AN INEQUALITY WITH REGARD TO CIRCUMSCRIPTIBLE SIMPLEX

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Dedicated to Professor Lu Yang on the occasion of his 71st birthday

Abstract. An $n-$simplex is circumscriptible if there is a sphere tangent to each of its $n(n+1)/2$ edges. We prove that the radius of the curcumscribed sphere is at least $\sqrt{\frac{2n}{n-1}}$ times the radius of the edge-tangent sphere in the circumscriptible $n-$simplex. This settles affirmatively a part of a problem posed by the authors.

1. Introduction and Main Results

Throughout the paper we denote $\{A_0, A_1, \cdots, A_n\}$ as the vertex set of an n-dimensional simplex Ω in the n-dimensional Euclidean space $\mathbb{E}^n$, V as the volume of Ω, and R, r, ℓ as the radius of the curcumscribed sphere, the inscribed sphere and the edge-tangent sphere of Ω, respectively.

Every $n-$simplex has a circumscribed sphere passing through its $n+1$ vertices and an inscribed sphere tangent to each of its $n+1$ faces. An $n-$simplex is circumscriptible if there is a sphere tangent to each of its $n(n+1)/2$ edges. We call this the edge-tangent sphere of the $n-$simplex. But it is not hard to see that not every $n-$simplex with $n \geq 3$ can have an edge-tangent sphere. However, the following sufficient and necessary condition for a simplex to have an edge-tangent sphere can be found [6] by Lin and Zhu.

Theorem 1.1. *Supposing the edge lengths of an $n-$simplex $\Omega = P_0P_1P_2\cdots P_n$ are $P_iP_j = a_{ij}$ for $0 \leq i < j \leq n$. The $n-$simplex has an edge-tangent sphere if and only if there exist $x_i > 0$ with $0 \leq i \leq n$ satisfying $a_{ij} = x_i + x_j$ for $0 \leq i < j \leq n$.*

A double inequality for the radius of the circumscriptible tetrahedron is proved in [1] and [2] as follows

$$R \geq \sqrt{3}\ell \geq 3r. \tag{1.1}$$

As a generalization of inequality (1.1), Wu and Zhang [2] posed an analogous conjecture for the circumscriptible $n-$simplex.

Conjecture 1.1. *In circumscriptible $n-$simplex Ω, prove or disprove that*

$$R \geq \sqrt{\frac{2n}{n-1}}\ell \geq nr. \tag{1.2}$$

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Recently, Wu and Zhang [3] solved the right hand of inequality (1.2). In this paper, we will settle the left hand of inequality (1.2) affirmatively and obtain the following result.

Theorem 1.2. *For circumscriptible $n-$simplex Ω, we have*

$$R^2 \geq \frac{2n}{n-1}\ell^2. \tag{1.3}$$

2. Preliminary Results

In order to prove Theorem 1.2, we require several lemmas. Throughout this section, let x_i for $0 \leq i \leq n$ be defined by Theorem 1.1, and

$$A = \begin{pmatrix} -2x_0^2 & 2x_1x_0 & 2x_2x_0 & \cdots & 2x_nx_0 \\ 2x_0x_1 & -2x_1^2 & 2x_2x_1 & \cdots & 2x_nx_1 \\ 2x_0x_2 & 2x_1x_2 & -2x_2^2 & \cdots & 2x_nx_2 \\ \vdots & \vdots & \vdots & & \vdots \\ 2x_0x_n & 2x_1x_n & 2x_2x_n & \cdots & -2x_n^2 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} x_0^2 & 1 \\ x_1^2 & 1 \\ x_2^2 & 1 \\ \vdots & \vdots \\ x_n^2 & 1 \end{pmatrix}, B_2 = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_0^2 & x_1^2 & x_2^2 & \cdots & x_n^2 \end{pmatrix},$$

$$D = \begin{pmatrix} 0 & (x_1+x_0)^2 & (x_2+x_0)^2 & \cdots & (x_n+x_0)^2 \\ (x_0+x_1)^2 & 0 & (x_2+x_1)^2 & \cdots & (x_n+x_1)^2 \\ (x_0+x_2)^2 & (x_1+x_2)^2 & 0 & \cdots & (x_n+x_2)^2 \\ \vdots & \vdots & \vdots & & \vdots \\ (x_0+x_n)^2 & (x_1+x_n)^2 & (x_2+x_n)^2 & \cdots & 0 \end{pmatrix}.$$

and let

$$M = \sum_{i=0}^{n} x_i, N = \sum_{i=0}^{n} {x_i}^2, P = \sum_{i=0}^{n} \frac{1}{x_i}, Q = \sum_{i=0}^{n} \frac{1}{x_i^2}.$$

Lemma 2.1. *We have*

$$|A| = (-1)^n (n-1) 2^{2n+1} \left(\prod_{i=0}^{n} x_i \right)^2, \tag{2.1}$$

and

$$A^{-1} = \begin{pmatrix} \frac{2-n}{4n-4} \cdot \frac{1}{x_0^2} & \frac{1}{4n-4} \cdot \frac{1}{x_1x_0} & \frac{1}{4n-4} \cdot \frac{1}{x_2x_0} & \cdots & \frac{1}{4n-4} \cdot \frac{1}{x_nx_0} \\ \frac{1}{4n-4} \cdot \frac{1}{x_0x_1} & \frac{2-n}{4n-4} \cdot \frac{1}{x_1^2} & \frac{1}{4n-4} \cdot \frac{1}{x_2x_1} & \cdots & \frac{1}{4n-4} \cdot \frac{1}{x_nx_1} \\ \frac{1}{4n-4} \cdot \frac{1}{x_0x_2} & \frac{1}{4n-4} \cdot \frac{1}{x_1x_2} & \frac{2-n}{4n-4} \cdot \frac{1}{x_2^2} & \cdots & \frac{1}{4n-4} \cdot \frac{1}{x_nx_2} \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{1}{4n-4} \cdot \frac{1}{x_0x_n} & \frac{1}{4n-4} \cdot \frac{1}{x_1x_n} & \frac{1}{4n-4} \cdot \frac{1}{x_2x_n} & \cdots & \frac{2-n}{4n-4} \cdot \frac{1}{x_n^2} \end{pmatrix}. \tag{2.2}$$

Proof. It is clear to see that

$$\begin{aligned}
|A| &= 2^{n+1}\left(\prod_{i=0}^{n} x_i\right)^2 \begin{vmatrix} -1 & 1 & 1 & \cdots & 1 \\ 1 & -1 & 1 & \cdots & 1 \\ 1 & 1 & -1 & \cdots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \cdots & -1 \end{vmatrix} \\
&= 2^{n+1}\left(\prod_{i=0}^{n} x_i\right)^2 \begin{vmatrix} -1 & 1 & 1 & \cdots & 1 \\ 2 & -2 & 0 & \cdots & 0 \\ 2 & 0 & -2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 2 & 0 & 0 & \cdots & -2 \end{vmatrix} \\
&= 2^{n+1}\left(\prod_{i=0}^{n} x_i\right)^2 \begin{vmatrix} n-1 & 1 & 1 & \cdots & 1 \\ 0 & -2 & 0 & \cdots & 0 \\ 0 & 0 & -2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & -2 \end{vmatrix} \\
&= 2^{n+1}\left(\prod_{i=0}^{n} x_i\right)^2 \cdot (-1)^n (n-1) 2^n \\
&= (-1)^n (n-1) 2^{2n+1}\left(\prod_{i=0}^{n} x_i\right)^2 .
\end{aligned}$$

Let us to compute A^{-1}. We know that the adjoint of A is

$$A^* = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1,n+1} \\ A_{21} & A_{22} & \cdots & A_{2,n+1} \\ \vdots & \vdots & & \vdots \\ A_{n+1,1} & A_{n+1,2} & \cdots & A_{n+1,n+1} \end{pmatrix},$$

where A_{ij} is the cofactor of the element $2x_{i-1}x_{j-1}(i \neq j)$ or $-2x_{i-1}^2(i = j)$ of A.

From the process of computing $|A|$ above, for $0 \leq j \leq n$, it is easily to obtain that

$$A_{jj} = (-1)^{n-1}(n-2)2^{2n-1}\left(\prod_{\substack{i=0,\\ i\neq j-1}}^{n} x_i\right)^2 .$$

Now, we compute A_{ij} for $0 \leq i \leq j \leq n$ because of $A_{ij} = A_{ji}$ with $A = A'$. That is

$$A_{ij} = (-1)^{i+j} \begin{vmatrix} -2x_0^2 & 2x_1x_0 & \cdots & 2x_{j-2}x_0 & 2x_jx_0 & \cdots & 2x_nx_0 \\ 2x_0x_1 & -2x_1^2 & \cdots & 2x_{j-2}x_1 & 2x_jx_1 & \cdots & 2x_nx_1 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 2x_0x_{i-2} & 2x_1x_{i-2} & \cdots & 2x_{j-2}x_{i-2} & 2x_jx_{i-2} & \cdots & 2x_nx_{i-2} \\ 2x_0x_i & 2x_1x_i & \cdots & 2x_{j-2}x_i & 2x_jx_i & \cdots & 2x_nx_i \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 2x_0x_n & 2x_1x_n & \cdots & 2x_{j-2}x_n & 2x_jx_n & \cdots & -2x_n^2 \end{vmatrix}$$

$$= (-1)^{i+j} 2^n \frac{\left(\prod_{i=0}^n x_i\right)^2}{x_{i-1}x_{j-1}} \begin{vmatrix} -1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & -1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & & & \vdots \\ 1 & 1 & & C_1 & & 1 \\ \vdots & \vdots & & & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & -1 \end{vmatrix}$$

$$= (-1)^{i+j} 2^n \frac{\left(\prod_{i=0}^n x_i\right)^2}{x_{i-1}x_{j-1}} \begin{vmatrix} -2 & & & & & \\ & -2 & & & & \\ & & \ddots & & & \\ & & & C_2 & & \\ & & & & \ddots & \\ & & & & & -2 \end{vmatrix}$$

$$= (-1)^{i+j} 2^n (-2)^{n-(j-i)} |C_2| \frac{\left(\prod_{i=0}^n x_i\right)^2}{x_{i-1}x_{j-1}}$$

$$= (-1)^{i+j} 2^n (-2)^{n-(j-i)} (-1)^{j-i+1} (-2)^{j-i-1} \frac{\left(\prod_{i=0}^n x_i\right)^2}{x_{i-1}x_{j-1}}$$

$$= (-1)^{2j+1} 2^n (-2)^{n-1} \frac{\left(\prod_{i=0}^n x_i\right)^2}{x_{i-1}x_{j-1}}$$

$$= (-1)^n 2^{2n-1} \frac{\left(\prod_{i=0}^n x_i\right)^2}{x_{i-1}x_{j-1}},$$

where

$$C_1 = \begin{pmatrix} 1 & -2 & 1 & \cdots & 1 \\ 1 & 1 & -2 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & -2 \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}_{(j-i)\times(j-i)},$$

and

$$C_2 = \begin{pmatrix} 0 & -2 & 0 & \cdots & 0 \\ 0 & 0 & -2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -2 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}_{(j-i)\times(j-i)}.$$

Hence,

$$A^{-1} = \frac{1}{|A|} A^* = \begin{pmatrix} \frac{2-n}{4n-4} \cdot \frac{1}{x_0^2} & \frac{1}{4n-4} \cdot \frac{1}{x_1 x_0} & \frac{1}{4n-4} \cdot \frac{1}{x_2 x_0} & \cdots & \frac{1}{4n-4} \cdot \frac{1}{x_n x_0} \\ \frac{1}{4n-4} \cdot \frac{1}{x_0 x_1} & \frac{2-n}{4n-4} \cdot \frac{1}{x_1^2} & \frac{1}{4n-4} \cdot \frac{1}{x_2 x_1} & \cdots & \frac{1}{4n-4} \cdot \frac{1}{x_n x_1} \\ \frac{1}{4n-4} \cdot \frac{1}{x_0 x_2} & \frac{1}{4n-4} \cdot \frac{1}{x_1 x_2} & \frac{2-n}{4n-4} \cdot \frac{1}{x_2^2} & \cdots & \frac{1}{4n-4} \cdot \frac{1}{x_n x_2} \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{1}{4n-4} \cdot \frac{1}{x_0 x_n} & \frac{1}{4n-4} \cdot \frac{1}{x_1 x_n} & \frac{1}{4n-4} \cdot \frac{1}{x_2 x_n} & \cdots & \frac{2-n}{4n-4} \cdot \frac{1}{x_n^2} \end{pmatrix}.$$

□

Lemma 2.2. *For $x_i > 0$ with $0 \le i \le n$,*

$$|D| = (-1)^n \frac{2^{2n-3}}{n-1} \left(\prod_{i=0}^{n} x_i \right)^2 \cdot \{[MP-(n-1)(n-3)]^2 - [M^2-(n-1)N][P^2-(n-1)Q]\}.$$

Proof. It is easily to find that

$$\begin{vmatrix} A & B_1 \\ -B_2 & E_2 \end{vmatrix} = \begin{vmatrix} A & B_1 \\ 0 & E_2 + B_2A^{-1}B_1 \end{vmatrix} = |A| \cdot |E_2 + B_2A^{-1}B_1|, \tag{2.3}$$

and

$$\begin{vmatrix} A & B_1 \\ -B_2 & E_2 \end{vmatrix} = \begin{vmatrix} A + B_1B_2 & B_1 \\ 0 & E_2 \end{vmatrix} = |A + B_1B_2|, \tag{2.4}$$

where $E_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Using (2.2), we obtain

$$E_2 + B_2A^{-1}B_1 = \begin{pmatrix} \frac{1}{4(n-1)}[MP-(n-3)(n-1)] & \frac{1}{4(n-1)}\left[P^2-(n-1)Q\right] \\ \frac{1}{4(n-1)}\left[M^2-(n-1)N\right] & \frac{1}{4(n-1)}[MP-(n-3)(n-1)] \end{pmatrix}. \tag{2.5}$$

From (2.1) and (2.3)–(2.5), it is deduced that

$$\begin{aligned} |D| &= |A + B_1B_2| \\ &= |A| \cdot |E_2 + B_2A^{-1}B_1| \\ &= (-1)^n \frac{2^{2n-3}}{n-1} \left(\prod_{i=0}^{n} x_i \right)^2 \\ &\quad \cdot \{[MP-(n-1)(n-3)]^2 - [M^2-(n-1)N][P^2-(n-1)Q]\}. \end{aligned}$$

□

Lemma 2.3. ([5, Corollary 2, p. 96]) *Let $P_iP_j = a_{ij} = x_i + x_j$ with $0 \le i \le j \le n$ be the edge lengths of circumscriptible $n-$simplex Ω. Then*

$$(n!)^2V^2R^2 = -\det\left(-\frac{1}{2}a_{ij}^2\right) = \frac{(-1)^n|D|}{2^{n+1}}.$$

Lemma 2.4. ([4, Theorem 4.10, and Theorem 7.2])*For any circumscriptible $n-$simplex Ω, we have*

$$P^2 - (n-1)Q > 0, \tag{2.6}$$

and

$$\ell^2 = \frac{2(n-1)}{P^2-(n-1)Q}.$$

Lemma 2.5. ([6, Corollary 1]) *Given a circumscriptible $n-$simplex Ω,*

$$(n!)^2V^2\ell^2 = 2^n(n-1)\left(\prod_{i=0}^{n} x_i\right)^2.$$

Lemma 2.6. *The radius R of the curcumscribed sphere of circumscriptible $n-$simplex Ω is given by*

$$\left(\frac{R}{\ell}\right)^2 = \frac{[MP-(n-1)(n-3)]^2-[M^2-(n-1)N][P^2-(n-1)Q]}{16(n-1)^2}, \tag{2.7}$$

and

$$R^2 = \frac{[MP-(n-1)(n-3)]^2-[M^2-(n-1)N][P^2-(n-1)Q]}{8(n-1)[P^2-(n-1)Q]}.$$

Proof. This follow straightforwardly from Lemmas 2.2–2.5 by standard arguments. □

3. The Proof of Theorem 1.2

Proof. If $M=\sum_{i=0}^n x_i, N=\sum_{i=0}^n {x_i}^2, P=\sum_{i=0}^n \frac{1}{x_i}$, and $Q=\sum_{i=0}^n \frac{1}{x_i^2}$ with $x_i>0$ for $0\le i\le n$, by using the well-known power mean inequality and Chuachy inequality, then we have $N\ge \frac{M^2}{n+1}, Q\ge \frac{P^2}{n+1}$, and $MP\ge (n+1)^2$. Further considering (2.6) follows that

$$\begin{aligned}[M^2-&(n-1)N][P^2-(n-1)Q]\\ &\le \left[M^2-\frac{(n-1)}{n+1}M^2\right]\left[P^2-\frac{(n-1)}{n+1}P^2\right]=\frac{4}{(n+1)^2}M^2P^2.\end{aligned} \tag{3.1}$$

From (3.1), $1-\frac{4}{(n+1)^2}>0$ and the function

$$y=\left[1-\frac{4}{(n+1)^2}\right]\left[x-\frac{(n-3)(n+1)^2}{n+3}\right]^2-\frac{4(n-1)(n-3)^2}{n+3}$$

is increasing on interval $\left(\frac{(n-3)(n+1)^2}{n+3},+\infty\right)$, and $MP\ge (n+1)^2\ge \frac{(n-3)(n+1)^2}{n+3}$ for $n\ge 2$, we obtain

$$\begin{aligned}&[MP-(n-1)(n-3)]^2-[M^2-(n-1)N][P^2-(n-1)Q]\\ &\quad\ge [MP-(n-1)(n-3)]^2-\frac{4}{(n+1)^2}M^2P^2\\ &\quad=\left[1-\frac{4}{(n+1)^2}\right]\left[MP-\frac{(n-3)(n+1)^2}{n+3}\right]^2-\frac{4(n-1)(n-3)^2}{n+3}\\ &\quad\ge\left[1-\frac{4}{(n+1)^2}\right]\left[(n+1)^2-\frac{(n-3)(n+1)^2}{n+3}\right]^2-\frac{4(n-1)(n-3)^2}{n+3}\\ &\quad=32n(n-1).\end{aligned} \tag{3.2}$$

According to (2.7) and (3.2), we get

$$\left(\frac{R}{\ell}\right)^2\ge\frac{32n(n-1)}{16(n-1)^2}=\frac{2n}{n-1}.$$

It is clear to show that inequality (1.3) holds.

Thus, the proof of Theorem 1.2 is completed. □

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