BOUNDS FOR THE DEVIATION OF A FUNCTION FROM A GENERALISED CHORD GENERATED BY ITS EXTREMITIES WITH APPLICATIONS

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Abstract. Bounds for the deviation of a real-valued function f defined on a compact interval $[a,b]$ to the generalised chord

$$\frac{v(b)-v(t)}{v(b)-v(a)}\cdot f(a)+\frac{v(t)-v(a)}{v(b)-v(a)}\cdot f(b),$$

where $v:[a,b]\to\mathbb{R}$ and $v(a)\neq v(b)$, that connects its end points $(a,f(a))$ and $(b,f(b))$ are given. Applications for normalised positive linear functionals are provided as well.

1. Introduction

Consider a function $f:[a,b]\to\mathbb{R}$ and assume that it is bounded on $[a,b]$. Denote by $\Phi_f(t)$ the error in approximating the function f by its (straight line) chord d_f which connects the points $(a,f(a))$ and $(b,f(b))$, i.e.,

$$\Phi_f(t):=\frac{b-t}{b-a}\cdot f(a)+\frac{t-a}{b-a}f(b)-f(t),\qquad t\in[a,b]. \tag{1.1}$$

In the recent paper [3], sharp error estimates for $\Phi_f(t)$ under various assumptions on the function f have been derived. We recall here some of them.

If there exist the constants $-\infty<m<M<\infty$ such that $m\leq f(t)\leq M$ for each $t\in[a,b]$, then $|\Phi_f(t)|\leq M-m$. The multiplication constant 1 in front of $(M-m)$ cannot be replaced by a smaller quantity. If $f:[a,b]\to\mathbb{R}$ is a convex function on $[a,b]$, then

$$\begin{aligned}0\leq\Phi_f(t)&\leq\frac{1}{b-a}(t-a)(b-t)\left[f'_-(b)-f'_+(a)\right]\\&\leq\frac{1}{4}(b-a)\left[f'_-(b)-f'_+(a)\right],\end{aligned} \tag{1.2}$$

for any $t\in[a,b]$. In the case where the lateral derivatives $f'_-(b)$ and $f'_+(a)$ are finite, then the second inequality and the constant $\frac{1}{4}$ are sharp.

Date: November 22, 2007.

1991 *Mathematics Subject Classification.* Primary 26D15.

Key words and phrases. Bounded variation, Stieltjes integral, Monotonicity, Normalised positive functionals.

If $f:[a,b]\to\mathbb{R}$ is a function of bounded variation, then

$$\text{(1.3)}\qquad |\Phi_f(t)| \le \frac{b-t}{b-a}\cdot\bigvee_a^t(f)+\frac{t-a}{b-a}\bigvee_t^b(f)$$
$$\le\begin{cases}\left[\frac{1}{2}+\left|t-\frac{a+b}{2}\right|\right]\bigvee_a^b(f);\\[2mm] \left[\left(\frac{b-t}{b-a}\right)^p+\left(\frac{t-a}{b-a}\right)^p\right]^{\frac{1}{p}}\left[\left(\bigvee_a^t(f)\right)^q+\left(\bigvee_t^b(f)\right)^q\right]^{\frac{1}{q}}\\ \qquad\qquad\text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2mm] \frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^t(f)-\bigvee_t^b(f)\right|.\end{cases}$$

The first inequality in (1.3) is sharp. The constant $\frac{1}{2}$ is best possible in the first and third branches.

In particular, if f is $L-$Lipschitzian on $[a,b]$, i.e., $|f(t)-f(s)|\le L|t-s|$ for any $t,s\in[a,b]$, then

$$\text{(1.4)}\qquad |\Phi_f(t)|\le\frac{2(b-t)(t-a)}{b-a}L\le\frac{1}{2}(b-a)L,$$

for any $t\in[a,b]$. The constants 2 and $\frac{1}{2}$ are best possible.

For extensions to n- time differentiable functions see [4].

In this paper we consider a natural generalisation of the above problem by introducing the error function for the approximation of $f(t)$ with $\frac{v(b)-v(t)}{v(b)-v(a)}\cdot f(a)+\frac{v(t)-v(a)}{v(b)-v(a)}\cdot f(b)$, where $v:[a,b]\to\mathbb{R}$ is another function with the property that $v(a)\neq v(b)$. Error bounds for different pairs of functions (f,v) are derived. Applications in obtaining error bounds in approximating the quantity $A(f\circ u)$ by the generalised trapezoid formula

$$\frac{A(v\circ u)-v(a)}{v(b)-v(a)}\cdot f(a)+\frac{v(b)-A(v\circ u)}{v(b)-v(a)}\cdot f(b),$$

where A is a normalised linear functional are also given.

2. Bounds for $\Phi_{f,v}$ when f,v are of Bounded Variation

For a function $p:[a,b]\to\mathbb{R}$ we define the kernel $Q_p:[a,b]^2\to\mathbb{R}$ by

$$\text{(2.1)}\qquad Q_p(t,s):=\begin{cases}p(t)-p(b) & \text{if } a\le s\le t\le b,\\ p(t)-p(a) & \text{if } a\le t<s\le b.\end{cases}$$

With this notation we have the following representation of the function $\Phi_{f,v}$, where

$$\Phi_{f,v}(t)=\frac{v(t)-v(a)}{v(b)-v(a)}\cdot f(b)+\frac{v(b)-v(t)}{v(b)-v(a)}\cdot f(a)-f(t)$$

with $t\in[a,b]$.

Lemma 1. *If $f,v:[a,b]\to\mathbb{R}$ are bounded functions on $[a,b]$, then*

$$\Phi_{f,v}(t)=\frac{1}{v(b)-v(a)}\int_a^b Q_v(t,s)\,df(s)=\frac{1}{v(b)-v(a)}\int_a^b Q_{-f}(t,s)\,dv(s) \tag{2.2}$$

provided $v(b)\neq v(a)$, where the integrals are taken in the Riemann-Stieltjes sense.

Proof. We have

$$\begin{aligned}\Phi_{f,v}(t)&=\frac{[v(t)-v(b)][f(t)-f(a)]+[v(t)-v(a)][f(b)-f(t)]}{v(b)-v(a)}\\&=\frac{[v(t)-v(b)]\int_a^t df(s)+[v(t)-v(a)]\int_t^b df(s)}{v(b)-v(a)}\\&=\frac{1}{v(b)-v(a)}\int_a^b Q_v(t,s)\,df(s).\end{aligned} \tag{2.3}$$

Also, by rearranging the terms in the first equality, we also have

$$\begin{aligned}\Phi_{f,v}(t)&=\frac{[f(a)-f(t)]\int_t^b dv(s)+[f(b)-f(t)]\int_a^t dv(s)}{v(b)-v(a)}\\&=\frac{1}{v(b)-v(a)}\int_a^b Q_{-f}(t,s)\,dv(s)\end{aligned} \tag{2.4}$$

and the representation (2.2) is proved. ∎

The following estimation result can be stated.

Theorem 1. *Assume that $f,v:[a,b]\to\mathbb{R}$ are bounded and $v(a)\neq v(b)$.*

(i) *If f is of bounded variation on $[a,b]$, then*

$$|\Phi_{f,v}(t)|\leq\left|\frac{v(b)-v(t)}{v(b)-v(a)}\right|\cdot\bigvee_a^t(f)+\left|\frac{v(t)-v(a)}{v(b)-v(a)}\right|\cdot\bigvee_t^b(f) \tag{2.5}$$

$$\leq\begin{cases}\max\left\{\left|\frac{v(b)-v(t)}{v(b)-v(a)}\right|,\left|\frac{v(t)-v(a)}{v(b)-v(a)}\right|\right\}\bigvee_a^b(f);\\[2mm]\left[\left|\frac{v(b)-v(t)}{v(b)-v(a)}\right|^p+\left|\frac{v(t)-v(a)}{v(b)-v(a)}\right|^p\right]^{\frac{1}{p}}\left\{\left[\bigvee_a^t(f)\right]^q+\left[\bigvee_t^b(f)\right]^q\right\}^{\frac{1}{q}},\\ \qquad\qquad \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2mm]\frac{|v(b)-v(t)|+|v(t)-v(a)|}{|v(b)-v(a)|}\left\{\frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^t(f)-\bigvee_t^b(f)\right|\right\}.\end{cases}$$

(ii) *If v is of bounded variation on $[a,b]$, then*

$$|\Phi_{f,v}(t)| \leq \left|\frac{f(b)-f(t)}{v(b)-v(a)}\right| \cdot \bigvee_a^t (v) + \left|\frac{f(t)-f(a)}{v(b)-v(a)}\right| \cdot \bigvee_t^b (v) \tag{2.6}$$

$$\leq \begin{cases} \max\left\{\left|\frac{f(b)-f(t)}{v(b)-v(a)}\right|, \left|\frac{f(t)-f(a)}{v(b)-v(a)}\right|\right\} \bigvee_a^b (v); \\ \left[\left|\frac{f(b)-f(t)}{v(b)-v(a)}\right|^p + \left|\frac{f(t)-f(a)}{v(b)-v(a)}\right|^p\right]^{\frac{1}{p}} \left\{\left[\bigvee_a^t (f)\right]^q + \left[\bigvee_t^b (f)\right]^q\right\}^{\frac{1}{q}}, \\ \qquad \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{|f(b)-f(t)|+|f(t)-f(a)|}{|v(b)-v(a)|} \left\{\frac{1}{2}\bigvee_a^b (v) + \frac{1}{2}\left|\bigvee_a^t (v) - \bigvee_t^b (v)\right|\right\}. \end{cases}$$

Proof. Utilising the equality (2.3) and taking the modulus, we have successively:

$$\begin{aligned} |\Phi_{f,v}(t)| &\leq \left|\frac{v(b)-v(t)}{v(b)-v(a)}\right| \cdot \left|\int_a^t df(s)\right| + \left|\frac{v(t)-v(a)}{v(b)-v(a)}\right| \cdot \left|\int_t^b df(s)\right| \\ &\leq \left|\frac{v(b)-v(t)}{v(b)-v(a)}\right| \cdot \bigvee_a^t (f) + \left|\frac{v(t)-v(a)}{v(b)-v(a)}\right| \cdot \bigvee_t^b (f) \\ &\leq \begin{cases} \max\left\{\left|\frac{v(b)-v(t)}{v(b)-v(a)}\right|, \left|\frac{v(t)-v(a)}{v(b)-v(a)}\right|\right\} \bigvee_a^b (f); \\ \left[\left|\frac{v(b)-v(t)}{v(b)-v(a)}\right|^p + \left|\frac{v(t)-v(a)}{v(b)-v(a)}\right|^p\right]^{\frac{1}{p}} \left\{\left[\bigvee_a^t (f)\right]^q + \left[\bigvee_t^b (f)\right]^q\right\}^{\frac{1}{q}}, \\ \qquad \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{|v(b)-v(t)|+|v(t)-v(a)|}{|v(b)-v(a)|} \left\{\frac{1}{2}\bigvee_a^b (f) + \frac{1}{2}\left|\bigvee_a^t (f) - \bigvee_t^b (f)\right|\right\}, \end{cases} \end{aligned}$$

where for the last inequality we have used the Hölder inequality.

The inequality (2.6) goes likewise by utilising the equality (2.4). ∎

Remark 1. *Since $v(a) \neq v(b)$, we can assume without loss the generality that $v(a) < v(b)$. Now, if we assume that*

$$v(a) \leq v(t) \leq v(b) \quad \text{for any} \quad t \in (a,b), \tag{2.7}$$

then from the first branch of (2.5) we get the inequality

$$|\Phi_{f,v}(t)| \leq \left[\frac{1}{2} + \frac{\left|v(t) - \frac{v(a)+v(b)}{2}\right|}{v(b)-v(a)}\right] \bigvee_a^b (f), \qquad t \in [a,b]. \tag{2.8}$$

The constant $\frac{1}{2}$ is sharp in (2.8).

To prove the sharpness of the constant we take in (2.8) $v(t) = t$ and then choose $t = \frac{a+b}{2}$. This produces the result:

$$\left|f\left(\frac{a+b}{2}\right) - \frac{f(a)+f(b)}{2}\right| \leq \frac{1}{2}\bigvee_a^b (f), \tag{2.9}$$

which is sharp since for $f(t) = \left|t - \frac{a+b}{2}\right|$, $t \in [a,b]$ we obtain in both sides of (2.9) the same quantity $\frac{b-a}{2}$.

Remark 2. *We also remark that, if v satisfies (2.7), then from the last inequality in (2.5) we get*

$$|\Phi_{f,v}(t)| \le \frac{1}{2}\bigvee_a^b(f) + \frac{1}{2}\left|\bigvee_a^t(f) - \bigvee_t^b(f)\right|, \qquad t \in [a,b] \tag{2.10}$$

for which the first constant $\frac{1}{2}$ is also best possible.

Remark 3. *If f satisfies the property that $f(a) \le f(t) \le f(b)$ for any $t \in [a,b]$, then from the first inequality in (2.6) we get*

$$|\Phi_{f,v}(t)| \le \left[\frac{1}{2}\cdot\frac{f(b)-f(a)}{|v(b)-v(a)|} + \left|\frac{f(t)-\frac{f(a)+f(b)}{2}}{v(b)-v(a)}\right|\right]\bigvee_a^b(f), \qquad t \in [a,b]. \tag{2.11}$$

With the same assumptions for f we have from the second inequality in (2.6) that

$$|\Phi_{f,v}(t)| \le \frac{f(b)-f(a)}{|v(b)-v(a)|}\left\{\frac{1}{2}\bigvee_a^b(v) + \frac{1}{2}\left|\bigvee_a^t(v) - \bigvee_t^b(v)\right|\right\}, \qquad t \in [a,b]. \tag{2.12}$$

The first constant $\frac{1}{2}$ in (2.12) is best possible.

Indeed, if we choose $v(t) = t$ and then $t = \frac{a+b}{2}$ in (2.12), we have

$$\left|\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right)\right| \le \frac{1}{2}[f(b)-f(a)]. \tag{2.13}$$

Now, for $f : [a,b] \to \mathbb{R}$, $f(t) = 0$ if $t \in [a,b]$ and $f(b) = k > 0$, we obtain on both sides the same quantity $\frac{k}{2}$.

3. Bounds for $\Phi_{f,v}$ when $v(a) < v(t) < v(b)$ $(f(a) < f(t) < f(b))$

The following result may be stated as well.

Theorem 2. *Assume that $f, v : [a,b] \to \mathbb{R}$ are bounded and $v(a) \ne v(b)$.*

(i) *If $v(a) < v(t) < v(b)$ for any $t \in (a,b)$, then*

$$|\Phi_{f,v}(t)| \le \frac{1}{4}[v(b)-v(a)]\left[\left|\frac{f(t)-f(a)}{v(t)-v(a)}\right| + \left|\frac{f(b)-f(t)}{|v(b)-v(t)|}\right|\right], \quad t \in [a,b]. \tag{3.1}$$

The constant $\frac{1}{4}$ is best possible.

(ii) *If $f(a) < f(t) < f(b)$ for $t \in (a,b)$, then*

$$|\Phi_{f,v}(t)| \le \frac{1}{4}\frac{[f(b)-f(a)]^2}{|v(b)-v(a)|}\left[\left|\frac{v(t)-v(a)}{f(t)-f(a)}\right| + \left|\frac{v(b)-v(t)}{|f(b)-f(t)|}\right|\right], \ t \in [a,b]. \tag{3.2}$$

Proof. (i) From the first equality in (2.3), we have:

$$\begin{aligned}
|\Phi_{f,v}(t)| &\le \frac{|[v(b)-v(t)][v(t)-v(a)]|}{|v(b)-v(a)|}\left[\left|\frac{f(t)-f(a)}{v(t)-v(a)}\right| + \left|\frac{f(b)-f(t)}{|v(b)-v(t)|}\right|\right] \\
&= \frac{[v(b)-v(t)][v(t)-v(a)]}{|v(b)-v(a)|}\left[\left|\frac{f(t)-f(a)}{v(t)-v(a)}\right| + \left|\frac{f(b)-f(t)}{|v(b)-v(t)|}\right|\right] \\
&\le \frac{1}{4}[v(b)-v(a)]\left[\left|\frac{f(t)-f(a)}{v(t)-v(a)}\right| + \left|\frac{f(b)-f(t)}{|v(b)-v(t)|}\right|\right]
\end{aligned}$$

since, for any $t \in (a,b)$,

$$[v(b)-v(t)][v(t)-v(a)] \le \frac{1}{4}[v(b)-v(a)]^2.$$

For the best constant, choose $v(t)=t$ and then $t=\frac{a+b}{2}$ in (3.1) to obtain

$$\left|\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right| \leq \frac{1}{2}\left[\left|f\left(\frac{a+b}{2}\right)-f(a)\right|+\left|f(b)-f\left(\frac{a+b}{2}\right)\right|\right]. \tag{3.3}$$

If we consider the function $f:[a,b]\to\mathbb{R}$,

$$f(t)=\begin{cases} 0 & \text{if } t\in[a,b) \\ k & \text{if } t=b,\ k>0, \end{cases}$$

then (3.3) becomes an equality with both terms $\frac{k}{2}$.

(ii) The proof goes likewise and the details are omitted. ∎

Remark 4.

(a) *Under the assumptions of (i) of Theorem 2 and if there exist* $L_a>0$, $L_b>0$, $\alpha,\beta\geq 0$ *such that*

$$\left|\frac{f(t)-f(a)}{v(t)-v(a)}\right|\leq L_a(t-a)^{\alpha}, \quad \left|\frac{f(b)-f(t)}{v(b)-v(t)}\right|\leq L_b(b-t)^{\beta}, \quad t\in(a,b), \tag{3.4}$$

then we have the inequality:

$$|\Phi_{f,v}(t)|\leq\frac{1}{4}[v(b)-v(a)]\left[L_a(t-a)^{\alpha}+L_b(b-t)^{\beta}\right], \qquad t\in(a,b). \tag{3.5}$$

(aa) *Under the assumptions of (ii) of Theorem 2 and if there exist the constants* $H_a,H_b>0$ *and* $\gamma,\delta\geq 0$ *such that*

$$\left|\frac{v(t)-v(a)}{f(t)-f(a)}\right|\leq H_a(t-a)^{\gamma}, \quad \left|\frac{v(b)-v(t)}{f(b)-f(t)}\right|\leq H_b(b-t)^{\delta}, \quad t\in(a,b), \tag{3.6}$$

then we have the inequality:

$$|\Phi_{f,v}(t)|\leq\frac{1}{4}\cdot\frac{[f(b)-f(a)]^2}{|v(b)-v(a)|}\left[H_a(t-a)^{\gamma}+H_b(b-t)^{\delta}\right], \qquad t\in(a,b). \tag{3.7}$$

The following corollary provides some uniform bounds in the case where the functions are differentiable.

Corollary 1. *Assume that* $f,v:[a,b]\to\mathbb{R}$ *are continuous on* $[a,b]$ *and differentiable on* (a,b) *with* $v(a)\neq v(b)$.

(i) *If* $v(a)<v(t)<v(b)$ *and* $v'(t)\neq 0$ *for* $t\in(a,b)$, *then*

$$|\Phi_{f,v}(t)|\leq\frac{1}{2}\cdot[v(b)-v(a)]\sup_{s\in(a,b)}\left|\frac{f'(s)}{v'(s)}\right|, \qquad t\in(a,b). \tag{3.8}$$

(ii) *If* $f(a)<f(t)<f(b)$ *and* $f'(t)\neq 0$ *for* $t\in(a,b)$, *then*

$$|\Phi_{f,v}(t)|\leq\frac{1}{2}\cdot\frac{[f(b)-f(a)]^2}{|v(b)-v(a)|}\sup_{s\in(a,b)}\left|\frac{v'(s)}{f'(s)}\right|, \qquad t\in(a,b). \tag{3.9}$$

Proof. (i) Applying Cauchy's mean value theorem, we deduce that for any $t\in(a,b)$ there exists an s between t and a such that

$$\frac{f(t)-f(a)}{v(t)-v(a)}=\frac{f'(s)}{v'(s)}.$$

Therefore,

$$\left|\frac{f(t)-f(a)}{v(t)-v(a)}\right| \leq \sup_{s\in(a,b)}\left|\frac{f'(s)}{v'(s)}\right|, \qquad t\in(a,b)$$

and in a similar manner,

$$\left|\frac{f(b)-f(t)}{v(b)-v(t)}\right| \leq \sup_{s\in(a,b)}\left|\frac{f'(s)}{v'(s)}\right|, \qquad t\in(a,b).$$

Utilising the inequality (2.13) we deduce (3.8).

The proof of (ii) goes likewise and we omit the details. ∎

4. Bounds for $\Phi_{f,v}$ when f,v are Lipschitzian

We can state the following result.

Theorem 3. *Assume that $f,v:[a,b]\to\mathbb{R}$ are bounded functions on $[a,b]$ and $v(a)\neq v(b)$.*

(i) *If there exist constants $M_a,M_b>0$ and $\alpha,\beta>0$ such that $|f(t)-f(a)|\leq M_a(t-a)^{\alpha}$, $|f(b)-f(t)|\leq M_b(b-t)^{\beta}$ for any $t\in[a,b]$ and $v:[a,b]\to\mathbb{R}$ is Riemann integrable on $[a,b]$, then*

$$|\Phi_{f,v}(t)| \leq M_a\left|\frac{v(b)-v(t)}{f(b)-f(t)}\right|(t-a)^{\alpha}+M_b\left|\frac{v(t)-v(a)}{f(t)-f(a)}\right|(b-t)^{\beta} \tag{4.1}$$

for any $t\in[a,b]$.

(ii) *If there exist constants $N_a,N_b>0$, $\gamma,\delta>0$ such that $|v(t)-v(a)|\leq N_a(t-a)^{\gamma}$, $|v(b)-v(t)|\leq N_b(b-t)^{\delta}$ for any $t\in[a,b]$, then*

$$|\Phi_{f,v}(t)| \leq N_b\left|\frac{f(t)-f(a)}{v(t)-v(a)}\right|(b-t)^{\delta}+N_a\left|\frac{f(b)-f(t)}{v(b)-v(t)}\right|(t-a)^{\gamma} \tag{4.2}$$

for any $t\in[a,b]$.

Proof. Utilising the representation (2.3) we have:

$$|\Phi_{f,v}(t)| \leq \frac{|f(t)-f(a)|\,|v(b)-v(t)|+|v(t)-v(a)|\,|f(b)-f(t)|}{|v(b)-v(a)|}$$

for any $t\in[a,b]$, which clearly produces the desired inequalities (4.1) and (4.2). ∎

We notice that, if more information is provided for f and v, then more specific bounds can be obtained. For instance, if f is as in (i) of Theorem 3 and $v(a)<v(t)<v(b)$ for each $t\in(a,b)$, then we get from (4.1) the following inequality:

$$|\Phi_{f,v}(t)| \leq \left[\frac{1}{2}+\left|\frac{v(t)-\frac{v(a)+v(b)}{2}}{v(b)-v(a)}\right|\right]\left[M_a(t-a)^{\alpha}+M_b(b-t)^{\beta}\right] \tag{4.3}$$

for any $t\in[a,b]$.

Similarly, if v satisfies condition (ii) of Theorem 3 and $f(a)<f(t)<f(b)$ for each $t\in(a,b)$, then

$$|\Phi_{f,v}(t)| \leq \left[\frac{1}{2}\cdot\frac{f(b)-f(a)}{|v(b)-v(a)|}+\left|\frac{f(t)-\frac{f(a)+f(b)}{2}}{v(b)-v(a)}\right|\right] \times\left[N_b(b-t)^{\delta}+N_a(t-a)^{\gamma}\right] \tag{4.4}$$

for any $t\in[a,b]$.

If f is $M-$Lipschitzian, then from (4.1) we get

$$\begin{aligned} (4.5)\quad \left|\Phi_{f,v}(t)\right| &\leq M\left[\left|\frac{v(b)-v(t)}{v(b)-v(a)}\right|(t-a)+\left|\frac{v(t)-v(a)}{v(b)-v(a)}\right|(b-t)\right] \\ &\leq M\left[\frac{1}{2}(b-a)+\left|t-\frac{a+b}{2}\right|\right]\left[\left|\frac{v(b)-v(t)}{v(b)-v(a)}\right|+\left|\frac{v(t)-v(a)}{v(b)-v(a)}\right|\right], \end{aligned}$$

for any $t\in[a,b]$.

Also, if v is $N-$Lipschitzian, then from (4.1) we get

$$\begin{aligned} (4.6)\quad \left|\Phi_{f,v}(t)\right| &\leq N\left[\left|\frac{f(t)-f(a)}{v(b)-v(a)}\right|(b-t)+\left|\frac{f(b)-f(t)}{v(b)-v(a)}\right|(t-a)\right] \\ &\leq N\left[\frac{1}{2}(b-a)+\left|t-\frac{a+b}{2}\right|\right]\left[\left|\frac{f(t)-f(a)}{v(b)-v(a)}\right|+\left|\frac{f(b)-f(t)}{v(b)-v(a)}\right|\right] \end{aligned}$$

for any $t\in[a,b]$.

Moreover, if f is $M-$Lipschitzian and $v(a)<v(t)<v(b)$ for any $t\in[a,b]$, then from (4.5) we get the simpler inequality:

$$(4.7)\qquad \left|\Phi_{f,v}(t)\right|\leq M\left[\frac{1}{2}(b-a)+\left|t-\frac{a+b}{2}\right|\right]$$

for any $t\in[a,b]$.

If v is $N-$Lipschitzian and $f(a)<f(t)<f(b)$, $v(a)<v(b)$, then from (4.6) we also have:

$$(4.8)\qquad \left|\Phi_{f,v}(t)\right|\leq N\cdot\frac{f(b)-f(a)}{v(b)-v(a)}\left[\frac{1}{2}(b-a)+\left|t-\frac{a+b}{2}\right|\right],$$

for each $t\in[a,b]$.

5. Applications for Positive Linear Functionals

Let L be a linear class of real-valued functions $g:E\to\mathbb{R}$ having the properties

(L1) $f,g\in L$ imply $(\alpha f+\beta g)\in L$ for all $\alpha,\beta\in\mathbb{R}$;
(L2) $\mathbf{1}\in L$, i.e., if $f_0(t)=1$, $t\in E$, then $f_0\in L$.

An isotonic linear functional $A:L\to\mathbb{R}$ is a functional satisfying

(A1) $A(\alpha f+\beta g)=\alpha A(f)+\beta A(g)$ for all $f,g\in L$ and $\alpha,\beta\in\mathbb{R}$;
(A2) If $f\in L$ and $f\geq 0$, then $A(f)\geq 0$;
(A3) The mapping A is *normalised* if $A(\mathbf{1})=1$.

For a function $u:E\to[a,b]$, we consider the function

$$\Phi_{f,v}(u):=\frac{v\circ u-v(a)}{v(b)-v(a)}\cdot f(b)+\frac{v(b)-v\circ u}{v(b)-v(a)}\cdot f(a)-f\circ u$$

and assume throughout this section that $\Phi_{f,v}(u)\in L$.

It is obvious that for a normalised linear functional $A:L\to\mathbb{R}$ we have

$$A(\Phi_{f,v}(u))=\frac{A(v\circ u)-v(a)}{v(b)-v(a)}\cdot f(b)+\frac{v(b)-A(v\circ u)}{v(b)-v(a)}\cdot f(a)-A(f\circ u)$$

and the inequalities in the previous section can be utilised to provide various upper bounds for the quantity

$$\left|A(\Phi_{f,v}(u))\right|.$$

For the sake of brevity we give here only some bounds that are simple and perhaps more useful for applications.

Proposition 1. *Let* $f : [a, b] \to \mathbb{R}$ *be of bounded variation on* $[a, b]$ *and* $v(a) < v(b)$, $v(a) \le v(t) \le v(b)$ *for each* $t \in [a, b]$. *If* $u \in L$ *so that* $\Phi_{f,v}(u) \in L$ *and* $A : L \to \mathbb{R}$ *is a normalised positive linear functional on* L, *then:*

$$\left|\frac{A(v \circ u) - v(a)}{v(b) - v(a)} \cdot f(b) + \frac{v(b) - A(v \circ u)}{v(b) - v(a)} \cdot f(a) - A(f \circ u)\right| \le \left[\frac{1}{2} + \frac{1}{v(b) - v(a)} A\left(\left|v \circ u - \frac{v(a) + v(b)}{2} \cdot \mathbf{1}\right|\right)\right] \bigvee_a^b (f). \tag{5.1}$$

Proof. Utilising the inequality (2.8) and the properties of the functional A, we have

$$\begin{aligned}|A(\Phi_{f,v}(u))| &\le A(|\Phi_{f,v}(u)|) \\ &\le A\left[\left(\frac{1}{2} + \left|\frac{v \circ u - \frac{v(a)+v(b)}{2}}{v(b) - v(a)}\right|\right) \bigvee_a^b (f)\right] \\ &= \bigvee_a^b (f) \left[\frac{1}{2} + \frac{1}{v(b) - v(a)} A\left(\left|v \circ u - \frac{v(a) + v(b)}{2} \cdot \mathbf{1}\right|\right)\right]\end{aligned}$$

and the inequality (5.1) is proved. ∎

Proposition 2. *Let* $f, v : [a, b] \to \mathbb{R}$ *be bounded and* $v(a) \ne v(b)$. *Also, assume that* $u \in L$ *such that* $\Phi_{f,v}(u) \in L$ *and* $A : L \to \mathbb{R}$ *is a normalised positive linear functional on* L.

(i) *If* $v(a) < v(t) < v(b)$ *for each* $t \in [a, b]$, *then*

$$\left|\frac{A(v \circ u) - v(a)}{v(b) - v(a)} \cdot f(b) + \frac{v(b) - A(v \circ u)}{v(b) - v(a)} \cdot f(a) - A(f \circ u)\right| \le \frac{1}{4}[v(b) - v(a)]\left[A\left(\left|\frac{f - f(a) \cdot \mathbf{1}}{v - v(a) \cdot \mathbf{1}}\right|\right) + A\left(\left|\frac{f(b) \cdot \mathbf{1} - f}{v(b) \cdot \mathbf{1} - v}\right|\right)\right], \tag{5.2}$$

provided $\frac{f - f(a) \cdot \mathbf{1}}{v - v(a) \cdot \mathbf{1}}, \frac{f(b) \cdot \mathbf{1} - f}{v(b) \cdot \mathbf{1} - v} \in L$;

(ii) *If* $f(0) < f(t) < f(b)$ *for* $t \in (a, b)$, *then*

$$\left|\frac{A(v \circ u) - v(a)}{v(b) - v(a)} \cdot f(b) + \frac{v(b) - A(v \circ u)}{v(b) - v(a)} \cdot f(a) - A(f \circ u)\right| \le \frac{1}{4} \cdot \frac{[f(b) - f(a)]^2}{|v(b) - v(a)|}\left[A\left(\left|\frac{v - v(a) \cdot \mathbf{1}}{f - f(a) \cdot \mathbf{1}}\right|\right) + A\left(\left|\frac{v(b) \cdot \mathbf{1} - v}{f(b) \cdot \mathbf{1} - f}\right|\right)\right], \tag{5.3}$$

provided $\frac{v - v(a) \cdot \mathbf{1}}{f - f(a) \cdot \mathbf{1}}, \frac{v(b) \cdot \mathbf{1} - v}{f(b) \cdot \mathbf{1} - f} \in L$.

Utilising Corollary 1 we can state the following result that can be utilised for applications.

Proposition 3. *Let* $f, v : [a, b] \to \mathbb{R}$ *be continuous on* $[a, b]$ *and differentiable on* (a, b). *Also, assume that* $u \in L$ *such that* $\Phi_{f,v}(u) \in L$ *and* $A : L \to \mathbb{R}$ *is a normalised positive functional on* L.

(i) *If v is strictly monotonic on $[a,b]$, then*

$$\left|\frac{A(v\circ u)-v(a)}{v(b)-v(a)}\cdot f(b)+\frac{v(b)-A(v\circ u)}{v(b)-v(a)}\cdot f(a)-A(f\circ u)\right| \leq \frac{1}{2}\left|v(b)-v(a)\right|\sup_{s\in(a,b)}\left|\frac{f'(s)}{v'(s)}\right|. \tag{5.4}$$

(ii) *If f is strictly monotonic on $[a,b]$, then*

$$\left|\frac{A(v\circ u)-v(a)}{v(b)-v(a)}\cdot f(b)+\frac{v(b)-A(v\circ u)}{v(b)-v(a)}\cdot f(a)-A(f\circ u)\right| \leq \frac{1}{2}\cdot\frac{\left[f(b)-f(a)\right]^2}{\left|v(b)-v(a)\right|}\sup_{s\in(a,b)}\left|\frac{v'(s)}{f'(s)}\right|, \tag{5.5}$$

provided $v(a)\neq v(b)$.

For other inequalities for isotonic linear functionals, see the papers [1], [2], [6] and the books [5] and [7].

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