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ON THE INVERSE OF AN INTEGRAL INEQUALITY

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ABSTRACT. In this short paper, an inverse of an integral inequality is posed and a conjecture is posed.

1. INTRODUCTION

Let $f : [a, b] \to [0, \infty)$ be continuous function and $g : [a, b] \to [0, \infty)$ be non-decreasing, differentiable on (a, b). Recently, the authors obtained an interesting inequality for these functions f and g in [4]. Precisely, by using integration by parts, they got

Theorem A. *Let be given $\alpha \geqq 1$. If the following inequality*

$$\int_x^b f^\alpha(t)\,\mathrm{d}t \geqq \int_x^b g^\alpha(t)\,\mathrm{d}t \tag{1}$$

holds for all $x \in [a, b]$ then

$$\int_x^b f^\beta(t)\,\mathrm{d}t \geqq \int_x^b g^\beta(t)\,\mathrm{d}t \tag{2}$$

holds for all $x \in [a, b]$ and for every $\beta \geqq \alpha$.

In this paper, we are interested in obtaining some reversed inequalities of (2). Precisely, our main result is following theorem.

Theorem 1. *Let be given $\alpha \geqq 1$. Let $f : [a, b] \to [0, \infty)$ be continuous function and $g : [a, b] \to (0, \infty)$ be non-decreasing, differentiable on (a, b) such that*

$$\int_a^x f^\alpha(t)\,\mathrm{d}t \leqq \int_a^x g^\alpha(t)\,\mathrm{d}t \tag{3}$$

holds for all $x \in [a, b]$. Then

$$\int_a^x f^\beta(t)\ \mathrm{d}t \leqq \int_a^x g^\beta(t)\ \mathrm{d}t \tag{4}$$

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holds for all $x \in [a,b]$ *and for every* $0 \leqq \beta \leqq \alpha$.

If replacing $f^\alpha(x)$ by $f(x)$ and $g^\alpha(x)$ by $g(x)$, then Theorem 1 can be simplified into the following Corollary.

Corollary 1. *Let* $f : [a,b] \to [0,\infty)$ *be continuous function and* $g : [a,b] \to (0,\infty)$ *be non-decreasing, differentiable on* (a,b) *such that*

$$\int_a^x f(t)\,\mathrm{d}t \leqq \int_a^x g(t)\,\mathrm{d}t \tag{5}$$

holds for all $x \in [a,b]$. *Then*

$$\int_a^x f^\beta(t)\ \mathrm{d}t \leqq \int_a^x g^\beta(t)\ \mathrm{d}t \tag{6}$$

holds for all $x \in [a,b]$ *and for every* $0 \leqq \beta \leqq 1$.

2. Proof of Theorem 1

Before proving our main Theorem, we need an essential lemma below.

Lemma 1. *If* (3) *holds true for all* $x \in [a,b]$ *then*

$$\int_a^x f^{\alpha-\gamma}(t) g^\gamma(t)\,\mathrm{d}t \leqq \int_a^x g^\alpha(t)\,\mathrm{d}t. \tag{7}$$

holds true for all $x \in [a,b]$ *and for every* $0 \leqq \gamma \leqq \alpha$.

Proof of Lemma 1. Cases either $\gamma = 0$ or $\gamma = \alpha$ are trivial. Consider $0 < \gamma < \alpha$. Indeed, by Holder inequality with the exponentials $\frac{\alpha}{\alpha-\gamma}$ and $\frac{\alpha}{\gamma}$, we obtain

$$\begin{aligned}
\int_a^x f^{\alpha-\gamma}(t)\, g^\gamma(t)\,\mathrm{d}t &\leqq \left(\int_a^x \left(f^{\alpha-\gamma}(t)\right)^{\frac{\alpha}{\alpha-\gamma}}\mathrm{d}t\right)^{\frac{\alpha-\gamma}{\alpha}} \left(\int_a^x \left(g^\gamma(t)\right)^{\frac{\alpha}{\gamma}}\mathrm{d}t\right)^{\frac{\gamma}{\alpha}} \\
&= \left(\int_a^x f^\alpha(t)\,\mathrm{d}t\right)^{\frac{\alpha-\gamma}{\alpha}} \left(\int_a^x g^\alpha(t)\,\mathrm{d}t\right)^{\frac{\gamma}{\alpha}} \\
&\leqq \left(\frac{\int_a^x f^\alpha(t)\,\mathrm{d}t}{\int_a^x g^\alpha(t)\,\mathrm{d}t}\right)^{\frac{\alpha-\gamma}{\alpha}} \left(\int_a^x g^\alpha(t)\,\mathrm{d}t\right) \\
&\leqq \int_a^x g^\alpha(t)\,\mathrm{d}t
\end{aligned}$$

which completes our proof. □

Proof of Theorem 1. Using Lemma 1 with $\gamma = \alpha - \beta$ we have

$$\int_a^x f^\beta(t)\, g^{\alpha-\beta}(t)\,\mathrm{d}t \leqq \int_a^x g^\alpha(t)\,\mathrm{d}t$$

which yields

$$F(x) := \int_a^x g^{\alpha-\beta}(t)\left(g^\beta(t) - f^\beta(t)\right)\mathrm{d}t \geqq 0$$

for all $x \in [a, b]$. From that we obtain

$$\begin{aligned}
\int_a^x \left(g^\beta(t) - f^\beta(t)\right) \mathrm{d}t &= \int_a^x \frac{1}{g^{\alpha-\beta}(t)} g^{\alpha-\beta}(t) \left(g^\beta(t) - f^\beta(t)\right) \mathrm{d}t \\
&= \int_a^x \frac{1}{g^{\alpha-\beta}(t)} \,\mathrm{d}F(t) \\
&= \frac{F(t)}{g^{\alpha-\beta}(t)} \Bigg|_{t=a}^{t=x} - \int_a^x F(t)\, d\left(\frac{1}{g^{\alpha-\beta}(t)}\right) \\
&= \frac{F(x)}{g^{\alpha-\beta}(x)} + (\alpha-\beta) \int_a^x \frac{F(t)}{g^{\alpha+1-\beta}(t)} g'(t)\,\mathrm{d}t \\
&\geqq 0
\end{aligned}$$

which completes our proof. □

3. Discussion

In [2], some integral inequalities were obtained and the following open problem was posed

Open Problem. Let f be a continuous function on $[0, 1]$ satisfying the following condition

$$\int_x^1 f(t)\,\mathrm{d}t \geqq \int_x^1 t\,\mathrm{d}t \tag{8}$$

for $x \in [0, 1]$. Under what conditions does the inequality

$$\int_0^1 f^{\alpha+\beta}(t)\,\mathrm{d}t \geqq \int_0^1 t^\alpha f^\beta(t)\,\mathrm{d}t \tag{9}$$

hold for α and β.

The Open Problem has attracted much attention from some mathematicians (see [1, 5, 6, 7]). Similar to the Open Problem, in the paper [1] L. Bougoffa considered the following reversed inequality of (9)

$$\int_0^1 f^{\alpha+\beta}(t)\,\mathrm{d}t \leqq \int_0^1 t^\alpha f^\beta(t)\,\mathrm{d}t \tag{10}$$

provided inequality (8) is reverse.

From our point of view, the reversed inequality of (9) should be

$$\int_0^1 t^\alpha f^\beta(t)\,\mathrm{d}t \leqq \int_0^1 t^{\alpha+\beta}\,\mathrm{d}t \tag{11}$$

and we should replace inequality (8) by the following inequality

$$\int_0^x f(t)\,\mathrm{d}t \leqq \int_0^x t\,\mathrm{d}t. \tag{12}$$

Now let us turn back to Corollary 1. If (12) holds true for all $x \in [0, 1]$ then

$$\int_0^x f^\alpha(t)\,\mathrm{d}t \leqq \int_0^x t^\alpha\,\mathrm{d}t \tag{13}$$

holds true for all $x \in [0,1]$ and for every $0 \leqq \alpha \leqq 1$.

Conjecture. Let $f, g, h : [0,1] \to [0,+\infty)$ be continuous function satisfying the following conditions

$$\int_0^x f(t)\,\mathrm{d}t \leqq \int_0^x g(t)\,\mathrm{d}t \tag{14}$$

for all $x \in [0,1]$. If h is non-decreasing convex function then the following inequality

$$\int_0^1 f(t)h(t)\,\mathrm{d}t \leqq \int_0^1 g(t)h(t)\,\mathrm{d}t \tag{15}$$

holds true for all $x \in [0,1]$.

If the conjecture holds true, then from the fact that function t^β is non-decreasing and convex with respect to t provided $\beta > 1$. This and (13) yield (11). Hence, (11) holds true provided $\alpha + \beta > 1$.

References

[1] L. Bougoffa, Note on an open problem, *JIPAM. J. Inequal. Pure Appl. Math.* **8** (2007), no. 2, Article 58, 4 pp, Available online at `http://jipam.vu.edu.au/article.php?sid=871`.

[2] Quoc Anh Ngo, Du Duc Thang, Tran Tat Dat, and Dang Anh Tuan, Notes on an Integral Inequality, *JIPAM. J. Inequal. Pure Appl. Math.* **7** (2006), no. 4, Article 120, 5 pp, Available online at `http://jipam.vu.edu.au/article.php?sid=737`.

[3] Quoc Anh Ngo and Feng Qi, Generalizations of an integral inequality, submitted.

[4] Sever S. Dragomir and Quoc Anh Ngo, On an integral inequality, preprint.

[5] Hu Jianjun, Some Remarks on an Integral Inequality, submitted.

[6] Yu Miao and Feng Qi, A Discrete version of an Open problem and several answers, submitted.

[7] Yu Miao and Feng Qi, An other answers to an Open problem, submitted.

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