SOME RESULTS RELATED TO THE CAUCHY-BUNYAKOVSKY-SCHWARZ INEQUALITY

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Abstract. Some refinements and reverses of the Cauchy-Bunyakovsky-Schwarz inequality for the Lebesgue integral in measurable spaces are given. Results for the discrete case are pointed out as well.

1. Introduction

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a σ-algebra of parts $\mathcal{A}$ and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R} \cup \{\infty\}$.

For a μ-measurable function $w \geq 0$ μ-a.e. on Ω and $p \geq 1$, we define the Lebesgue space

$$L_{p,w}(\Omega, \mathcal{A}, \mu) := \left\{ f : \Omega \to \mathbb{K} | \text{ } f \text{ is } \mu\text{-measurable, } \int_\Omega w(x) |f(x)|^p d\mu(x) < \infty \right\},$$

where $\mathbb{K} = \mathbb{C}, \mathbb{R}$.

For $p = \infty$, we defined the space

$$L_{\infty,w}(\Omega, \mathcal{A}, \mu) := \left\{ f : \Omega \to \mathbb{K} | \text{ } f \text{ is } \mu\text{-measurable, } ess \sup_{x \in \Omega} [w(x) |f(x)|] < \infty \right\}.$$

It is known that for $p \in [1, \infty]$, the spaces $L_{p,w}(\Omega, \mathcal{A}, \mu)$ together with the usual norms

$$\|f\|_{w,p} := \begin{cases} ess \sup\limits_{x \in \Omega} [w(x) |f(x)|] & \text{if } f \in L_{\infty,w}(\Omega, \mathcal{A}, \mu) \\ \left(\int_\Omega w(x) |f(x)|^p d\mu(x) \right)^{\frac{1}{p}}, & p \geq 1 \end{cases}$$

are Banach spaces.

If $p = 2$, then $L_{2,w}(\Omega, \mathcal{A}, \mu)$ is a Hilbert space. Its norm $\|\cdot\|_{w,2}$ is generated by the inner product

$$\langle f, g \rangle_{w,2} := \int_\Omega w(x) f(x) \overline{g(x)} d\mu(x),$$

where $\overline{g(x)}$ is the complex conjugate of $g(x)$.

Date: 20 August, 2007.

2000 *Mathematics Subject Classification.* 26D15, 26D10.

Key words and phrases. Cauchy-Bunyakovsky-Schwarz inequality, Lebesgue integral, Measurable spaces.

The following inequality, that holds for any $f, g \in L_{w,2}(\Omega, \mathcal{A}, \mu)$, is well known in the literature as the Cauchy-Bunyakovsky-Schwarz inequality:

$$\left| \int_\Omega w(x) f(x) g(x) \, d\mu(x) \right| \leq \left(\int_\Omega w(x) |f(x)|^2 \, d\mu(x) \right)^{\frac{1}{2}} \left(\int_\Omega w(x) |g(x)|^2 \, d\mu(x) \right)^{\frac{1}{2}}. \tag{1.1}$$

Actually, the above inequality has a stronger form, namely:

$$\int_\Omega w(x) |f(x) g(x)| \, d\mu(x) \leq \left(\int_\Omega w(x) |f(x)|^2 \, d\mu(x) \right)^{\frac{1}{2}} \left(\int_\Omega w(x) |g(x)|^2 \, d\mu(x) \right)^{\frac{1}{2}}. \tag{1.2}$$

The main aim of this present note is to provide some upper and lower bounds for the quantity

$$\begin{aligned} (0 \leq) \, & \|f\|_{w,2} \|g\|_{w,2} - \|fg\|_{w,1} \\ &= \left(\int_\Omega w(x) |f(x)|^2 \, d\mu(x) \right)^{\frac{1}{2}} \left(\int_\Omega w(x) |g(x)|^2 \, d\mu(x) \right)^{\frac{1}{2}} \\ &\quad - \int_\Omega w(x) |f(x) g(x)| \, d\mu(x) \end{aligned} \tag{1.3}$$

under the assumption that there exist constants $0 < m < M < \infty$ such that

$$0 \leq m \leq |f(x) g(x)| \leq M < \infty \quad \text{for} \quad \mu - \text{a.e.} \quad x \in \Omega. \tag{1.4}$$

Some reverses of the Cauchy-Bunyakovsky-Schwarz inequality are also given.

For some recent results related to the Cauchy-Bunyakovsky-Schwarz inequality see also [5], [6], [7] and [8].

2. The Results

Throughout this section, we assume that the nonnegative weight w, considered above, is Lebesgue integrable on Ω and $\int_\Omega w(x) \, d\mu(x) > 0$.

Theorem 1. *Let $f, g \in L_{w,2}(\Omega, \mathcal{A}, \mu)$ such that f/g, $g/f \in L_{w,1}(\Omega, \mathcal{A}, \mu)$ and that there exist constants $0 < m < M < \infty$ with the property that:*

$$m \leq |f(x) g(x)| \leq M \quad \textit{for} \quad \mu - \textit{a.e.} \quad x \in \Omega. \tag{2.1}$$

It then follows that,

$$\begin{aligned} (0 \leq) & \frac{1}{2} m \left[\frac{\|g\|_{w,2}}{\|f\|_{w,2}} \cdot \left\| \frac{f}{g} \right\|_{w,1} + \frac{\|f\|_{w,2}}{\|g\|_{w,2}} \cdot \left\| \frac{g}{f} \right\|_{w,1} - 2 \cdot \|\mathbf{1}\|_{w,1} \right] \\ &\leq \|f\|_{w,2} \|g\|_{w,2} - \|fg\|_{w,1} \\ &\leq \frac{1}{2} M \left[\frac{\|g\|_{w,2}}{\|f\|_{w,2}} \cdot \left\| \frac{f}{g} \right\|_{w,1} + \frac{\|f\|_{w,2}}{\|g\|_{w,2}} \cdot \left\| \frac{g}{f} \right\|_{w,1} - 2 \cdot \|\mathbf{1}\|_{w,1} \right], \end{aligned} \tag{2.2}$$

where $\mathbf{1}(x) = 1$, $x \in \Omega$.

Proof. Note the elementary identity:

$$\frac{u^2+v^2}{2} - uv = \frac{1}{2}uv \cdot \left(\sqrt{\frac{u}{v}} - \sqrt{\frac{v}{u}}\right)^2, \tag{2.3}$$

that holds for any $u, v \in (0, \infty)$.

Writing (2.3) for $u = \frac{|f(x)|}{\|f\|_{w,2}}$, $v = \frac{|g(x)|}{\|g\|_{w,2}}$, $x \in \Omega$ and utilising the assumption (2.1), we obtain

$$\begin{aligned}
&\frac{1}{2}m\left(\frac{\|g\|_{w,2}^{1/2}}{\|f\|_{w,2}^{1/2}} \cdot \sqrt{\left|\frac{f(x)}{g(x)}\right|} + \frac{\|f\|_{w,2}^{1/2}}{\|g\|_{w,2}^{1/2}} \cdot \sqrt{\left|\frac{g(x)}{f(x)}\right|}\right)^2 \cdot \frac{1}{\|f\|_{w,2} \cdot \|g\|_{w,2}} \\
&\leq \frac{1}{2}\left[\frac{|f(x)|^2}{\|f\|_{w,2}^2} + \frac{|g(x)|^2}{\|g\|_{w,2}^2}\right] - \frac{|f(x)g(x)|}{\|f\|_{w,2} \cdot \|g\|_{w,2}} \\
&\leq \frac{1}{2}M\left(\frac{\|g\|_{w,2}^{1/2}}{\|f\|_{w,2}^{1/2}} \cdot \sqrt{\left|\frac{f(x)}{g(x)}\right|} + \frac{\|f\|_{w,2}^{1/2}}{\|g\|_{w,2}^{1/2}} \cdot \sqrt{\left|\frac{g(x)}{f(x)}\right|}\right)^2 \cdot \frac{1}{\|f\|_{w,2} \cdot \|g\|_{w,2}}
\end{aligned} \tag{2.4}$$

for μ-a.e. $x \in \Omega$.

Since

$$\begin{aligned}
&\left(\frac{\|g\|_{w,2}^{1/2}}{\|f\|_{w,2}^{1/2}} \cdot \sqrt{\left|\frac{f(x)}{g(x)}\right|} + \frac{\|f\|_{w,2}^{1/2}}{\|g\|_{w,2}^{1/2}} \cdot \sqrt{\left|\frac{g(x)}{f(x)}\right|}\right)^2 \\
&\qquad = \frac{\|g\|_{w,2}}{\|f\|_{w,2}} \cdot \left|\frac{f(x)}{g(x)}\right| + \frac{\|f\|_{w,2}}{\|g\|_{w,2}} \cdot \left|\frac{g(x)}{f(x)}\right| - 2
\end{aligned}$$

then, by (2.4) we get

$$\begin{aligned}
&\frac{1}{2}m\left[\frac{1}{\|f\|_{w,2}^2} \cdot \left|\frac{f}{g}\right| + \frac{1}{\|g\|_{w,2}^2} \cdot \left|\frac{g}{f}\right| - \frac{2}{\|f\|_{w,2} \cdot \|g\|_{w,2}}\right] \\
&\leq \frac{1}{2}\left[\frac{|f(x)|^2}{\|f\|_{w,2}^2} + \frac{|g(x)|^2}{\|g\|_{w,2}^2}\right] - \frac{|fg|}{\|f\|_{w,2} \cdot \|g\|_{w,2}} \\
&\leq \frac{1}{2}M\left[\frac{1}{\|f\|_{w,2}^2} \cdot \left|\frac{f}{g}\right| + \frac{1}{\|g\|_{w,2}^2} \cdot \left|\frac{g}{f}\right| - \frac{2}{\|f\|_{w,2} \cdot \|g\|_{w,2}}\right]
\end{aligned} \tag{2.5}$$

μ-almost everywhere in Ω.

On multiplying (2.5) by $w \geq 0$ and integrating on Ω, we get

$$\begin{aligned}
&\frac{1}{2}m\left[\frac{1}{\|f\|_{w,2}^2} \cdot \left\|\frac{f}{g}\right\|_{w,1} + \frac{1}{\|g\|_{w,2}^2} \cdot \left\|\frac{g}{f}\right\|_{w,1} - \frac{2 \cdot \|\mathbf{1}\|_{w,1}}{\|f\|_{w,2} \cdot \|g\|_{w,2}}\right] \\
&\leq 1 - \frac{\|fg\|_{w,1}}{\|f\|_{w,2} \cdot \|g\|_{w,2}} \\
&\leq \frac{1}{2}M\left[\frac{1}{\|f\|_{w,2}^2} \cdot \left\|\frac{f}{g}\right\|_{w,1} + \frac{1}{\|g\|_{w,2}^2} \cdot \left\|\frac{g}{f}\right\|_{w,1} - \frac{2 \cdot \|\mathbf{1}\|_{w,1}}{\|f\|_{w,2} \cdot \|g\|_{w,2}}\right],
\end{aligned}$$

which is clearly equivalent to (2.2). ∎

Consider now the sequence $w_j \geq 0$ with $\sum_{j=1}^{\infty} w_j < \infty$ and define the Banach spaces $\ell_w^p(\mathbb{K})$ by

$$\ell_w^2(\mathbb{K}) := \left\{ x = (x_i)_{i\in\mathbb{N}} \left| \sum_{j=1}^{\infty} w_j |x_j|^p < \infty \right. \right\},$$

where

$$\|x\|_{w,p} := \sum_{j=1}^{\infty} w_j |x_j|^p .$$

With the above assumptions, we have the following discrete inequality.

Corollary 1. *If $x = (x_i)_{i\in\mathbb{N}}$, $y = (y_i)_{i\in\mathbb{N}} \in \ell_w^2(\mathbb{K})$ are such that $x_i, y_j \neq 0$, $j \in \mathbb{N}$, $\left(\frac{x_j}{y_j}\right)_{j\in\mathbb{N}}, \left(\frac{y_j}{x_j}\right)_{j\in\mathbb{N}} \in \ell_w^1(\mathbb{K})$ and that there exist constants $M, m \in \mathbb{R}$ such that*

$$0 < m \leq |x_j y_j| \leq M < \infty \quad \text{for each} \quad j \in \mathbb{N}, \tag{2.6}$$

then,

$$\begin{aligned} \frac{1}{2} m \Bigg\{ & \left[\frac{\sum_{j=1}^{\infty} w_j |y_j|^2}{\sum_{j=1}^{\infty} w_j |x_j|^2} \right]^{\frac{1}{2}} \cdot \sum_{j=1}^{\infty} w_j \left| \frac{x_j}{y_j} \right| \\ & + \left[\frac{\sum_{j=1}^{\infty} w_j |x_j|^2}{\sum_{j=1}^{\infty} w_j |y_j|^2} \right]^{\frac{1}{2}} \cdot \sum_{j=1}^{\infty} w_j \left| \frac{y_j}{x_j} \right| - 2 \cdot \sum_{j=1}^{\infty} w_j \Bigg\} \\ \leq & \left(\sum_{j=1}^{\infty} w_j |x_j|^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^{\infty} w_j |y_j|^2 \right)^{\frac{1}{2}} - \sum_{j=1}^{\infty} w_j |x_i y_j| \\ \leq \frac{1}{2} M \Bigg\{ & \left[\frac{\sum_{j=1}^{\infty} w_j |y_j|^2}{\sum_{j=1}^{\infty} w_j |x_j|^2} \right]^{\frac{1}{2}} \cdot \sum_{j=1}^{\infty} w_j \left| \frac{x_j}{y_j} \right| \\ & + \left[\frac{\sum_{j=1}^{\infty} w_j |x_j|^2}{\sum_{j=1}^{\infty} w_j |y_j|^2} \right]^{\frac{1}{2}} \cdot \sum_{j=1}^{\infty} w_j \left| \frac{y_j}{x_j} \right| - 2 \cdot \sum_{j=1}^{\infty} w_j \Bigg\}. \end{aligned} \tag{2.7}$$

3. Reverses of the CBS-Inequality

Before we state a reverse of the Cauchy-Bunyakovsky-Schwarz (CBS)-inequality, which can be naturally derived from Theorem 1, we present some known results for complex functions.

Assume that $f, g \in L_w^2(\Omega, \mathcal{A}, \mu)$ and that there exist real (complex) numbers $a, A \in \mathbb{K}$ such that

$$\operatorname{Re}\left[(A g(x) - f(x)) \left(\overline{f(x)} - \bar{a}\overline{g(x)} \right) \right] \geq 0 \tag{3.1}$$

for μ-a.e. $x \in \Omega$, then [1] (see also [3, p. 7]):

$$(0 \leq) \int_\Omega w(x)|f(x)|^2 d\mu(x) \int_\Omega w(x)|g(x)|^2 d\mu(x) - \left|\int_\Omega w(x) f(x)\overline{g(x)} d\mu(x)\right|^2 \leq \frac{1}{4}\cdot|A-a|^2\left(\int_\Omega w(x)|g(x)|^2 d\mu(x)\right)^2. \tag{3.2}$$

With the assumption (3.1), the following result also holds [2] (see also [3, p. 26]):

$$\int_\Omega w(x)|f(x)|^2 d\mu(x) \int_\Omega w(x)|g(x)|^2 d\mu(x) \leq \frac{1}{4}\cdot\frac{|A+a|^2}{\operatorname{Re}(A\bar{a})}\left|\int_\Omega w(x) f(x)\overline{g(x)} d\mu(x)\right|^2, \tag{3.3}$$

provided $\operatorname{Re}(A\bar{a}) > 0$.

Finally, if $A \neq a$ and the condition (3.1) holds true, then [3] (see also [4, p. 32])

$$(0 \leq) \left[\int_\Omega w(x)|f(x)|^2 d\mu(x) \int_\Omega w(x)|g(x)|^2 d\mu(x)\right]^{\frac{1}{2}} - \left|\int_\Omega w(x) f(x)\overline{g(x)} d\mu(x)\right| \leq \frac{1}{4}\cdot\frac{|A-a|^2}{|A+a|^2}\int_\Omega w(x)|g(x)|^2 d\mu(x). \tag{3.4}$$

We give now our new result which provide a different reverse for the CBS-inequality than the inequalities mentioned above:

Theorem 2. *Let $f, g \in L_{w,2}(\Omega, \mathcal{A}, \mu)$ be such that there exist constants $0 < M < \infty$ and $0 < n < N < \infty$ with the properties that:*

$$|f(x) g(x)| \leq M, \qquad n \leq \left|\frac{f(x)}{g(x)}\right| \leq N \quad \text{for } \mu\text{-a.e. } x \in \Omega, \tag{3.5}$$

then we have the reverse of the CBS-inequality:

$$(0 \leq) \left[\int_\Omega w(x)|f(x)|^2 d\mu(x) \int_\Omega w(x)|g(x)|^2 d\mu(x)\right]^{\frac{1}{2}} - \int_\Omega w(x)|f(x) g(x)| d\mu(x) \leq M\left(\frac{N}{n}-1\right)\int_\Omega w(x) d\mu(x). \tag{3.6}$$

Proof. From the second condition in (3.5),

$$\left|\frac{f(x)}{g(x)}\right| \leq N \quad \text{and} \quad \left|\frac{g(x)}{f(x)}\right| \leq \frac{1}{n}, \quad \text{for } \mu\text{-a.e. } x \in \Omega, \tag{3.7}$$

which implies that

$$\left\|\frac{f}{g}\right\|_{w,1} \leq N\int_\Omega w(x) d\mu(x) \quad \text{and} \quad \left\|\frac{g}{f}\right\|_{w,1} \leq \frac{1}{n}\int_\Omega w(x) d\mu(x). \tag{3.8}$$

Also, from (3.8) we have $|f(x)| \leq N|g(x)|$ and $|g(x)| \leq \frac{1}{n}|f(x)|$ for μ-a.e. $x \in \Omega$, which imply

$$\|f\|_{w,2} \leq N\|g\|_{w,2} \quad \text{and} \quad \|g\|_{w,2} \leq \frac{1}{n}\|f\|_{w,2}. \tag{3.9}$$

Utilising the second inequality in (2.2) and the inequalities (3.8) and (3.9), we deduce

$$\begin{aligned}\|f\|_{w,2}\|g\|_{w,2} - \|fg\|_{w,1} &\leq \frac{1}{2}M\left[\frac{N}{n} + \frac{M}{n} - 2\right]\int_\Omega w(x)\,d\mu(x) \\ &= M\left(\frac{N}{n} - 1\right)\int_\Omega w(x)\,d\mu(x)\end{aligned}$$

and the proof is complete. ∎

Corollary 2. *Assume that f, g are measurable and such that:*

$$0 < m_1 \leq |f(x)| \leq M_1 < \infty, \qquad 0 < m_2 \leq |g(x)| \leq M_2 < \infty \tag{3.10}$$

for μ-a.e. $x \in \Omega$, then

$$\begin{aligned}(0 \leq) &\left[\int_\Omega w(x)|f(x)|^2\,d\mu(x)\int_\Omega w(x)|g(x)|^2\,d\mu(x)\right]^{\frac{1}{2}} \\ &\qquad - \int_\Omega w(x)|f(x)g(x)|\,d\mu(x) \\ &\leq M_1M_2\left(\frac{M_1M_2}{m_1m_2} - 1\right)\int_\Omega w(x)\,d\mu(x).\end{aligned} \tag{3.11}$$

The proof is obvious by Theorem 2 on noticing that $|f(x)g(x)| \leq M_1M_2$ and

$$\frac{m_1}{M_2} \leq \left|\frac{f(x)}{g(x)}\right| \leq \frac{M_1}{m_2}$$

for μ-a.e. $x \in \Omega$.

Remark 1. *The discrete case can be stated as follows. Assume that $x = (x_i)_{i\in\mathbb{N}}$, $y = (y_i)_{i\in\mathbb{N}} \in \ell_w^2(\mathbb{K})$ are such that $x_i, y_i \neq 0$, $i \in \mathbb{N}$ and that there exist constants $0 < M < \infty$ and $0 < n < N < \infty$ with*

$$|x_jy_j| \leq M \quad \text{and} \quad n \leq \left|\frac{x_j}{y_j}\right| \leq N \quad \text{for each } j \in \mathbb{N}. \tag{3.12}$$

It follows that

$$\begin{aligned}0 &\leq \left(\sum_{j=1}^\infty w_j|x_j|^2 \cdot \sum_{j=1}^\infty w_j|y_j|^2\right)^{\frac{1}{2}} - \sum_{j=1}^\infty w_j|x_jy_j| \\ &\leq M\left(\frac{N}{n} - 1\right)\sum_{j=1}^\infty w_j.\end{aligned} \tag{3.13}$$

Also, if

$$0 < m_1 \leq |x_j| \leq M_1 < \infty, \qquad 0 < m_2 \leq |y_j| \leq M_2 < \infty, \text{ for each } j \in \mathbb{N} \tag{3.14}$$

then

$$0 \leq \left(\sum_{j=1}^{\infty} w_j \left| x_j \right|^2 \cdot \sum_{j=1}^{\infty} w_j \left| y_j \right|^2 \right)^{\frac{1}{2}} - \sum_{j=1}^{\infty} w_j \left| x_j y_j \right| \leq M_1 M_2 \left(\frac{M_1 M_2}{m_1 m_2} - 1 \right) \sum_{j=1}^{\infty} w_j. \tag{3.15}$$

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