MONOTONICITY RESULTS OF INTEGRAL MEAN AND APPLICATION TO EXTENSION OF THE SECOND GAUTSCHI-KERSHAW'S INEQUALITY

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Abstract. Let $f \in C^{\infty}(I)$, and let $x, x+s, x+t \in I$ and h_n be defined by

$$h_n(x) = I_{f^{(n)}}(x+s, x+t) - x, \quad n = 0, 1, 2, \ldots.$$

The monotonicity results of this function are considered, where $I_f(s,t)$ is integral f-mean of s and t. This generalities results given by N. Elezović and J. Pečarić (2000), and N. Batir (2007). This result is applied to extension of the second Gautschi-Kershaw's inequality.

1. Introduction

A function f is said to be completely monotonic on an interval I, if f has derivatives of all orders on I and satisfies

$$(-1)^n f^{(n)}(x) \geq 0 \quad \text{for all} \quad x \in I \quad \text{and} \quad n = 0, 1, 2, \ldots. \tag{1}$$

If the inequality (1) is strict, then f is said to be strictly completely monotonic on I. It is known (Bernstein's Theorem) that f is completely monotonic on $(0, \infty)$ if and only if

$$f(x) = \int_0^{\infty} e^{-xt} \, \mathrm{d}\mu(s),$$

where μ is a nonnegative measure on $[0, \infty)$ such that the integral converges for all $x > 0$. See [34, p.161]. Completely monotonic functions have remarkable applications in different branches. For instance, they play a role in potential theory, probability theory, physics,, numerical and asymptotic analysis, and combinatorics. A detailed collection of the most important properties of completely monotonic functions can be found in [34, Chapter IV], and in an abstract in [5]. Recall [1, 27] that a function f is said to be completely monotonic on an interval I, if f has derivatives of all orders on I such that

$$(-1)^n f^{(n)}(x) \geq 0 \tag{2}$$

holds for all nonnegative integer n.

In 1959 W. Gautschi [15] presented the remarkable inequality:

$$n^{1-s} < \frac{\Gamma(n+1)}{\Gamma(n+s)} < \exp[(1-s)\psi(n+1)], \quad 0 < s < 1, \, n = 1, 2, \ldots, \tag{3}$$

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where $\psi = \Gamma'/\Gamma$ denotes the logarithmic derivative of the gamma function. In 1983 D. Kershaw [21] gave the following closer bounds:

$$\left(x+\frac{s}{2}\right)^{1-s} < \frac{\Gamma(x+1)}{\Gamma(x+s)} < \left(x-\frac{1}{2}+\sqrt{x+\frac{1}{4}}\right)^{1-s}, \tag{4}$$

$$\exp[(1-s)\psi(x+\sqrt{s})] < \frac{\Gamma(x+1)}{\Gamma(x+s)} < \exp\left[(1-s)\psi\left(x+\frac{s+1}{2}\right)\right] \tag{5}$$

for real $x>0$ and $0<s<1$. Inequalities (4) and (5) are called in the literature the first and the second Gautschi-Kershaw's inequality, respectively.

Inequalities for the ratio $\Gamma(x+1)/\Gamma(1+\lambda)$ $(x>0; \lambda\in(0,1))$ have a remarkable application, they can be used to obtain estimates for ultraspherical polynomials. The left-hand inequality of (4) was considered in 1984 by L. Lorch [23]. He obtained the following results for integer $x>0$:

$$\frac{\Gamma(x+1)}{\Gamma(x+s)} > \left(x+\frac{s}{2}\right)^{1-s} \quad \text{for} \quad 0<s<1 \quad \text{or} \quad s>2, \tag{6}$$

$$\frac{\Gamma(x+1)}{\Gamma(x+s)} < \left(x+\frac{s}{2}\right)^{1-s} \quad \text{for} \quad 1<s<2. \tag{7}$$

Lorch [23] used (6) to prove a sharpened inequality for ultraspherical polynomials of degree $n=0,1,2\ldots$, and prameter $0<\lambda<1$,

$$(\sin\theta)^\lambda |P_n^{(\lambda)}(\cos\theta)| < \frac{2^{1-\lambda}}{\Gamma(\lambda)}(n+\lambda)^{\lambda-1}, \quad 0\le\theta\le\pi, \tag{8}$$

which refines the customary Bernstein inequality [6, p. 171]

$$(\sin\theta)^\lambda |P_n^{(\lambda)}(\cos\theta)| < \frac{2^{1-\lambda}}{\Gamma(\lambda)}n^{\lambda-1}, \quad 0\le\theta\le\pi. \tag{9}$$

In (9) the constant $2^{1-\lambda}/\Gamma(\lambda)$ can not be replaced by a smaller one, but it is possible to improve the factor $n^{\lambda-1}$. In 1992, A. Laforgia [22] proved in (9) the term $n^{\lambda-1}$ can be replaced by $\Gamma(n+\lambda)/\Gamma(n+1)$. Since $\Gamma(n+\lambda)/\Gamma(n+1) < n^{\lambda-1}$, see [15], this provides another refinement of Bernstein inequality, but does not improve Lorch's result because $(n+\lambda)^{\lambda-1} < \Gamma(n+\lambda)/\Gamma(n+1)$ for $0<\lambda<1$, see [16, 23]. In 1994, Y. Chow et al. [10] showed that (9) holds with $\Gamma(n+2\lambda)(\Gamma(n+1))^{-1}(n+\lambda)^{-\lambda}$ instead of $n^{\lambda-1}$. This sharpens Lorch's result for all $\lambda\in(0,1/2)$, since the inequality $\Gamma(n+2\lambda)(\Gamma(n+1))^{-1}(n+\lambda)^{-\lambda} < (n+\lambda)^{\lambda-1}$ is valid for all $\lambda\in(0,1/2)$, see [2, 16]. In 1997, H. Alzer [2] proved the following inequality

$$(\sin\theta)^\lambda |P_n^{(\lambda)}(\cos\theta)| < \frac{2^{1-\lambda}}{\Gamma(\lambda)}\frac{\Gamma(n+\frac{3}{2}\lambda)}{\Gamma(n+1+\frac{1}{2}\lambda)}, \quad 0\le\theta\le\pi, \tag{10}$$

and showed that the inequality (10) refines results given by Bernstein, Lorch and Laforgia. In [2] Alzer showed also that the bound $\Gamma(n+\frac{3}{2}\lambda)/\Gamma(n+1+\frac{1}{2}\lambda)$ can not be compared with the bound given by Chow et al., that is, with $\Gamma(n+2\lambda)[\Gamma(n+1)]^{-1}(n+\lambda)^{-\lambda}$. Earlier, Durand [11] generalized the standard result (9). As a consequence of (23) in [11], there follows the bound, for $0<\lambda<1$, $n\ge 0$,

$$(\sin\theta)^\lambda |P_n^{(\lambda)}(\cos\theta)| < \frac{\Gamma(n/2+\lambda)}{\Gamma(\lambda)\Gamma(n/2+1)}, \quad 0\le\theta\le\pi. \tag{11}$$

In [2] Alzer does not compare (10) with Durand's bound (11). In [16] C, Giordano and A. Laforgia showed that (11) is sharper than (10).

C. Giordano et al. [17] and B. Palumbo [26] gave a unified treatment and some extensions of Gautschi-Kershaw type inequalities. For each $s > 0$, $x > 0$, the inequality (4) is valid for $0 < s < 1$ or $s > 2$, while the reverse inequality is valid for $1 < s < 2$; and the inequality (5) is valid for $0 < s < 1$, while the reverse inequality is valid for $s > 1$.

Let us denote that

$$f_1(x) = \frac{\Gamma(x+1)}{\Gamma(x+s)}(x+\alpha)^{s-1},$$
$$f_2(x) = \frac{\Gamma(x+1)}{\Gamma(x+s)}\exp[(1-s)\psi(x+\beta)],$$

respectively, for all $x > 0$ and $s, \alpha, \beta > 0$. Since

$$\lim_{x\to\infty} f_1(x) = 1 \quad \text{and} \quad \lim_{x\to\infty} f_2(x) = 1,$$

the lower bound and upper bound in the Gautschi-Kershaw type inequalities actually follow from the monotonicity properties of the functions f_1 and f_2 respect to x, fixed α and β, respectively, [7, 18, 20, 25]. Bustoz and Ismail [7] proved the following more general results related to Gautschi-Kershaw's inequalities. The functions

$$f_1(x) = \frac{\Gamma(x+1)}{\Gamma(x+s)}(x+s/2)^{s-1},$$
$$f_2(x) = \frac{\Gamma(x+1)}{\Gamma(x+s)}\exp\left[(1-s)\psi\left(x+\frac{s+1}{2}\right)\right]$$

are completely monotonic on $(0,\infty)$ for $0 \le s \le 1$. Other classes of completely monotonic functions were presented in [25] and [3]. As a consequence, new upper and lower bounds are derived for the ratio $\Gamma(x+1)/\Gamma(x+s)$. Recently, logarithmically completely monotonic functions involving the ratio $\Gamma(x+t)/\Gamma(x+s)$ were presented in [29, 30, 31], as a consequence, new upper and lower bounds are derived for the ratio $\Gamma(x+t)/\Gamma(x+s)$. In [19] the inequalities for the product of gamma functions were presented.

It was shown in [8, 13, 28] that let $s, t > 0, r = \min\{s,t\}$. Then

$$z(x) = \left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)} - x$$

is strictly convex and decreasing function on $(-r,\infty)$ for $|t-s| < 1$, and concave and increasing on the same interval for $|t-s| > 1$. This implies the following result: For all $x > 0$ and $s, t > 0$, the inequalities

$$\left(x+\frac{s+t-1}{2}\right)^{t-s} < \frac{\Gamma(x+t)}{\Gamma(x+s)} < \left(x+\left[\frac{\Gamma(t)}{\Gamma(s)}\right]^{1/(t-s)}\right)^{t-s} \tag{12}$$

holds for $s < t < s+1$ and $t < s-1$, and with reversed sign for $s+1 < t$ and $s-1 < t < s$. The bounds are the best possible.

The inequality (5) can be written as

$$\psi(x+\sqrt{s}) < \frac{\log\Gamma(x+1) - \log\Gamma(x+s)}{1-s} < \psi\left(x+\frac{s+1}{2}\right) \tag{13}$$

N. Elezović and J. Pečarić [13] considered more general form

$$\psi(x+\alpha) < \frac{\log\Gamma(x+t)-\log\Gamma(x+s)}{t-s} < \psi\left(x+\beta\right) \tag{14}$$

for $x>0, s, t>0$, and proved that

$$\alpha = I_{\psi}(s,t), \quad \beta = \frac{s+t}{2}$$

are the best possible [13, Theorem 4], where

$$I_{\psi}(s,t) = \psi^{-1}\left(\frac{1}{t-s}\int_s^t \psi(u)du\right)$$

is integral ψ-mean of s and t, ψ^{-1} denotes the inverse function of ψ.

Stolarsky's mean $S_p(a,b)$ of two positive numbers a, b is defined in [33] for $a=b$ by $S_p(a,b)=a$ and for $a\neq b$ by

$$\begin{aligned} S_p(a,b) &= \left(\frac{b^p-a^p}{p(b-a)}\right)^{1/(p-1)}, \quad p\neq 0,\ 1;\\ S_0(a,b) &= \frac{b-a}{\ln b-\ln a} = L(a,b);\\ S_1(a,b) &= \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)} = I(a,b). \end{aligned}$$

Clearly,

$$S_2(a,b) = A(a,b), \quad S_{-1}(a,b) = G(a,b).$$

Where A, G, L, I are arithmetic, geometric, logarithmic and identric mean. It is known that $S_p(a,b)$ for $a\neq b$ is a strictly increasing function of p. Clearly, for $a\neq b$,

$$G(a,b) < L(a,b) < I(a,b) < A(a,b).$$

It was shown [12, Lemma 1] that for $s,t>0$,

$$\psi(L(s,t)) < \frac{\log\Gamma(t)-\log\Gamma(s)}{t-s} < \psi(A(s,t)). \tag{15}$$

N. Batir [4, Theorem 2.7] established an extended form of (15): Let x and y be positive real numbers and n be a positive integer. Then

$$\begin{aligned} (-1)^n\psi^{(n+1)}\left(\frac{x+y}{2}\right) &< \frac{(-1)^n[\psi^{(n)}(x)-\psi^{(n)}(y)]}{x-y}\\ &< (-1)^n\psi^{(n+1)}\left(S_{-(n+1)}(x,y)\right). \end{aligned} \tag{16}$$

Recently, F. Qi [32] presented closer lower bound:

$$(-1)^n\psi^{(n+1)}\left(I(x,y)\right) < \frac{(-1)^n[\psi^{(n)}(x)-\psi^{(n)}(y)]}{x-y}. \tag{17}$$

Now let us give main theorems in this article. The following Theorem 1 generalities Theorem 4 and Theorem 5 given by N. Elezović and J. Pečarić [12], and Lemma 1.5 due to N. Batir [4].

Theorem 1. *Let $f \in C^\infty(I)$ and satisfies*

$$(-1)^k f^{(k+1)}(x) > 0, k = 0, 1, 2, \dots. \tag{18}$$

Let $x, x+s, x+t \in I$ and h_n be defined by

$$h_n(x) = I_{f^{(n)}}(x+s, x+t) - x, \quad n = 0, 1, 2, \dots. \tag{19}$$

Then,
(a) $x \mapsto h_n(x)$ is strictly increasing and strictly concave on I.
(b) $\min\{s,t\} < h_n(x) < \frac{s+t}{2}$ for all $x \in I$ and all $n = 0, 1, 2, \dots$.
(c) $n \mapsto h_n(x)$ is strictly decreasing.

As direct consequences of Theorem 1, an extended form of the second Gautschi-Kershaw's inequality is deduced as follows.

Theorem 2. *Let $s, t > 0$ be given constants, then for all $x > 0$ and all integers $n \geq 0$,*

$$\begin{aligned}(-1)^n \psi^{(n)}(x+\alpha) &< \frac{(-1)^n[\psi^{(n-1)}(x+t) - \psi^{(n-1)}(x+s)]}{t-s} \\ &< (-1)^n \psi^{(n)}(x+\beta)\end{aligned} \tag{20}$$

holds with the best possible constants

$$\alpha = I_{\psi^{(n)}}(s,t), \quad \beta = \frac{s+t}{2},$$

where let us denote $\psi^{(-1)}(x) = \log\Gamma(x)$.

Remark 1. In particular, when $n = 0$ (20) become (14).

2. Proofs of theorems

In order to prove our main results, we need the following results. Let $I \subset \mathbb{R}$ be an open interval, $f : I \to \mathbb{R}$ strictly convex (concave) differentiable function, $s, t \in I$. Then there exists the unique $\alpha \in [s,t]$ for which

$$f'(\alpha) = \frac{f(t) - f(s)}{t-s}.$$

α is called differential f-mean of s and t, and donoted by

$$D_f = D_f(s,t) = (f')^{-1}\left(\frac{f(t) - f(s)}{t-s}\right). \tag{21}$$

If $f : I \to \mathbb{R}$ is strictly monotone, then there exists the unique $\beta \in [s,t]$ for which

$$\frac{1}{t-s}\int_s^t f(u)du = f(\beta).$$

β is called integral f-mean of s and t, and donoted by

$$I_f = I_f(s,t) = f^{-1}\left(\frac{1}{t-s}\int_s^t f(u)du\right). \tag{22}$$

Obviously, for convex (concave) differentiable function f, it holds

$$I_{f'} = D_f. \tag{23}$$

Lemma 1. *Let $f \in C^2(I)$. Suppose*

$$\begin{cases} (i) & f'(x) > 0, \\ (ii) & f''(x) < 0, \\ (iii) & \left(\frac{f''(x)}{f'(x)}\right)' > 0, \end{cases} \quad \textit{or} \quad \begin{cases} (i)' & f'(x) < 0, \\ (ii)' & f''(x) > 0, \\ (iii)' & \left(\frac{f''(x)}{f'(x)}\right)' > 0. \end{cases}$$

Then,

(i) $D_f < I_f$.

(ii) The function $h_0(x) = I_f(x+s, x+t) - x$ is strictly increasing.

Lemma 2. *Suppose $f : I \to \mathbb{R}$ satisfies*

$$\begin{cases} (i) & f'(x) > 0, \\ (ii) & f''(x) < 0, \\ (iii) & f'''(x) > 0, \\ (iv) & \left(\frac{f''(x)}{f'(x)}\right)' > 0, \ \left(\frac{f'''(x)}{f''(x)}\right)' > 0, \end{cases} \quad \textit{or} \quad \begin{cases} (i)' & f'(x) < 0, \\ (ii)' & f''(x) > 0, \\ (iii)' & f'''(x) < 0, \\ (iv)' & \left(\frac{f''(x)}{f'(x)}\right)' > 0, \ \left(\frac{f'''(x)}{f''(x)}\right)' > 0. \end{cases}$$

Then the function $h_0(x) = I_f(x+s, x+t) - x$ is strictly concave.

Lemma 1 and Lemma 2 were established in [12].

Now we are in position to prove our Theorem 1 and Theorem 2.

Proof of Theorem 1. It is well known [34] that if ϕ is strictly completely monotonic on I, then

$$\phi^{(k+1)}(x)\phi^{(k-1)}(x) > \phi^{(k)}(x)^2, \quad x \in I, k = 1, 2, \ldots,$$

this yields

$$\left(\frac{\phi^{(k)}(x)}{\phi^{(k-1)}(x)}\right)' = \frac{\phi^{(k+1)}(x)\phi^{(k-1)}(x) - \phi^{(k)}(x)^2}{\phi^{(k-1)}(x)^2} > 0.$$

The choice $\phi = f'$ leads to

$$\left(\frac{f^{(k+1)}(x)}{f^{(k)}(x)}\right)' > 0, \quad x \in I, k = 1, 2, \ldots.$$

If n is an even number, then

$$\begin{cases} f^{(n+1)}(x) > 0, \\ f^{(n+2)}(x) < 0, \\ f^{(n+3)}(x) > 0. \end{cases}$$

The conditions $(i) - (iv)$ of Lemma 1 and Lemma 2 are satisfied.

If n is an odd number, then

$$\begin{cases} f^{(n+1)}(x) < 0, \\ f^{(n+2)}(x) > 0, \\ f^{(n+3)}(x) < 0. \end{cases}$$

The conditions $(i)' - (iv)'$ of Lemma 1 and Lemma 2 are satisfied. Hence, $x \mapsto h_n(x)$ is strictly increasing and strictly concave on I. This proves (a).

(19) can be written as

$$f^{(n-1)}(x+t) - f^{(n-1)}(x+s) = f^{(n)}(x + h_n(x))(t-s).$$

This implies by mean value theorem for differentiation that

$$\min\{s,t\} < h_n(x) < \max\{s,t\}.$$

Let us denote $y = f^{(n)}(x)$, $x = (f^{(n)})^{-1}(y)$, then

$$((f^{(n)})^{-1}(y))'' = -\frac{f^{(n+2)}(x)}{f^{(n+1)}(x)}\frac{1}{f^{(n+1)}(x)^2} > 0,$$

the function $(f^{(n)})^{-1}$ is strictly convex. By Jensen's integral inequality, we get

$$h_n(x) < \frac{1}{t-s}\int_{x+s}^{x+t} u du - x = \frac{s+t}{2}.$$

Hence,

$$\min\{s,t\} < h_n(x) < \frac{s+t}{2}.$$

which proves (b).

It is easy to see that

$$h_{n+1}(x) < h_n(x)$$

is equivalent to

$$D_{f^{(n)}} < I_{f^{(n)}}.$$

If n is an even number, then the conditions $(i) - (iii)$ of Lemma 1 are satisfied. If n is an odd number, then the conditions $(i)' - (iii)'$ of Lemma 1 are satisfied. This was proved in the proof of (a), hence, $D_{f^{(n)}} < I_{f^{(n)}}$ is true by Lemma 1. This completes the proof of Theorem 1. □

Remark 2. From the proof of the Theorem 1, we easily see that if f is strictly completely monotonic, then Theorem 1 holds too.

Proof of Theorem 2. It is well known [24, p.16] that

$$(-1)^m\psi^{(m+1)}(x) = \int_0^\infty \frac{t^m}{1-e^{-t}}e^{-xt}\,\mathrm{d}t > 0$$

for $x > 0$ and $m = 0, 1, 2, \ldots$. By Theorem 1, we have for $x > 0, s, t > 0$ and all integers $n \geq 0$,

$$I_{\psi^{(n)}}(s,t) < I_{\psi^{(n)}}(x+s, x+t) - x < \frac{s+t}{2}. \tag{24}$$

Since the fuontion $x \mapsto I_{\psi^{(n)}}(x+s, x+t) - x$ is strictly increasing $(0,\infty)$, the constant $I_{\psi^{(n)}}(s,t)$ in (24) is the best possible.

The right inequality of (16) can be written as

$$(-1)^n\psi^{(n)}\left(S_{-n}(x,y)\right) < \frac{(-1)^n[\psi^{(n-1)}(x) - \psi^{(n-1)}(y)]}{x-y} \tag{25}$$

for $x, y > 0$ and all integers $n \geq 2$. In [9], the inequality (25) was extended to all integers $n \geq 0$, let us denote $\psi^{(-1)}(x) = \log\Gamma(x)$. (25) can be written as

$$S_{-n}(x,y) < I_{\psi^{(n)}}(x,y),$$

this implies

$$S_{-n}(x+s, x+t) - x < I_{\psi^{(n)}}(x+s, x+t) - x < \frac{s+t}{2}.$$

N.Batir proved in [4, Lemma 1.4] that

$$\lim_{x\to\infty}[S_{-n}(x+s,x+t)-x]=\frac{s+t}{2},$$

and then

$$\lim_{x\to\infty}\left[I_{\psi^{(n)}}(x+s,x+t)-x\right]=\frac{s+t}{2},$$

hence, the constant $\frac{s+t}{2}$ in (24) is the best possible. Write (24) as

$$x+I_{\psi^{(n)}}(s,t)<(\psi^{(n)})^{-1}\left(\frac{1}{t-s}\int_{x+s}^{x+t}\psi^{(n)}(u)du\right)<x+\frac{s+t}{2}. \tag{26}$$

Since function $x\mapsto(-1)^n\psi^{(n)}(x)$ is strictly increasing, we obtain

$$\begin{aligned}(-1)^n\psi^{(n)}\left(x+I_{\psi^{(n)}}(s,t)\right)&<\frac{(-1)^n[\psi^{(n-1)}(x+t)-\psi^{(n-1)}(x+s)]}{t-s}\\&<(-1)^n\psi^{(n)}\left(x+\frac{s+t}{2}\right)\end{aligned}$$

for $x,s,t>0$ and all integers $n\ge 0$. This completes the proof of Theorem 2. □

Observing the Lemma 1 and Lemma 2, it is natural to propose the following conjecture.

Conjecture 1. Suppose $f:I\to\mathbb{R}$ satisfies

$$\begin{cases}(i)\quad (-1)^k f^{(k+1)}(x)>0, & k=0,1,2,3,\\ (ii)\quad \left(\frac{f^{(k+1)}(x)}{f^{(k)}(x)}\right)'>0, & k=1,2,3.\end{cases}$$

Then

$$h_n'''(x)>0,$$

where h_n is defined by (19).

Further, we propose the following conjecture: Let $f\in C^\infty(I)$ and satisfies

$$(-1)^k f^{(k+1)}(x)>0,\quad x\in I,\quad k=0,1,2,\ldots.$$

Then, for all integers $n\ge 0$ and all integers $k\ge 0$,

$$(-1)^k h_n^{(k+1)}(x)>0,\quad x\in I.$$

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