TWO CLASSES OF LOGARITHMICALLY COMPLETELY MONOTONIC FUNCTIONS ASSOCIATED WITH THE GAMMA FUNCTION

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Abstract. The logarithmically complete monotonicity of the functions $x \mapsto \frac{e^x\Gamma(x+b)}{(x+b)^{x+a}}$ in $(-b,\infty)$ for $a \in \mathbb{R}$ and $b \geq 0$, and $x \mapsto \frac{\Gamma(x+\beta)}{x^\alpha \Gamma(x)}$ in $(0,\infty)$ for $\alpha \in \mathbb{R}, 0 < \beta < 1$ are considered and the corresponding results by M. Merkle are generalized. As applications of these results, some inequalities for ratio of gamma functions by J. D. Kečkić and P. M. Vasić are extended and refined.

1. Introduction

A function f is said to be completely monotonic on an interval I, if f has derivatives of all orders on I and satisfies

$$(-1)^n f^{(n)}(x) \geq 0 \quad \text{for all} \quad x \in I \quad \text{and} \quad n = 0, 1, 2, \ldots. \tag{1}$$

If the inequality (1) is strict, then f is said to be strictly completely monotonic on I. Let $\mathcal{C}$ denote the set of completely monotonic functions. It is known (Bernstein's Theorem) that f is completely monotonic on $(0,\infty)$ if and only if

$$f(x) = \int_0^\infty e^{-xt}\,\mathrm{d}\mu(t),$$

where μ is a nonnegative measure on $[0,\infty)$ such that the integral converges for all $x > 0$. See [39, p.161]. A detailed collection of the most important properties of completely monotonic functions can be found in [39, Chapter IV].

A positive function f is said to be logarithmically completely monotonic on an interval I if its logarithm $\ln f$ satisfies

$$(-1)^n [\ln f(x)]^{(n)} \geq 0 \tag{2}$$

for all $x \in I$ and $n = 1, 2, \ldots$. If the inequality (2) is strict for all $x \in I$ and $n = 1, 2, \ldots$, then f is said to be strictly logarithmically completely monotonic. Let $\mathcal{L}$ on $(0,\infty)$ stand for the set of logarithmically completely monotonic functions.

A function f on $(0,\infty)$ is called a Stieltjes transform if it can be written in the form

$$f(x) = a + \int_0^\infty \frac{\mathrm{d}\mu(s)}{s+x},$$

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where a is a nonnegative number and μ a nonnegative measure on $[0,\infty)$ satisfying

$$\int_0^\infty \frac{1}{1+s}\,\mathrm{d}\mu(s) < \infty.$$

The set of Stieltjes transforms is denoted by $\mathcal{S}$.

Relation $\mathcal{L} \subset \mathcal{C}$ between the completely monotonic functions and the logarithmically completely monotonic functions was proved in [12, 33, 34]. Motivated by the papers [34, 35], among other things, it is proved in [9] that $\mathcal{S} \setminus \{0\} \subset \mathcal{L} \subset \mathcal{C}$. The class of logarithmically completely monotonic functions can be characterized as the infinitely divisible completely monotonic functions which are established by Horn in [23, Theorem 4.4] and restated in [9, Theorem 1.1]. There have been a lot of literature about the (logarithmically) completely monotonic functions, for example, [5, 6, 8, 9, 10, 11, 13, 14, 19, 27, 31, 33, 34, 35, 39] and the references therein.

The classical gamma function

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,\mathrm{d}t \quad (x > 0)$$

is one of the most important functions in analysis and its applications. The psi or digamma function, the logarithmic derivative of the gamma function, and the polygamma functions can be expressed [28, p. 16] as

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} = -\gamma + \int_0^\infty \frac{e^{-t}-e^{-xt}}{1-e^{-t}}\,\mathrm{d}t, \tag{3}$$

$$\psi^{(n)}(x) = (-1)^{n+1}\int_0^\infty \frac{t^n}{1-e^{-t}}e^{-xt}\,\mathrm{d}t \tag{4}$$

for $x > 0$ and $n \in \mathbb{N}$, where $\gamma = 0.57721566490153286\ldots$ is the Euler-Mascheroni constant.

There exists a very extensive literature on these functions. In particular, inequalities, monotonicity and complete monotonicity properties for these functions have been published. Please refer to the papers [2, 3, 4] and the references therein.

The ratio of gamma functions

$$T(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma^2((x+y)/2)}, \quad x, y > 0 \tag{5}$$

was researched in Join Gurland's paper [22], where the inequality

$$\frac{\Gamma(x)\Gamma(x+2\beta)}{\Gamma^2(x+\beta)} \geq 1 + \frac{\beta^2}{x}, \quad x > 0, x+2\beta > 0$$

was presented. Since then, there has appeared a considerable number of papers about Gurland's ratio and its properties [15, 16, 25, 26, 29]. For example, the inequality

$$\frac{x^x y^y}{((x+y)/2)^{x+y}} \leq T(x,y) \quad \text{for} \quad x > 0, y > 0 \tag{6}$$

was obtained in [25], using convexity of the function $x \mapsto \log\Gamma(x) - x\log x$ on $(0,\infty)$; the same result was earlier obtained in [15] by other means. The following upper bound was also obtained in [25]:

$$T(x,y) \leq \frac{(x-1)^{x-1}(y-1)^{y-1}}{((x+y-2)/2)^{x+y-2}} \quad \text{for} \quad x > 1, y > 1, \tag{7}$$

using the fact that for $x > 1$ the function $x \mapsto \log\Gamma(x) - (x-1)\log(x-1)$ is concave. Ratio (5) is related to also well-investigated Gautschi's ratio [18]

$$Q(x,\beta) = \frac{\Gamma(x+\beta)}{\Gamma(x)}, \quad x > 0,$$

where usually $\beta \in [0,1]$, see a survey in [30] or the bibliography in [3]. In fact, there is the following connection between the two ratios:

$$T(x, x+2\beta) = \frac{Q(x+\beta,\beta)}{Q(x,\beta)}.$$

M. Merkle [29] studied the monotonicity behaviour of the functions

$$f_{a,b}(x) = \frac{e^x\Gamma(x+b)}{(x+b)^{x+a}}, \quad a \in \mathbb{R}, b \geq 0$$

and

$$g_{\alpha,\beta}(x) = \frac{\Gamma(x+\beta)}{x^\alpha\Gamma(x)}, \quad \alpha \in \mathbb{R}, 0 < \beta < 1.$$

Merkle in 2005 proved that the function $f_{1/2,1}(x) = e^x\Gamma(x+1)/(x+1)^{x+1/2}$ is logarithmically convex on $(0,\infty)$ and $f_{0,1}(x) = e^x\Gamma(x+1)/(x+1)^x$ is logarithmically concave on $(0,\infty)$, and used these results to conclude the double inequality for $x > 0, y > 0$,

$$\frac{4(x+1)^x(y+1)^y}{xy(x+y)^2((x+y)/2+1)^{x+y}} \leq T(x,y) \leq \frac{4(x+1)^{x+1/2}(y+1)^{y+1/2}}{xy(x+y)^2((x+y)/2+1)^{x+y}}. \tag{8}$$

Moreover, the author showed that the left inequality in (8) is sharper than (6) for $x, y > 0$, the right inequality in (8) is sharper than (7) for $x, y > 1$.

Also in [29] Merkle proved that the function $x \mapsto \log g_{1,\beta}(x) = \log Q(x,\beta) - \log x$ is convex on $(0,\infty)$ and $x \mapsto \log g_{\beta,\beta}(x) = \log Q(x,\beta) - \beta\log x$ is concave on $(0,\infty)$, and used these results to obtain the following two double bounds for $\beta \in [0,1]$,

$$\frac{(x-1+2\beta)^\beta(x+\beta)^{1-\beta}}{x} \leq T(x,x+2\beta) \leq \frac{(x+\beta)^{1+\beta}}{x(x+1)^\beta}, \tag{9}$$

$$\frac{(x+\beta)^{2\beta}}{x^{\beta(2-\beta)}(x+1)^{\beta^2}} \leq T(x,x+2\beta) \leq \frac{(x-1+2\beta)^\beta(x+\beta)^{\beta(1-\beta)}}{x^\beta(x-1+\beta)^{\beta(1-\beta)}}, \tag{10}$$

with equality on both sides if and only if $\beta = 0$ or $\beta = 1$. The left inequality in (9) and the right inequality in (10) hold for $x > 1-\beta$, and other two inequalities hold for all $x > 0$.

In this article, as generalizations of some results of [29], we are about to consider the logarithmically complete monotonicity property of the function $f_{a,b}(x)$ on $(-b,\infty)$ and $g_{\alpha,\beta}(x)$ on $(0,\infty)$. Our main results are as follows.

Theorem 1. *Let $a \in \mathbb{R}$ and $b \geq 0$ be real numbers, define for $x > -b$,*

$$f_{a,b}(x) = \frac{e^x\Gamma(x+b)}{(x+b)^{x+a}}.$$

Then, the function $x \mapsto f_{a,b}(x)$ is strictly logarithmically completely monotonic on $(-b,\infty)$ if and only if $b - a \leq \frac{1}{2}$. So is the function $x \mapsto [f_{a,b}(x)]^{-1}$ if and only if $b - a \geq 1$.

Theorem 2. *Let $\alpha \in \mathbb{R}, 0 < \beta < 1$ and define for $x > 0$,*

$$g_{\alpha,\beta}(x) = \frac{\Gamma(x+\beta)}{x^\alpha \Gamma(x)}.$$

Then, the function $x \mapsto g_{\alpha,\beta}(x)$ is strictly logarithmically completely monotonic on $(0,\infty)$ if and only if $\alpha \geq 1$. So is the function $x \mapsto [g_{\alpha,\beta}(x)]^{-1}$ if and only if $\alpha \leq \beta$.

2. Remarks

Before proving our main theorems, let us present some remarks about this topic.

Remark 1. In [7] it has been shown that the function $f_{-1/2,0}(x) = e^x\Gamma(x)/x^{x-1/2}$ is strictly decreasing and logarithmically convex from $(0,\infty)$ onto $(\sqrt{2\pi}, \infty)$ and the function $f_{-1,0}(x) = e^x\Gamma(x)/x^{x-1}$ is strictly increasing and logarithmically concave from $(0,\infty)$ onto $(1,\infty)$. By the monotonicity of the functions $f_{-1/2,0}(x)$ and $f_{-1,0}(x)$ we conclude the double inequality

$$\frac{b^{b-1}}{a^{a-1}}e^{a-b} < \frac{\Gamma(b)}{\Gamma(a)} < \frac{b^{b-1/2}}{a^{a-1/2}}e^{a-b}, \quad b > a > 0, \tag{11}$$

which extends the range of a result by J. D. Kečkić and P. M. Vasić in [25], which says that inequality (11) holds for $b > a \geq 1$, see also [2, p. 342].

Since the function $f_{0,1}(x) = e^x\Gamma(x+1)/(x+1)^x$ is strictly increasing, and $f_{1/2,1}(x) = e^x\Gamma(x+1)/(x+1)^{x+1/2}$ is strictly decreasing on $(0,\infty)$, we conclude the double inequality

$$\frac{a(b+1)^b}{b(a+1)^a}e^{a-b} < \frac{\Gamma(b)}{\Gamma(a)} < \frac{a(b+1)^{b+1/2}}{b(a+1)^{a+1/2}}e^{a-b}, \quad b > a > 0. \tag{12}$$

The inequality (12) is sharper than inequality (11) from the fact that the function $(1+1/x)^x$ is strictly increasing on $(0,\infty)$, and $(1+1/x)^{x+1/2}$ is strictly decreasing on $(0,\infty)$.

Denote by

$$I(a,b) = \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)}, \quad a > 0, b > 0, a \neq b,$$

the so-called identric mean, then inequality (11) yields the following bounds for the $(b-a)$th power of $I(a,b)$:

$$\left(\frac{b}{a}\right)^{1/2}\frac{\Gamma(b)}{\Gamma(a)} < I(a,b)^{b-a} < \frac{b}{a}\frac{\Gamma(b)}{\Gamma(a)}, \quad b > a > 0. \tag{13}$$

H.Alzer proved in [2] that the inequality

$$\left(\frac{b}{a}\right)^r\frac{\Gamma(b)}{\Gamma(a)} < [I(a,b)]^{b-a} < \left(\frac{b}{a}\right)^s\frac{\Gamma(b)}{\Gamma(a)} \tag{14}$$

is valid for all real numbers $b > a \geq 1$ if and only if $r \leq 1/2$ and $s \geq \gamma$, where $\gamma = 0.577\ldots$ stands for the Euler-Mascheroni constant.

H.Alzer [3] proved that the functions

$$F(x) = \log\Gamma(x) - \left(x - \frac{1}{2}\right) + x - \frac{1}{2}\log(2\pi) - \sum_{i=1}^{2n} \frac{B_{2i}}{2i(2i-1)x^{2i-1}},$$

$$G(x) = -\log\Gamma(x) + \left(x - \frac{1}{2}\right) - x + \frac{1}{2}\log(2\pi) + \sum_{i=1}^{2n+1} \frac{B_{2i}}{2i(2i-1)x^{2i-1}}$$

are strictly completely monotonic on $(0,\infty)$, where $n \geq 0$ is an integer, $B_i(i = 1, 2, \ldots)$ are Bernoulli numbers, defined by

$$\frac{t}{e^t - 1} = \sum_{i=0}^{\infty} B_i \frac{t^i}{i!}.$$

The monotonicity of the functions F and G implies the following inequality:

$$\begin{aligned} \left(\frac{b}{a}\right)^{1/2} \frac{\Gamma(b)}{\Gamma(a)} \exp\left[\sum_{i=1}^{2n} \frac{B_{2i}}{2i(2i-1)}\left(\frac{1}{a^{2i-1}} - \frac{1}{b^{2i-1}}\right)\right] &< I(a,b)^{b-a} \\ < \left(\frac{b}{a}\right)^{1/2} \frac{\Gamma(b)}{\Gamma(a)} \exp&\left[\sum_{i=1}^{2n+1} \frac{B_{2i}}{2i(2i-1)}\left(\frac{1}{a^{2i-1}} - \frac{1}{b^{2i-1}}\right)\right] \end{aligned} \tag{15}$$

for all integers $n \geq 0$ and all real numbers $b > a > 0$.

Remark 2. Inspired by Stirling's formula, Muldoon [32] studied the monotonicity behaviour of the function

$$H_{a,b}(x) = [x^a (e/x)^x \Gamma(x)]^b, \quad a, b \in \mathbb{R}; \quad b \neq 0. \tag{16}$$

He proved in 1978: if $a \leq 1/2$ and $b > 0$, then $H_{a,b}$ is completely monotonic. Moreover, he used this result to present an interesting characterization of the gamma function via the notation of complete monotonicity. In 1986, Ismail, Lorch and Muldoon [24] showed: if $a \geq 1$ and $b = -1$, then $H_{a,b}$ is completely monotonic. In 2006 Alzer complemented these results: $H_{a,b}$ is completely monotonic if and only if either $a \leq 1/2$ and $b > 0$ or $a \geq 1$ and $b < 0$.

Remark 3. Let $\alpha \in \mathbb{R}$ and $\beta \geq 0$ be real numbers, define

$$F_{\alpha,\beta}(x) = \frac{e^x \Gamma(x+\beta)}{x^{x+\beta-\alpha}} \tag{17}$$

in $(0,\infty)$. It was shown in [13] that the function $F_{\alpha,\beta}$ is logarithmically completely monotonic in $(0,\infty)$ if $2\alpha \leq 1 \leq \beta$. The function $F_{\alpha,1}(x)$ is logarithmically completely monotonic in $(0,\infty)$ if and only if $2\alpha \leq 1$. So is the function $[F_{\alpha,1}(x)]^{-1}$ if and only if $\alpha \geq 1$. Guo et.al [20] presented some supplements to these results: Let $\alpha \in \mathbb{R}$ and $\beta > 0$. If $\beta > 0$ and $\alpha \leq 0$, then $F_{\alpha,\beta}$ is logarithmically completely monotonic in $(0,\infty)$; If $F_{\alpha,\beta}$ is logarithmically completely monotonic in $(0,\infty)$, then $\alpha \leq \min\{\beta, \frac{1}{2}\}$; If $\beta \geq 1$, then $F_{\alpha,\beta}$ is logarithmically completely monotonic in $(0,\infty)$ if and only if $\alpha \leq \frac{1}{2}$.

Remark 4. For $\beta \in \mathbb{R}$, let

$$g_\beta(x) = \frac{e^x \Gamma(x+1)}{(x+\beta)^{x+\beta}}$$

in the $I_\beta \triangleq (\max\{0, -\beta\}, \infty)$. Guo et al. [21] showed that the function $g_\beta(x)$ is logarithmically completely monotonic if and only if $\beta \geq 1$ and the function $[g_\beta(x)]^{-1}$

is logarithmically completely monotonic if and only if $\beta \le \frac{1}{2}$. The monotonicity of the function $g_1(x)$ and $g_{1/2}(x)$ implies the following inequality:

$$\frac{a(b+1/2)^{b+1/2}}{b(a+1/2)^{a+1/2}}e^{a-b} < \frac{\Gamma(b)}{\Gamma(a)} < \frac{a(b+1)^{b+1}}{b(a+1)^{a+1}}e^{a-b}, \quad b > a > 0. \tag{18}$$

It is easy to see that the right inequality in (12) is sharper than the right inequality in (18), while the left inequality in (18) is sharper than the left inequality in (12).

Remark 5. Let $a > 0$ be given real number, define for $x > 0$,

$$f_a(x) = \frac{\Gamma(x+a)}{x^a\Gamma(x)}.$$

In [37] S.-L.Qiu and M.Vuorinen proved that the function $f_{1/2}(x) = \frac{\Gamma(x+\frac{1}{2})}{\sqrt{x}\Gamma(x)}$ is strictly increasing and log-concave from $(0,\infty)$ onto $(0,1)$. Qi et al. [36] proved the following results: $\lim_{x\to\infty} f_a(x) = 1$ for any $a \in (0,\infty)$; For $a > 1$, the function $f_a(x)$ is logarithmically completely monotonic on $(0,\infty)$ and $\lim_{x\to 0+} f_a(x) = \infty$; For $0 < a < 1$, the function $[f_a(x)]^{-1}$ is logarithmically completely monotonic on $(0,\infty)$ and $\lim_{x\to 0+} f_a(x) = 0$.

3. Proofs of theorems

Proof of Theorem 1. Using the representations [38, p. 153] (also see [17, p. 824])

$$\psi(x) = -\frac{1}{2x} + \ln x - \int_0^\infty \left(\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2}\right) e^{-xt}\,\mathrm{d}t, \quad x > 0,$$

$$\frac{1}{x+s} = \int_0^\infty e^{-(x+s)t}\,\mathrm{d}t, \quad x > 0, s \ge 0,$$

we conclude that

$$(\ln f_{a,b}(x))' = \psi(x+b) - \log(x+b) + \frac{b-a}{x+b}$$
$$= -\int_0^\infty [\delta(t) - (b-a-1)]\, t^{n-1}e^{-(x+b)t}\,\mathrm{d}t,$$

where

$$\delta(t) = \frac{1}{e^t-1} - \frac{1}{t}, \quad t > 0,$$

and then, for $n \ge 1$

$$(-1)^n[\ln f_{a,b}(x)]^{(n)} = \int_0^\infty [\delta(t) - (b-a-1)]\, t^{n-1}e^{-(x+b)t}\,\mathrm{d}t. \tag{19}$$

Differentation gives

$$t^2(e^t-1)^2\delta'(t) = -t^2e^t + e^{2t} - 2e^t + 1 = \sum_{k=4}^{\infty}[2^k - 2 - k(k-1)]\frac{t^k}{k!} > 0,$$

this yields

$$-\frac{1}{2} = \lim_{t\to 0}\delta(t) < \delta(t) < \lim_{t\to\infty}\delta(t) = 0. \tag{20}$$

Combining (19) with (20) implies that for $x > -b, n \ge 1$, if $b - a \le \frac{1}{2}$, then $(-1)^n(\ln f_{a,b}(x))^{(n)} > 0$; if $b - a \ge 1$, then $(-1)^n\left(\ln\frac{1}{f_{a,b}(x)}\right)^{(n)} > 0$.

Conversely, if the function $x \mapsto f_{a,b}(x)$ is strictly logarithmically completely monotonic on $(-b,\infty)$, then

$$(\ln f_{a,b}(x))' = \psi(x+b) - \log(x+b) + \frac{b-a}{x+b} < 0,$$

which is equivalent to

$$b-a < (x+b)[\log(x+b) - \psi(x+b)]. \tag{21}$$

It was proved [7] that the function $f(x) = x[\log x - \psi(x)]$ is strictly decreasing $(0,\infty)$. Moreover, $\lim_{x\to 0} f(x) = 1$ and $\lim_{x\to\infty} f(x) = \frac{1}{2}$. In (21) let x tend to ∞, then we conclude from the asymptotic formulas of ψ [1, pp. 259-260] that $b-a \le \frac{1}{2}$.

If the function $x \mapsto [f_{a,b}(x)]^{-1}$ is strictly logarithmically completely monotonic on $(-b,\infty)$, then

$$\left(\ln \frac{1}{f_{a,b}(x)}\right)' = -\psi(x+b) + \log(x+b) - \frac{b-a}{x+b} < 0,$$

which is equivalent to

$$(x+b)[\log(x+b) - \psi(x+b)] < b-a. \tag{22}$$

In (22) let x tend to $-b$, then we obtain $1 \le b-a$. The proof of Theorem 1 is complete. □

Proof of Theorem 2. Using the representations (3), (4) and

$$\frac{(n-1)!}{x^n} = \int_0^\infty t^{n-1} e^{-xt}\,\mathrm{d}t, \quad x>0,\ n=1,2,\dots,$$

we obtain that for $n \ge 1$,

$$\begin{aligned}(-1)^n(\ln g_{\alpha,\beta}(x))^{(n)} &= (-1)^n\left[\psi^{(n-1)}(x+\beta) - \psi^{(n-1)}(x) - \frac{\alpha(-1)^{n-1}(n-1)!}{x^n}\right]\\ &= \int_0^\infty [\varphi(t) - (1-\alpha)]\, t^{n-1} e^{-xt}\,\mathrm{d}t,\end{aligned}$$

where

$$\varphi(t) = \frac{e^{(1-\beta)t}-1}{e^t-1}, \quad t>0.$$

Differentation yields

$$\begin{aligned}(e^t-1)^2\varphi'(t) &= -\beta e^{(2-\beta)t} - (1-\beta)e^{(1-\beta)t} + e^t\\ &= -\sum_{k=2}^{\infty}[\beta(2-\beta)^k + (1-\beta)^{k+1} - 1]\frac{t^k}{k!}.\end{aligned} \tag{23}$$

Set $p = 1-\beta$, then for $k \ge 2$,

$$\begin{aligned}&\beta(2-\beta)^k + (1-\beta)^{k+1} - 1\\ &= (1-p)(1+p)^k + p^{k+1} - 1\\ &= (1-p)\sum_{i=0}^{k}\binom{k}{i}p^i + (p-1)\sum_{i=0}^{k}p^i\\ &= (1-p)\sum_{i=1}^{k-1}\left[\binom{k}{i} - 1\right]p^i > 0,\end{aligned} \tag{24}$$

Combining (23) with (24) implies $\varphi'(t) < 0$ for $t > 0$. Moreover

$$0 = \lim_{t\to\infty} \varphi(t) < \varphi(t) < \lim_{t\to 0+} \varphi(t) = 1 - \beta.$$

It is easy to see that for $x > 0, n \geq 1$, if $\alpha \geq 1$, then $(-1)^n(\ln g_{\alpha,\beta}(x))^{(n)} > 0$, if $\alpha \leq \beta$, then $(-1)^n \left(\ln \frac{1}{g_{\alpha,\beta}(x)}\right)^{(n)} > 0$.

Conversely, if the function $x \mapsto g_{\alpha,\beta}(x)$ is strictly logarithmically completely monotonic on $(0, \infty)$, then

$$(\ln g_{\alpha,\beta}(x))' = \psi(x+\beta) - \psi(x) - \frac{\alpha}{x} < 0. \tag{25}$$

Note that $\psi(x+1) = \psi(x) + \frac{1}{x}$, (25) can be written as

$$\begin{aligned} \alpha > x[\psi(x+\beta) - \psi(x)] &= x\left[\psi(x+\beta) - \psi(x) - \frac{\beta}{x}\right] + \beta \\ &= x[\psi(x+\beta) - \psi(x+1)] + 1. \end{aligned} \tag{26}$$

It was proved [14, Theorem 1] that the function

$$x \mapsto x\left[\psi(x+\beta) - \psi(x) - \frac{\beta}{x}\right]$$

is strictly decreasing $(0, \infty)$. In (26) let $x \to 0$, we get $\alpha \geq 1$.

If the function $x \mapsto [g_{\alpha,\beta}(x)]^{-1}$ is strictly logarithmically completely monotonic on $(0, \infty)$, then

$$\left(\ln \frac{1}{g_{a,b}(x)}\right)' = -\psi(x+\beta) + \psi(x) + \frac{\alpha}{x} < 0,$$

which is equivalent to

$$\alpha < x[\psi(x+\beta) - \psi(x)] \tag{27}$$

In (27) let x tend to ∞, then we conclude from the asymptotic formulas of ψ [1, pp. 259-260] that $\alpha \leq \beta$. The proof of Theorem 2 is complete. □

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