big)\Big)(1-q_1)^{\alpha/s}M^{\alpha}_{n-1,s}(\mathbf{a}''; \mathbf{q}''), \end{aligned}$$

which is equivalent to

$$\frac{\Big(1 - \lambda_{r,s}\big((1-q)^{\alpha}\big)\Big)\lambda_{r,s}\big((1-q_1)^{\alpha}\big)}{\Big(1 - \lambda_{r,s}\big((1-q_1)^{\alpha}\big)\Big)\lambda_{r,s}\big((1-q)^{\alpha}\big)}M^{\alpha}_{n-1,s} \leq M^{\alpha}_{n-1,r}.$$

Now the above inequality follows from $M_{n-1,s} \leq M_{n-1,r}$ and

$$\Big(1 - \lambda_{r,s}\big((1-q)^{\alpha}\big)\Big)\lambda_{r,s}\big((1-q_1)^{\alpha}\big) \leq \Big(1 - \lambda_{r,s}\big((1-q_1)^{\alpha}\big)\Big)\lambda_{r,s}\big((1-q)^{\alpha}\big),$$

since $\lambda_{r,s}(x)$ is an increasing function of $x$.

Thus $f_n(\mathbf{a}; \mathbf{q}_n, q) \geq 0$ follows from $f_{n-1}(\mathbf{a}''; \mathbf{q}'', q) \geq 0$. Moreover, on writing $q'' = \min(q_2/(1-q_1), \ldots, q_n/(1-q_1))$ and noticing that $q'' > q$, we deduce that $f_{n-1}(\mathbf{a}''; \mathbf{q}'', q) \geq f_{n-1}(\mathbf{a}''; \mathbf{q}'', q'')$. Hence the the determination of $f_n \geq 0$ can be reduced to the determination of $f_{n-1} \geq 0$ with different weights.

It remains to show the case $a_1 > 0$, so that $\mathbf{a}$ is an interior point of $[0,1]^{n-1}$. In this case we have

$$\nabla f_n(\mathbf{a}; \mathbf{q}, q) = 0.$$

Thus $a_1, \dots, a_{n-1}$ solve the equation

$$g(x) = \lambda_{r,s}\big((1-q)^{\alpha}\big)M_{n,r}^{\alpha-r}x^{r-1} + \Big(1-\lambda_{r,s}\big((1-q)^{\alpha}\big)\Big)M_{n,s}^{\alpha-s}x^{s-1} - A_n^{\alpha-1} = 0.$$

Note that

$$f_n(\mathbf{a};\mathbf{q},q) = \sum_{i=1}^{n} q_i a_i g(a_i) = q_n a_n g(a_n).$$

Thus if $g(a_n) \geq 0$ then $f_n \geq 0$. If $g(a_n) < 0$, we note that $g(x) = 0$ can have at most two roots in $(0, a_n]$ since it is easy to see that $g'(x) = 0$ can have at most one positive root. As $\lim_{x\to 0^+} g(x) = +\infty$, this further implies that $g(x) = 0$ has only one root in $(0, a_n]$, so that we only need to show $f_n \geq 0$ for the case $n = 2$ and this completes the proof. □

In what follows, we will apply Theorem 3.1 to establish (1.5) for certain $r, s, \alpha$'s. Before we proceed, we note that there is a natural condition to be satisfied by $r, s, \alpha$ in order for (1.5) to hold. Namely, if we take $n = 2$ and rewrite it as

$$\frac{M_{2,r}^{\alpha} - M_{2,s}^{\alpha}}{M_{2,r}^{\alpha} - A_2^{\alpha}} \leq \frac{1}{1-\lambda_{r,s}\big((1-q)^{\alpha}\big)}.$$

On taking $x_1 \to x_2$, we conclude that

$$\frac{r-s}{r-1} \leq \frac{1}{1-\lambda_{r,s}\big((1-q)^{\alpha}\big)}. \tag{3.1}$$

Before we prove our next result, we need two lemmas:

**Lemma 3.1.** *Fix $u < 0$, the function*

$$f(t) = \frac{1-t}{1-t^u}, \quad 0 < t \neq 1$$

*is concave for $u < -1$ and is convex for $-1 < u < 0$.*

*Proof.* Calculation yields that

$$f''(t) = \frac{ut^{u-2}}{(1-t^u)^3}g(t),$$

where

$$g(t) = -(u-1)t^{u+1} + (1+u)t^u - (1+u)t + u - 1.$$

Now

$$g''(t) = u(u+1)(u-1)t^{u-2}(1-t).$$

Thus if $-1 < u < 0$, then $g''(t) > 0$ for $0 < t < 1$ and $g''(t) < 0$ for $t > 1$. Since $g'(1) = 0$, this implies that $g'(t) < 0$ for $0 < t \neq 1$. As $g(1) = 0$, we then conclude that $g(t) > 0$ for $0 < t < 1$ and $g(t) < 0$ for $t > 1$. It follows from this that $f(t)$ is convex for $-1 < u < 0$ and the other assertion can be shown similarly, which completes the proof. □

**Lemma 3.2.** *For $r > 1, 0 < q_1, q_2 < 1, q_1 + q_2 = 1$, the function*

$$f(t) = (q_1 t^{1/(r-1)} + q_2)^{\alpha-1}(q_1 t^{r/(r-1)} + q_2)^{1-\alpha/r}$$

*is convex for $t > 0$ when $r \geq 2, 1 > \alpha > 0$ or $\alpha = 1, r > 1$.*

*Proof.* Direct calculation shows that

$$f''(t) = \frac{q_1}{r-1}(q_1 t^{1/(r-1)} + q_2)^{\alpha-3}(q_1 t^{r/(r-1)} + q_2)^{-\alpha/r-1}t^{\frac{1}{r-1}-2}g(t)$$

where

$$\begin{aligned} g(t) &= \frac{(1-\alpha)(r-2)}{r-1}q_1^2q_2t^{2+\frac{2}{r-1}}+\frac{(1-\alpha)(r-\alpha)}{r-1}q_1q_2^2t^{2+\frac{1}{r-1}}+\frac{r-\alpha}{r-1}q_1^2q_2t^{1+\frac{2}{r-1}} \\ &+2\Big(1-\frac{(1-\alpha)^2}{r-1}\Big)q_1q_2^2t^{1+\frac{1}{r-1}}+\frac{r-\alpha}{r-1}q_2^3t+\frac{(1-\alpha)(r-\alpha)}{r-1}q_1q_2^2t^{\frac{1}{r-1}}+\frac{(1-\alpha)(r-2)}{r-1}q_2^3. \end{aligned}$$

One then easily deduce the assertion of the lemma from the above expression of $g(t)$ and this completes the proof. □

**Theorem 3.2.** *Inequality* (1.5) *holds for the cases* $r\geq 2, 1>\alpha>0$ *or* $\alpha=1, r>1, r+s\geq 2$, *provided that* (3.1) *holds with strict inequality.*

*Proof.* By Theorem 3.1, it suffices to prove the theorem for the case $n=2$. We write $\lambda=\lambda_{r,s}\big((1-q)^\alpha\big)$ for short in this proof. What we need to prove is the following:

$$(q_1x_1+q_2x_2)^\alpha\leq\lambda(q_1x_1^r+q_2x_2^r)^{\alpha/r}+(1-\lambda)(q_1x_1^s+q_2x_2^s)^{\alpha/s}.$$

Without loss of generality, we may assume that $q_1=q, q_2=1-q$ and define

$$f(t)=(q_1t^s+q_2)^{-\alpha/s}\Big((q_1t+q_2)^\alpha-\lambda(q_1t^r+q_2)^{\alpha/r}\Big),$$

so that we need to show that $f(t)\leq f(1)$ for $t\geq 0$. We have

$$\begin{aligned} f'(t) &= \alpha q_1q_2(q_1t^s+q_2)^{-\alpha/s-1}\Big((q_1t+q_2)^{\alpha-1}(1-t^{s-1})+\lambda(q_1t^r+q_2)^{\alpha/r-1}(t^{s-1}-t^{r-1})\Big) \\ &= \alpha q_1q_2(q_1t^s+q_2)^{-\alpha/s-1}(q_1t^r+q_2)^{\alpha/r-1}(1-t^{s-1})g(t^{r-1}), \end{aligned}$$

where

$$g(t)=(q_1t^{1/(r-1)}+q_2)^{\alpha-1}(q_1t^{r/(r-1)}+q_2)^{1-\alpha/r}+\lambda\frac{1-t}{1-t^{(s-1)/(r-1)}}-\lambda.$$

By Lemma 3.1 and Lemma 3.2, we see that $g(t)$ is a strictly convex function for $r\geq 2, 1>\alpha>0$ or $\alpha=1, r>1, r+s\geq 2$. Note that by our assumption (3.1) with strict inequality,

$$\lim_{t\to 1}g(t)=1+\lambda\frac{r-s}{s-1}<0.$$

On the other hand, note that $\lambda$ satisfies

$$(1-q)^\alpha=\lambda(1-q)^{\alpha/r}+(1-\lambda)(1-q)^{\alpha/s},$$

so that

$$\lim_{t\to 0^+}g(t)=(1-q)^{\alpha-\alpha/r}-\lambda=(1-\lambda)(1-q)^{\alpha/s-\alpha/r}>0.$$

As $g(t)$ is strictly convex, this implies that there are exactly two roots $t_1, t_2$ of $g(t)=0$ satisfying that $t_1\in(0,1)$ and $t_2>1$. Note further that $f(0)=f(1)$ and $\lim_{t\to 0^+}f'(t)<0$, which implies that $f(t)\leq f(1)$ for $0\leq t\leq 1$. Similarly, we note that $f'(t)<0$ for $t\in(1,1+\epsilon)$ with $\epsilon>0$ small enough. This combined with the observation that $\lim_{t\to+\infty}f(t)\leq f(1)$ as $\lambda_{r,s}(x)$ is an increasing function of $x$ implies that $f(t)\leq f(1)$ for $t\geq 1$ which completes the proof. □

We remark here that if condition (3.1) is satisfied for some $r, s, \alpha$ then it is also satisfied for $r, s, \alpha'$ with $0<\alpha'<\alpha$. Thus it is not surprising to expect a result like Theorem 3.2 to hold.

To end this section, we prove a variant of (1.1) which is motivated by the following inequality due to Mesihović [11] (with $q_i=1/n$ here) :

$$(1-\frac{1}{n})A_n+\frac{1}{n}G_n\geq\Big(\frac{1}{n}\sum_{i=1}^n x_i^{1-1/n}\Big)\Big(\frac{1}{n}\sum_{i=1}^n x_i^{1/n}\Big).$$

We now generalize the above inequality to the arbitrary weight case:

**Theorem 3.3.** *For* $0 \leq x_1 \leq x_2 \leq \cdots \leq x_n$,

$$(1-q)A_n + qG_n \geq \Big(\sum_{i=1}^n q_i x_i^{1-q}\Big)\Big(\sum_{i=1}^n q_i x_i^{q}\Big),$$

*with equality holding if and only if* $x_1 = \cdots = x_n$ *or* $x_1 = 0, q_1 = q, x_2 = \cdots = x_n$ *or* $n = 2, q = 1/2$.

*Proof.* Define

$$D_n(\mathbf{x}) = (1-q)A_n + qG_n - \Big(\sum_{i=1}^n q_i x_i^{1-q}\Big)\Big(\sum_{i=1}^n q_i x_i^{q}\Big).$$

We need to show $D_n \geq 0$ and we have

$$\frac{1}{q_n}\frac{\partial D_n}{\partial x_n} = 1 - q + q\frac{G_n}{x_n} - (1-q)\Big(x_n^{-q}\sum_{i=1}^n q_i x_i^{q}\Big) - q\Big(x_n^{q-1}\sum_{i=1}^n q_i x_i^{1-q}\Big). \tag{3.2}$$

By a change of variables: $\frac{x_i}{x_n} \rightarrow x_i, 1 \leq i \leq n$, we may assume $0 \leq x_1 \leq x_2 \leq \cdots \leq x_n = 1$ in (3.2) and rewrite it as

$$g_n(x_1, \ldots, x_{n-1}) := 1 - q + qG_n - (1-q)\Big(\sum_{i=1}^n q_i x_i^{q}\Big) - q\Big(\sum_{i=1}^n q_i x_i^{1-q}\Big). \tag{3.3}$$

We want to show $g_n \geq 0$. Let $\mathbf{a} = (a_1, \ldots, a_{n-1}) \in [0,1]^{n-1}$ be the point in which the absolute minimum of $g_n$ is reached. We may assume $a_1 \leq a_2 \leq \cdots \leq a_{n-1}$. If $a_1 = 0$, then $\mathbf{a}$ is a boundary point of $[0,1]^{n-1}$, and in this case we have

$$\begin{aligned} g_n(a_1 = 0, \ldots, a_{n-1}) &= 1 - q - (1-q)\Big(\sum_{i=2}^n q_i a_i^{q}\Big) - q\Big(\sum_{i=2}^n q_i a_i^{1-q}\Big) \\ &\geq 1 - q - (1-q)(1-q_1) - q(1-q_1) = q_1 - q \geq 0, \end{aligned}$$

with equality holding if and only if $q_1 = q, a_2 = \cdots = a_n = 1$. Now suppose $a_1 > 0$ and $a_{m-1} < a_m = \cdots = a_n = 1$ for some $1 \leq m \leq n$, then $a_1, \ldots, a_{m-1}$ solve the equation

$$\nabla g_n(a_1, \ldots, a_{n-1}) = 0.$$

Equivalently, $a_1, \ldots, a_{m-1}$ solve the equation

$$G_n = (1-q)(x^q + x^{1-q}).$$

As the right-hand side expression above is an increasing function of $x$, the above equation has at most one root (regarding $G_n$ as constant). So we only need to show $g_n \geq 0$ for the case $a_1 = \cdots = a_{m-1} = x < a_m = \cdots = a_n = 1$ in (3.3) for some $1 \leq m \leq n$. In this case we regard $g_n$ as a function of $x$ and we recast it as

$$\begin{aligned} h(\omega, x) &= 1 - q + qx^{\omega} - (1-q)\Big(\omega x^q + 1 - \omega\Big) - q\Big(\omega x^{1-q} + 1 - \omega\Big) \\ &= \omega - q + qx^{\omega} - (1-q)\omega x^q - q\omega x^{1-q}. \end{aligned}$$

Here $0 < x \leq 1$ and $q \leq \omega \leq 1 - q$. Note first that when $q = 1/2$, $h(\omega, x) = 0$ so that we may now assume $0 < q < 1/2$. We have

$$\frac{\partial h}{\partial \omega}\Big|_{\omega = q} = 1 + qx^q \ln x - (1-q)x^q - qx^{1-q} := d(x).$$

Now

$$d'(x) = qx^{q-1}e(x),$$

where

$$e(x) = 1 + q\ln x - (1-q) - (1-q)x^{1-2q}.$$

Note that $e'(x) = 0$ has one root $(1-q)(1-2q)x_0^{1-2q} = q$ so that if $0 < x_0 < 1$, then at this point

$$e(x_0) = q \ln x_0 + q - \frac{q}{1-2q} < 0.$$

Note also that $\lim_{x \to 0^+} e(x) < 0, e(1) = 2q - 1 < 0$. This implies that $e(x) < 0$ for $0 < x \le 1$. Hence $d(x) \ge d(1) = 0$ for $0 < x \le 1$. As it is easy to see that $h(\omega, x)$ is a convex function of $\omega$ for fixed $x$, we conclude that $h(\omega, x)$ is an increasing function of $q \le \omega \le 1-q$ for fixed $x$. Thus for $0 < x \le x, q \le \omega \le 1-q$,

$$h(\omega, x) \ge h(q, x) = q^2(x^q - x^{1-q}) \ge 0,$$

with equality holding if and only if $x = 1$.

Thus we have shown $g_n \ge 0$, hence $\frac{\partial D_n}{\partial x_n} \ge 0$ with equality holding if and only if $x_1 = \cdots = x_n$ or $x_1 = 0, q_1 = q, x_2 = \cdots = x_n$ or $n = 2, q_1 = 1/2$. By letting $x_n$ tend to $x_{n-1}$, we have $D_n \ge D_{n-1}$ (with weights $q_1, \ldots, q_{n-2}, q_{n-1} + q_n$) with equality holding if and only if $x_n = x_{n-1}$ or $n = 2$ and either $q = 1/2$ or $x_1 = 0, q_1 = q$. It follows by induction that $D_n \ge 0$ with equality holding if and only if $x_1 = \cdots = x_n$ or $x_1 = 0, q_1 = q, x_2 = \cdots = x_n$ or $n = 2, q = 1/2$ and this completes the proof. □

We remark here that if we define

$$S(\beta) = (\sum_{i=1}^n q_i x_i^{1-\beta}\Big)\Big(\sum_{i=1}^n q_i x_i^{\beta}\Big).$$

Then for $1 \le \beta \le 1/2$,

$$\frac{dS}{d\beta} = \sum_{1 \le i \le j \le n} q_i q_j x_i^\beta x_j^\beta \Big(x_j^{1-2\beta} - x_i^{1-2\beta}\Big) \ln\Big(\frac{x_i}{x_j}\Big) \le 0.$$

Hence Theorem 3.3 improves (1.1) for the case $\{1, 1/2, 0\}$, $\alpha = 1$.

## 4. Inequalities Involving the Symmetric Means

In this section, we set $q_i = 1/n, 1 \le i \le n$. As an analogue of (1.5) (or (1.1)), we first consider

$$A_n^\alpha \le \lambda_{\alpha,r,k}(n) M_{n,r}^\alpha + \Big(1 - \lambda_{\alpha,r,k}(n)\Big) P_{n,k}^\alpha \tag{4.1}$$

where $\alpha > 0, r > 1, n \ge k \ge 2$ and

$$\lambda_{\alpha,r,k}(n) = \frac{(1-\frac{1}{n})^\alpha - \Big(\frac{n-k}{n}\Big)^{\alpha/k}}{(1-\frac{1}{n})^{\alpha/r} - \Big(\frac{n-k}{n}\Big)^{\alpha/k}}. \tag{4.2}$$

The case $r = 2, \alpha = k$ in (4.1) is just Theorem 1.3. In what follows, we will give a proof of Theorem 1.3 by combining the methods in [9] and [10]. Before we prove our result, we would like to first recast (4.1) for the case $\alpha = k$ as

$$\Big(\sum_{i=1}^n x_i\Big)^k \le \Big(n^k - \tilde{\lambda}_{r,k}(n)\binom{n}{k}\Big) M_{n,r}^k + \tilde{\lambda}_{r,k}(n) E_{n,k}, \tag{4.3}$$

where $\tilde{\lambda}_{r,k}(n)$ is defined as in the statement of Theorem 1.3. Now we need two lemmas:

**Lemma 4.1.** *For $2 \le r \le k \le n-1$,*

$$\tilde{\lambda}_{r,k}(n) \le \tilde{\lambda}_{r,k}(n-1). \tag{4.4}$$

*Proof.* We follow the method in the proof of Lemma 2 in [10]. We write $\tilde{\lambda}_{r,k}(n)$ as

$$\tilde{\lambda}_{r,k}(n)=\frac{k!\Big(n^k(1-\frac{1}{n})^{k/r}-(n-1)^k\Big)}{\Big(\prod_{i=1}^{k-1}(n-i)\Big)\Big(n(1-\frac{1}{n})^{k/r}-n+k\Big)}.$$

From the above we see that in order for (4.4) to hold, it suffices to show that

$$f(t)=\frac{k!\Big((1-t)^{k/r}-(1-t)^k\Big)}{\Big(\prod_{i=1}^{k-1}(1-it)\Big)\Big((1-t)^{k/r}-1+kt\Big)}$$

is increasing on $(0,1/k)$. The logarithmic derivative of $f(t)$ is

$$\frac{f'(t)}{f(t)}=\frac{k}{r(1-t)}\Big(1-r+\frac{r-1}{1-(1-t)^{k(1-1/r)}}-\frac{r-1+(k-r)t}{(1-t)^{k/r}-1+kt}\Big)+\sum_{i=1}^{k-1}\frac{i}{1-it}.$$

Note that for $0<t<1$, we have

$$(1-t)^{k/r}-1+kt\geq 1-(1-t)^{k(1-1/r)}>0,$$

by considering the Taylor expansions to the order of $t^2$. It follows from this that

$$\begin{aligned}\frac{f'(t)}{f(t)} &\geq \frac{k}{r(1-t)}\Big(1-r+\frac{r-1}{1-(1-t)^{k(1-1/r)}}-\frac{r-1+(k-r)t}{1-(1-t)^{k(1-1/r)}}\Big)+\sum_{i=1}^{k-1}\frac{i}{1-t}\\ &= \frac{k}{1-t}\Big(\frac{1-r}{r}-\frac{(k-r)t/r}{1-(1-t)^{k(1-1/r)}}+\frac{k-1}{2}\Big).\end{aligned}$$

It is easy to see that the function

$$t\mapsto\frac{t}{1-(1-t)^{k(1-1/r)}}$$

is an increasing function for $0<t\leq 1$. Hence

$$\frac{1-r}{r}-\frac{(k-r)t/r}{1-(1-t)^{k(1-1/r)}}+\frac{k-1}{2}\geq\frac{1-r}{r}-\frac{k-r}{r}+\frac{k-1}{2}=(k-1)(\frac{1}{2}-\frac{1}{r})\geq 0.$$

□

**Lemma 4.2.** *Inequality* (4.1) *holds for all* $\mathbf{x}$ *in the case* $n\geq k\geq r\geq 2,\alpha=k$ *if it holds for* $\mathbf{x}=(a,b,\dots,b)$ *with* $0<a\leq b$.

*Proof.* In this proof we assume that $0\leq x_1\leq x_2\leq\ldots\leq x_n$. We prove the lemma by induction on $n$. When $n=k$, the assertion holds as a special case of (1.3). Now assume the assertion holds for $n-1$, and we proceed to show it also holds for $n$. If $x_1=0$, we use the equivalent form (4.3) of (4.1) for the case $\alpha=k$ to see that what we need to prove is:

$$\Big(\sum_{i=2}^{n-1}x_i\Big)^k\leq\Big(n^k-\tilde{\lambda}_{r,k}(n)\binom{n}{k}\Big)\Big(\frac{n-1}{n}\Big)^{k/r}M^k_{n-1,r}(x_2,\dots,x_n)+\tilde{\lambda}_{r,k}(n)E_{n-1,k}(x_2,\dots,x_n). \quad (4.5)$$

By the induction case $n-1$, we have

$$\Big(\sum_{i=2}^{n-1}x_i\Big)^k\leq\Big((n-1)^k-\tilde{\lambda}_{r,k}(n-1)\binom{n-1}{k}\Big)M^k_{n-1,r}(x_2,\dots,x_n)+\tilde{\lambda}_{r,k}(n-1)E_{n-1,k}(x_2,\dots,x_n). \quad (4.6)$$

Note that

$$\Big(n^k-\tilde{\lambda}_{r,k}(n)\binom{n}{k}\Big)\Big(\frac{n-1}{n}\Big)^{k/r}=(n-1)^k-\tilde{\lambda}_{r,k}(n)\binom{n-1}{k}.$$

Using this with Lemma 4.1 together with the observation that

$$E_{n-1,k}/\binom{n-1}{k} = P^k_{n-1,k} \leq M^k_{n-1,r},$$

we see that inequality (4.5) follows from (4.6).

Thus from now on we may focus on the case $x_1 > 0$. Since both sides of (4.1) are homogeneous functions, it suffices to show that

$$\min_{\mathbf{x}\in\Delta}\left\{\lambda_{k,r,k}(n)M^k_{n,r} + \Big(1-\lambda_{k,r,k}(n)\Big)P^k_{n,k}\right\} \geq 1, \tag{4.7}$$

where

$$\Delta = \{\mathbf{x} = (x_1, \ldots, x_n) : x_i \geq 0, 1 \leq i \leq n, \sum_{i=1}^n x_i = n\}.$$

Assume that $\lambda_{k,r,k}(n)M^k_{n,r} + \Big(1-\lambda_{k,r,k}(n)\Big)P^k_{n,k}$ attains its minimum at some point $(a_1, \ldots, a_n)$ with $a_i > 0, 1 \leq i \leq n$. If $a_1 = a_2 = \ldots = a_n$, then (4.7) holds. Furthermore, if $n = 2$, then (4.7) also holds, being a special case of (1.3). Thus without loss of generality, we may assume $n \geq 3$ and $a_n > a_{n-1} \geq a_{n-2}$ here. We may also assume that when $r = 2$, $k > 2$ since otherwise inequality (4.1) becomes an indentity. Consider the function

$$f(x,y) := \lambda_{k,r,k}(n)M^k_{n,r}\big(a_1, \ldots, a_{n-2}, x, y\big) + \Big(1-\lambda_{k,r,k}(n)\Big)P^k_{n,k}\big(a_1, \ldots, a_{n-2}, x, y\big).$$

on the set

$$\{(x,y) : x \geq 0, y \geq 0, x + y = n - \sum_{i=1}^{n-2} a_i\}.$$

It is minimized at $(a_{n-1}, a_n)$. It is easy to see that $f$ has the form

$$\lambda_{k,r,k}(n)\Big(\frac{1}{n}(x^r + y^r) + B\Big)^{k/r} + C\big(a_1, \ldots, a_{n-2}\big)xy + D$$

where $B, C, D$ are non-negative constants with $C$ depends on $a_1, \ldots, a_{n-2}$. We now set $x + y = c$ and $xy = z$ with $0 \leq z = xy \leq (x+y)^2/4 = c^2/4$. Note here equality holds if and only if $xy = 0$ or $x = y$. We regard the above function as a function of $z = xy$ and recast it as

$$h(z) = \lambda_{k,r,k}(n)\Big(\frac{1}{n}\Big(\big(\frac{c+\sqrt{c^2-4z}}{2}\big)^r + \big(\frac{c-\sqrt{c^2-4z}}{2}\big)^r\Big) + B\Big)^{k/r} + C\big(a_1, \ldots, a_{n-2}\big)z + D.$$

For $y > x > 0$, calculation yields

$$h'(xy) = -\frac{\lambda_{k,r,k}(n)k}{n}\Big(\frac{1}{n}(x^r + y^r) + B\Big)^{k/r-1}\Big(\frac{y^{r-1} - x^{r-1}}{y - x}\Big) + C\big(a_1, \ldots, a_{n-2}\big).$$

Since $a_n > a_{n-1} > 0$, we must have $h'(a_{n-1}a_n) = 0$ and we can further recast this as

$$C\big(a_1, \ldots, a_{n-2}\big) = \frac{\lambda_{k,r,k}(n)k}{n}M^{k-r}_{n,r}\big(a_1, \ldots, a_n\big)\Big(\frac{a^{r-1}_n - a^{r-1}_{n-1}}{a_n - a_{n-1}}\Big).$$

Now if $a_{n-2} > 0$, we can repeat the same argument for the pair $(a_{n-1}, a_{n-2})$. By a slightly abuse of notation, we obtain

$$h'(a_{n-2}a_{n-1}) = C\big(a_1, \ldots, a_{n-3}, a_n\big) - \frac{\lambda_{k,r,k}(n)k}{n}M^{k-r}_{n,r}\big(a_1, \ldots, a_n\big)\Big(\frac{a^{r-1}_{n-1} - a^{r-1}_{n-2}}{a_{n-1} - a_{n-2}}\Big).$$

It's easy to see that (since we assume $a_i > 0$ and $k \geq 3$ when $r = 2$)

$$C\big(a_1, \ldots, a_{n-3}, a_n\big) > C\big(a_1, \ldots, a_{n-2}\big).$$

Moreover, one checks easily that the function

$$(x,y) \mapsto \frac{y^{r-1}-x^{r-1}}{y-x}$$

is increasing with respect to each variable $x, y > 0$ when $r \geq 2$. It follows that when $r \geq 2$,

$$h'(a_{n-2}a_{n-1}) > h'(a_{n-1}a_n) = 0,$$

which implies by decreasing the value of $a_{n-2}a_{n-1}$ while keeping $a_{n-2}+a_{n-1}$ fixed, one is able to get a smaller value for $\lambda_{k,r,k}(n)M^k_{n,r} + \Big(1-\lambda_{k,r,k}(n)\Big)P^k_{n,k}$, contradicting the assumption that it attains its minimum at $(a_1, \ldots, a_n)$.

Hence we conclude that $\lambda_{k,r,k}(n)M^k_{n,r} + \Big(1-\lambda_{k,r,k}(n)\Big)P^k_{n,k}$ is minimized at $(a, b, \ldots, b)$ with $0 < a \leq b$ satisfying $a + (n-1)b = n$. In this case (4.7) holds by our assumption which completes the proof. □

Now we are ready to prove a slightly generalization of Theorem 1.3:

**Corollary 4.1.** *Inequality* (4.1) *holds in the cases* $n \geq k \geq r = 2, \alpha = k$ *and* $n \geq \alpha = r = k \geq 2$.

*Proof.* The first case is just Theorem 1.3 and by Lemma 4.2, it suffices to show that inequality (4.1) holds for the case $\mathbf{x} = (a, b, \ldots, b)$ with $0 < a \leq b$ and this has been already treated in the proof of Theorem 2 in [10]. For the case $n \geq \alpha = r = k \geq 2$, by Lemma 4.2 again, we only need to check the case $\mathbf{x} = (a, b, \ldots, b)$ with $0 < a \leq b$. In this case we define

$$f(a,b) = \lambda_{k,k,k}(n)(\frac{a^k}{n} + \frac{(n-1)b^k}{n}) + \Big(1-\lambda_{k,k,k}(n)\Big)(\frac{n-k}{n}b^k + \frac{k}{n}ab^{k-1}) - (\frac{a}{n} + \frac{(n-1)b}{n})^k.$$

As in the proof of Theorem 1.2, it suffices to show that

$$\frac{n}{k(n-1)b^{k-1}}\frac{\partial f}{\partial b} = \lambda_{k,k,k}(n) + \Big(1-\lambda_{k,k,k}(n)\Big)\Big(\frac{n-k}{n-1} + \frac{k-1}{n-1}\frac{a}{b}\Big) - (\frac{a}{nb} + \frac{n-1}{n})^{k-1} \geq 0. \tag{4.8}$$

By a change of variables $a/b \to a$, we can recast inequality (4.8) as

$$g(a) = \lambda_{k,k,k}(n) + \Big(1-\lambda_{k,k,k}(n)\Big)\Big(\frac{n-k}{n-1} + \frac{k-1}{n-1}a\Big) - \Big(\frac{n-1}{n} + \frac{a}{n}\Big)^{k-1}.$$

As $g(1) = 0$ and $g(0) = 0$, we conclude that $g(a) \geq 0$ for $0 \leq a \leq 1$ as $g(a)$ is a concave function for $0 \leq a \leq 1$ and this completes the proof. □

We recall a result of Kuczma [9]:

**Theorem 4.1.** *For* $n \geq 3, 1 \leq k \leq n-1$, $P_{n,k} \leq M_{n,r(k)}$ *with*

$$r(k) = k(\ln n - \ln(n-1))/(\ln n - \ln(n-k)),$$

*and the result is best possible.*

The above theorem combined with Corollary 4.1 immediately yields the following

**Corollary 4.2.** *Let* $q_i = 1/n$ *with* $n \geq 1$ *an integer. Then for any integer* $n-1 \geq k \geq r = 2$ *or* $n-1 \geq k = r \geq 2$,

$$A^k_n \leq \lambda_{k,r,k}(n)M^k_{n,r} + \Big(1-\lambda_{k,r,k}(n)\Big)M^k_{n,r(k)},$$

*where* $\lambda_{k,r,k}(n)$ *is defined as in* (4.2) *and* $r(k)$ *is defined as in Theorem 4.1.*

Next, we consider the following inequality

$$P_{n,l}^{\alpha} \leq \mu_{\alpha,k,l}(n) A_n^{\alpha} + \Big(1-\mu_{\alpha,k,l}(n)\Big) P_{n,k}^{\alpha} \tag{4.9}$$

where $\alpha > 0, n \geq k > l \geq 2$ and

$$\mu_{\alpha,k,l}(n) = \frac{\left(\frac{n-l}{n}\right)^{\alpha/l} - \left(\frac{n-k}{n}\right)^{\alpha/k}}{\left(\frac{n-1}{n}\right)^{\alpha} - \left(\frac{n-k}{n}\right)^{\alpha/k}}.$$

We note here it is easy to check that the function

$$x \mapsto \left(\frac{n-x}{n}\right)^{1/x}$$

is a decreasing function for $1 \leq x < n$ so that we have $0 < \mu_{\alpha,k,l}(n) < 1$.

As an analogue of Theorem 1.1, one can show similarly that

**Proposition 4.1.** *Let $n = k > l \geq 2$, if (4.9) holds for $\alpha_0 > 0$, then it also holds for any $0 < \alpha < \alpha_0$.*

The case $n = k, \alpha = 1$ of (4.9) was established in [13]. In the case of $n = k$, one possible way of establishing (4.9) is to combine Theorem 4.1 and Theorem 1.2 together. However, this is not always applicable as one checks via certain change of variables that one needs to have $1/r(k) \geq 2$ in order to apply Theorem 1.2, a condition which is not always satisfied. We now proceed directly to show that

**Theorem 4.2.** *Inequality (4.9) holds for $n = k > l \geq 2$ and $0 < \alpha \leq n$.*

*Proof.* In view of Proposition 4.1, it suffices to prove the theorem for the case $\alpha = n$. We write $\mu(n) = \mu_{n,n,l}(n)$ in this proof and note that since both sides of (4.9) are homogeneous functions, it suffices to show that

$$\max_{\mathbf{x} \in \Delta} \left\{ P_{n,l}^{n} - \Big(1-\mu(n)\Big) G_n^n \right\} \leq \mu(n),$$

where

$$\Delta = \{\mathbf{x} = (x_1, \ldots, x_n) : x_i \geq 0, 1 \leq i \leq n, \sum_{i=1}^{n} x_i = n\}.$$

Assume that $P_{n,l}^{n} - \Big(1-\mu(n)\Big) G_n^n$ attains its maximum at some point $(a_1, \ldots, a_n)$ with $a_i \geq 0, 1 \leq i \leq n$. Consider the function

$$f(x,y) := P_{n,l}^{n}\big(x, y, a_3, \ldots, a_n\big) - xy\Big(1-\mu(n)\Big) \prod_{i=3}^{n} a_i$$

on the set

$$\{(x,y) : x \geq 0, y \geq 0, x + y = n - \sum_{i=3}^{n} a_i\}.$$

It is maximized at $(a_1, a_2)$. It is easy to see that $f$ has the form

$$\Big(Axy + B\Big)^{n/l} - Cxy$$

where $A, B, C$ are non-negative constants. The above function is certainly convex with respect to $xy$. As $0 \leq xy \leq (x+y)^2/4$ with equality holding if and only if $xy = 0$ or $x = y$, $f$ is maximized at $xy = 0$ or $x = y$. Repeating the same argument for other pairs $(a_i, a_j)$, we conclude that in order to show (4.9) for $\alpha = n = k$, it suffices to check it holds for $\mathbf{x}$ being of the following form

$(0,\dots,0,a\dots,a)$ or $(a,\dots,a)$ for some positive constant $a$. It is easy to see that (4.9) holds when $\mathbf{x}$ is of the second form and when $\mathbf{x}$ is of the first form, let $m$ denote the number of $a$'s in $\mathbf{x}$, if $m<l$, then it is easy to see that (4.9) holds. So we may now assume that $l\le m\le n=k$ and we need to show that

$$\Big(\frac{\binom{m}{l}^{1/l}}{m}\Big)^k \le \mu_{k,k,l}(n)\Big(\frac{\binom{n}{l}^{1/l}}{n}\Big)^k.$$

Certainly the left-hand side above increases with $m$, hence one only needs to verify the above inequality for the case $m=n-1$, which becomes an identity and this completes the proof. □

We note here that Alzer [1] has shown that for $n\ge 3$,

$$P^{n-1}_{n,n-1}\le \frac{n}{n+1}A^{n-1}_n+\frac{1}{n+1}G^{n-1}_n.$$

The case $\alpha=l=n-1$ of Theorem 4.2 now improves the above result, namely, for $n\ge 3$,

$$P^{n-1}_{n,n-1}\le \frac{n^{n-2}}{(n-1)^{n-1}}A^{n-1}_n+(1-\frac{n^{n-2}}{(n-1)^{n-1}})G^{n-1}_n,$$

as one checks easily that for $n\ge 3$,

$$\frac{n^{n-2}}{(n-1)^{n-1}}\le\frac{n}{n+1}.$$

Similar to Theorem 4.2, we have

**Theorem 4.3.** *For $1\le k\le n, q_i=1/n$,*

$$P^k_{n,k}\le \frac{n-k}{n-1}M^k_{n,k}+\frac{k-1}{n-1}G^k_n.$$

*Proof.* We define

$$f(\mathbf{x})=\frac{n-k}{n-1}M^k_{n,k}+\frac{k-1}{n-1}G^k_n-P^k_{n,k}.$$

We now set

$$\tilde{P}_{n-1,k-1}=P_{n-1,k-1}(x_1,\dots,x_{n-1}),\quad \tilde{G}_{n-1}=G_{n-1}(x_1,\dots,x_{n-1}),$$

so that

$$\frac{\partial f}{\partial x_n}=\frac{k}{n}\frac{n-k}{n-1}x^{k-1}_n+\frac{k}{n}\frac{k-1}{n-1}\frac{G^k_n}{x_n}-\frac{k}{n}\tilde{P}^{k-1}_{n-1,k-1},$$

where we have also used the following relation

$$P^k_{n,k}=\frac{k}{n}x_nP^{k-1}_{n-1,k-1}(x_1,\dots,x_{n-1})+\frac{n-k}{n}P^k_{n-1,k}(x_1,\dots,x_{n-1}).$$

Similar to the proof of Theorem 1.2, it suffices to show that $\frac{\partial f}{\partial x_n}\ge 0$. By a change of variables $x_i\to x_i/x_n$, it suffices to show that for all $0\le x_i\le 1$,

$$\tilde{P}^{k-1}_{n-1,k-1}\le\frac{n-k}{n-1}+\frac{k-1}{n-1}\tilde{G}^{\frac{(n-1)k}{n}}_{n-1}. \tag{4.10}$$

As a consequence of Lemma 3.2 in [7], one checks easily that

$$\tilde{P}^{k-1}_{n-1,k-1}\le\frac{n-k}{n-1}+\frac{k-1}{n-1}\tilde{G}^{n-1}_{n-1}.$$

The above inequality then implies (4.10) and we then conclude that $\frac{\partial f}{\partial x_n}\ge 0$ which completes the proof. □

We remark here that once again one may hope to establish the above theorem by combining Theorem 4.1 and Theorem 1.2 together. However, (1.3) is not applicable in this case since one checks readily that $k/r(k) \leq n$ is not satisfied in general.

We now want to establish some inequalities involving the symmetric means in the forms similar to (1.2). Before we state our result, let's first recall that for two real finite decreasing sequences $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $\mathbf{y} = (y_1, y_2, \dots, y_n)$, $\mathbf{x}$ is said to be majorized by $\mathbf{y}$ (which we denote by $\mathbf{x} \leq_{maj} \mathbf{y}$) if

$$\begin{aligned} x_1 + x_2 + \cdots + x_j &\leq y_1 + y_2 + \cdots + y_j \quad (1 \leq j \leq n-1), \\ x_1 + x_2 + \cdots + x_n &= y_1 + y_2 + \cdots + y_n. \end{aligned}$$

For a fixed positive sequence $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and a non-negative sequence $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, we define

$$F(\alpha) = \frac{1}{n!} \sum_{\sigma} x_{\sigma(1)}^{\alpha_1} x_{\sigma(2)}^{\alpha_2} \cdots x_{\sigma(n)}^{\alpha_n}, \tag{4.11}$$

where the sum is over all the permutations of $\mathbf{x}$. A well-known result of Muirhead states:

**Theorem 4.4** ([8, Theorem 45]). *Let $\alpha$ and $\alpha'$ be two non-negative decreasing sequences. Then $F(\alpha) \leq F(\alpha')$ for any positive sequence if and only if $\alpha \leq_{maj} \alpha'$.*

We now use the above result to show

**Theorem 4.5.** *For $1 \leq k \leq n, q_i = 1/n$,*

$$A_n^k \geq \frac{1}{n^{k-1}} M_{n,k}^k + \Big(1 - \frac{1}{n^{k-1}}\Big) P_{n,k}^k.$$

*Proof.* On expanding $A_n^k$ out, we can write it as

$$A_n^k \geq \frac{1}{n^{k-1}} M_{n,k}^k + \text{linear combinations of various terms of the form } F(\alpha), \tag{4.12}$$

where $F(\alpha)$ is defined as in (4.11) with $\alpha_i \geq 1, \sum_{i=1}^n \alpha_i = n$. It is then easy to see via Theorem 4.4 that for any such $\alpha$ appearing in (4.12), we have $P_{n,k}^k \leq F(\alpha)$. Hence one deduces that

$$A_n^k \geq \frac{1}{n^{k-1}} M_{n,k}^k + c P_{n,k}^k,$$

for some constant $c$, which can be easily identified to be $1 - \frac{1}{n^{k-1}}$ by taking $\mathbf{x} = (1, \dots, 1)$ and noticing that we get identities in all the steps above. □

Our next result gives a generalization of the above one and we shall need the following two lemmas in our next proof.

**Lemma 4.3** (Hadamard's inequality). *Let $f(x)$ be a convex function on $[a, b]$, then*

$$f\Big(\frac{a+b}{2}\Big) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}.$$

The next lemma is similar to that in [9].

**Lemma 4.4.** *Let $A, B, C, D > 0$ be arbitrary constants and let $k \geq r = 2$ or $k \geq r \geq 3$. The maximum value of $f(x, y) = A(x^r + y^r + B)^{k/r} + Cxy$ on the set $\{(x, y) : x \geq 0, y \geq 0, x + y = 2D\}$ is attained either when $x = y$ or $xy = 0$.*

*Proof.* We set $z = xy$ and note that $0 \leq z \leq D^2$ with equality holding if and only if $x = y$ or $xy = 0$. Moreover,

$$x, y = D \pm \sqrt{D^2 - z}.$$

This allows us to rewrite $f(x,y) = Ag(z) + CZ$ where

$$g(z) = \Big((D + \sqrt{D^2 - z})^r + (D - \sqrt{D^2 - z})^r + B\Big)^{k/r}.$$

It suffices to show that $g(z)$ is convex for $0 \leq z \leq D^2$. Note that $2g'(z) = k \cdot h(\sqrt{D^2 - z})$ where $h(w) = p(w)q(w)$ with

$$p(w) = \Big((D + w)^r + (D - w)^r + B\Big)^{k/r-1}, \quad q(w) = \Big((D - w)^{r-1} - (D + w)^{r-1}\Big)w^{-1}.$$

As the derivative of $\sqrt{D^2 - z}$ is negative for $0 \leq z < D^2$, it suffices to show $h'(w) \leq 0$ for $0 < w < D$. Note that $p(w) \geq 0, q(w) \leq 0$ and it's easy to check that $p'(w) \geq 0$ for $0 < w < D$. Hence it suffices to show that $q'(w) \leq 0$ for $0 < w < D$. Calculation shows that

$$\begin{aligned} w^2 q'(w) &= -(r-1)\Big((D + w)^{r-2} + (D - w)^{r-2}\Big)w - (D - w)^{r-1} + (D + w)^{r-1} \\ &= (r-1)\left(\int_{D-w}^{D+w} u^{r-2}du - w\left((D + w)^{r-2} + (D - w)^{r-2}\right)\right) \leq 0, \end{aligned}$$

by Lemma 4.3. This completes the proof. □

Now we are ready to prove

**Theorem 4.6.** *Let $q_i = 1/n$ and let $r$ be a real number, $r = 2$ or $r \geq 3$. Then for integers $n, k$, $n \geq k \geq r$,*

$$A_n^k \geq \frac{1}{n^{k - \frac{k}{r}}} M_{n,r}^k + \Big(1 - \frac{1}{n^{k - \frac{k}{r}}}\Big) P_{n,k}^k. \tag{4.13}$$

*Proof.* Since both sides of (4.13) are homogeneous functions, it suffices to show that

$$\max_{\mathbf{x} \in \Delta} \left\{ \frac{1}{n^{k - \frac{k}{r}}} M_{n,r}^k + \Big(1 - \frac{1}{n^{k - \frac{k}{r}}}\Big) P_{n,k}^k \right\} \leq 1,$$

where

$$\Delta = \{\mathbf{x} = (x_1, \ldots, x_n) : x_i \geq 0, 1 \leq i \leq n, \sum_{i=1}^n x_i = n\}.$$

Assume that $\frac{1}{n^{k - \frac{k}{r}}} M_{n,r}^k + \Big(1 - \frac{1}{n^{k - \frac{k}{r}}}\Big) P_{n,k}^k$ attains its maximum at some point $(a_1, \ldots, a_n)$ with $a_i \geq 0, 1 \leq i \leq n$. Consider the function

$$f(x,y) := \frac{1}{n^{k - \frac{k}{r}}} M_{n,r}^k(x, y, a_3, \ldots, a_n) + \Big(1 - \frac{1}{n^{k - \frac{k}{r}}}\Big) P_{n,k}^k(x, y, a_3, \ldots, a_n)$$

on the set

$$\{(x,y) : x \geq 0, y \geq 0, x + y = n - \sum_{i=3}^n a_i\}.$$

It is maximized at $(a_1, a_2)$. Clearly, $f$ has the form

$$A(x^r + y^r + B)^{k/r} + Cxy + (\text{constant}),$$

where $A, B, C$ are non-negative constants. By Lemma 4.4, $f$ attains its maximum value at either $x = 0$ or $y = 0$ or $x = y$. Repeating the same argument for other pairs $(a_i, a_j)$, we conclude that in order to show (4.13), it suffices to check it holds for $\mathbf{x}$ being of the following form $(0, \ldots, 0, a \ldots, a)$ or $(a, \ldots, a)$ for some positive constant $a$. It is easy to see that (4.13) holds when $\mathbf{x}$ is of the second form and when $\mathbf{x}$ is of the first form, let $m$ denote the number of $a$'s in $\mathbf{x}$, if $m < k$, then it is easy to see that (4.13) holds. So we may now assume that $k \leq m \leq n$ and we need to show that

$$(\frac{m}{n})^k \geq \frac{1}{n^{k-\frac{k}{r}}}(\frac{m}{n})^{\frac{k}{r}} + \Big(1 - \frac{1}{n^{k-\frac{k}{r}}}\Big)\frac{\binom{m}{k}}{\binom{n}{k}}.$$

Equivalently, we need to show

$$\frac{1}{n^k} \geq \frac{1}{n^k} m^{\frac{k}{r}-k} + \Big(1 - \frac{1}{n^{k-\frac{k}{r}}}\Big)\frac{\binom{m}{k} m^{-k}}{\binom{n}{k}}. \tag{4.14}$$

Note that

$$\binom{m}{k} m^{-k} = \frac{1}{k!}\prod_{i=1}^{k-1}(1 - \frac{i}{m}).$$

If we define for $1/n \leq u \leq 1/(k-1)$,

$$g(u) = \frac{1}{n^k} u^{k-\frac{k}{r}} + \Big(1 - \frac{1}{n^{k-\frac{k}{r}}}\Big)\frac{1}{\binom{n}{k}}\frac{1}{k!}\prod_{i=1}^{k-1}(1 - iu),$$

then it is easy to check that $g''(u) \geq 0$ for $r \geq k/(k-1)$. Further note that

$$g(1/n) = \frac{1}{n^k}, \quad g(\frac{1}{k-1}) = \frac{1}{n^k}(\frac{1}{k-1})^{k-\frac{k}{r}} \leq \frac{1}{n^k}.$$

This implies that $g(u) \leq 1/n^k$ for $1/n \leq u \leq 1/(k-1)$ and hence it follows that (4.14) holds for $k \leq m \leq n$ and this completes the proof. □

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