A REMARK ON A SUM ON PRIMES COUNTING FUNCTION

M. HASSANI AND H. MOSHTAGH

Abstract. We introduce some explicit lower and upper bounds for the summation $\sum_{2\leq n\leq x}\frac{1}{\pi(n)}$.

1. Introduction and Motivation

Let $\pi(x)$ is the number of primes not exceeding x. In 1980 Jean-Marie De Koninck and Aleksandar Ivić [3] showed

$$\sum_{2\leq n\leq x}\frac{1}{\pi(n)}=\frac{1}{2}\log^2 x+O(\log x),$$

as a result of the Prime Number Theorem.

In 2000, L. Panaitopol [5] proved that

$$\frac{1}{\pi(x)}=\frac{1}{x}\Big\{\log x-1-\sum_{i=1}^{m}\frac{k_i}{\log^i x}+O\Big(\frac{1}{\log^{m+1} x}\Big)\Big\},$$

where $m\geq 1$ and $\sum_{i=0}^{n-1} i!k_{n-i}=n.n!$, and using this with $m=2$, improved above result of Koninck and Ivić as follows

$$\sum_{2\leq n\leq x}\frac{1}{\pi(n)}=\frac{1}{2}\log^2 x-\log x-\log\log x+O(1).$$

Two years later, Ivić [2] got the following

$$\sum_{2\leq n\leq x}\frac{1}{\pi(n)}=\frac{1}{2}\log^2 x-\log x-\log\log x+C+\sum_{i=2}^{m}\frac{k_i}{(i-1)\log^{i-1} x}+O\Big(\frac{1}{\log^m x}\Big),$$

where C is an absolute constant.

As we know, all approximations of the summation $\sum_{2\leq n\leq x}\frac{1}{\pi(n)}$ have O-terms with effective constants, but there is no known computed one. In this note, following some carefull computations, we find some explicit approximations for this summation. Our main result is:

Theorem 1. *For $x\geq 2$ we have*

$$-1.51\log\log x+0.8994<\sum_{2\leq n\leq x}\frac{1}{\pi(n)}-\Big(\frac{1}{2}\log^2 x-\log x\Big)<-0.79\log\log x+6.4888.$$

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Before proof, we mention two remarks:

1. The factors of $\log\log -$terms and constants appeared in this theorem are not optimal, and one can improve them. But, these are best with our methods and computational tools.

2. A nice question is explicit determining the value of C in Ivić's expansion of the summation under study, which requires studying the sequence $\{C_N\}_{N\geq 2}$ defined by

$$C_N = \sum_{n=2}^{N} \frac{1}{\pi(n)} - \frac{1}{2}\log^2 N + \log N + \log\log N.$$

Ivić's result, guarantees that $C_N \to C$, and computations comment $C \approx 6.9$.

2. Proof of the Theorem

First we need Euler-Maclaurin summation formula to get some lemmas; suppose $B_n(t)$ denotes Bernoulli polynomilas and $B_n = B_n(0)$ denotes Bernoulli numbers; for example $B_2(t) = t^2 - t + 1/6$ and $B_2 = 1/6$. By $\{t\}$ we denote the fractional part of t. Considering the inequality $|B_{2m}(\{t\})| \leq B_{2m}$, and using Euler-Maclaurin summation formula (see for example [4], page 27) with $m = 1$, we get the following lemmas.

Lemma 2. *For $N \geq 2$ we have*

$$\Sigma_{-1}(N) := \sum_{n=2}^{N} \frac{1}{n\log n} = \log\log N + C_{-1} + R_{-1},$$

say, where $C_{-1} = 0.794678645452\ldots$ and $|R_{-1}| \leq (3N\log N + \log N + 1)/(6N^2\log^2 N)$.

Lemma 3. *For $N \geq 1$ we have*

$$\Sigma_0(N) := \sum_{n=1}^{N} \frac{1}{n} = \log N + C_0 + R_0,$$

say, where $C_0 = 0.577215664901\ldots$, and $|R_0| \leq (3N+1)/(6N^2)$. Note that $C_0 = \gamma$, the Euler constant.

Lemma 4. *For $N \geq 1$ we have*

$$\Sigma_1(N) := \sum_{n=1}^{N} \frac{\log n}{n} = \frac{1}{2}\log^2 N + C_1 + R_1,$$

say, where $C_1 = -0.072815845483\ldots$, and $|R_1| \leq (6N\log N + 13\log N + 25)/(12N^2)$.

Let $\mathfrak{S}(y,x) = \sum_{y<n\leq x} \frac{1}{\pi(n)}$ and $\mathfrak{S}(x) = \mathfrak{S}(1,x)$. To find upper bound, we consider Theorem 2.1 of [1], which asserts that for every $0 < \varepsilon \leq \frac{21}{20}$, the inequality

$$\frac{x}{\log x - 1 - \frac{\frac{4}{5}-\varepsilon}{\log x}} \leq \pi(x),$$

holds for all

$$x \geq \max\left\{32299, \left\lfloor e^{\frac{13-5\varepsilon+\sqrt{169+14\varepsilon-155\varepsilon^2}}{10\varepsilon}}\right\rfloor\right\} := \mathfrak{M},$$

say. We write $\mathfrak{S}(x) = \mathfrak{S}(\mathfrak{M}-1) + \mathfrak{S}(\mathfrak{M}-1, x)$. The value of $\mathfrak{S}(\mathfrak{M}-1)$ is absolute and one can find it after determining exact value of ε. We write

$$\mathfrak{S}(\mathfrak{M}-1, x) \leq \sum_{\mathfrak{M}-1<n\leq x} \frac{\log n - 1 - \frac{\frac{4}{5}-\varepsilon}{\log n}}{n} = \sum_{\mathfrak{M}-1<n\leq x} \frac{\log n}{n} - \sum_{\mathfrak{M}-1<n\leq x} \frac{1}{n} - \Big(\frac{4}{5}-\varepsilon\Big) \sum_{\mathfrak{M}-1<n\leq x} \frac{1}{n\log n}.$$

Thus (we restrict our self to integre values of x, but we note that we won't lose some thing by this restriction) for $x \geq \mathfrak{M}$, we have

$$\mathfrak{S}(x) \leq \mathfrak{S}(\mathfrak{M}-1) + \Sigma_1(x) - \Sigma_1(\mathfrak{M}-1) - \Sigma_0(x) + \Sigma_0(\mathfrak{M}-1) - \Big(\frac{4}{5}-\varepsilon\Big)\Big(\Sigma_{-1}(x) - \Sigma_{-1}(\mathfrak{M}-1)\Big).$$

Now, considering above lemmas and restriction $0 < \varepsilon < 4/5$, for $x \geq \mathfrak{M}$ we obtain

$$\mathfrak{S}(x) \leq \frac{1}{2}\log^2 x - \log x - \Big(\frac{4}{5}-\varepsilon\Big)\log\log x + C_u + R_u,$$

where

$$C_u = \mathfrak{S}(\mathfrak{M}-1) - \Sigma_1(\mathfrak{M}-1) + \Sigma_0(\mathfrak{M}-1) + \Big(\frac{4}{5}-\varepsilon\Big)\Sigma_{-1}(\mathfrak{M}-1) + C_1 - C_0 - \Big(\frac{4}{5}-\varepsilon\Big)C_{-1},$$

and

$$R_u = \frac{6x\log x + 13\log x + 25}{12x^2} + \frac{3x+1}{6x^2} + \Big(\frac{4}{5}-\varepsilon\Big)\frac{3x\log x + \log x + 1}{6x^2\log^2 x}.$$

Now, we put $\varepsilon = 1/12423$, which gives $\mathfrak{M} = 32299$. Then $C_u = 6.488573380484\ldots$, and R_u is a decreasing function of x for $x \geq \mathfrak{M}$, which yields $R_u \leq 0.000177415261\ldots$. Therfore, we obtain

$$\mathfrak{S}(x) \leq \frac{1}{2}\log^2 x - \log x - 0.799919504145\log\log x + 6.488750795746 \qquad (x \geq 32299),$$

which holds true for $2 \leq x \leq 32298$ by Maple computations (see Appendix). Similarly, we find lower bound; Corollary 2.4 of [1], asserts that for every $x \geq 7$, we have

$$\pi(x) \leq \frac{x}{\log x - 1 - \frac{151}{100\log x}}.$$

This result and above lemmas, yield for every $x \geq 7$ that

$$\mathfrak{S}(x) = \mathfrak{S}(6) + \mathfrak{S}(6, x) \geq \mathfrak{S}(6) + \sum_{6<n\leq x}\frac{\log n}{n} - \sum_{6<n\leq x}\frac{1}{n} - \frac{151}{100}\sum_{6<n\leq x}\frac{1}{n\log n},$$

and our above Lemmas give $\mathfrak{S}(x) \geq \frac{1}{2}\log^2 x - \log x - 1.51\log\log x + C_l + R_l$ for every $x \geq 7$, with

$$C_l = \mathfrak{S}(6) - \Sigma_1(6) + \Sigma_0(6) + 1.51\Sigma_{-1}(6) + C_1 - C_0 - 1.51C_{-1} = 3.734603144333\ldots,$$

and

$$R_l = -\frac{6x\log x + 13\log x + 25}{12x^2} - \frac{3x+1}{6x^2} - 1.51\frac{3x\log x + \log x + 1}{6x^2\log^2 x} \geq -0.358785749003\ldots,$$

where the last inequality holds for $x \geq 7$. Therefore, we obtain

$$\mathfrak{S}(x) \geq \frac{1}{2}\log^2 x - \log x - 1.51\log\log x + 3.375817395331 \qquad (x \geq 7),$$

which holds true for $x = 6$, too. Also, reducing the value of constant, Maple computations verfies the following

$$\mathfrak{S}(x) \geq \frac{1}{2}\log^2 x - \log x - 1.51\log\log x + 0.8994 \qquad (x \geq 2).$$

This completes the proof. □

References

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Appendix. Maple program for verifying lower bounds (or upper bounds by a little chang) for $\sum_{2\leq n\leq x}\frac{1}{\pi(n)}$ for the range $y \leq x \leq m$.

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for x from m by -1 while
evalf(sum(1/numtheory[pi](n),n=2..x)-(lower bound))>0
do x end do;
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The out put are those numbers $\leq x$ with bound holding and the last number in out put is y.

Mehdi Hassani,
Department of Mathematics, Institute for Advanced Studies in Basic Sciences, P.O. Box 45195-1159, Zanjan, Iran
E-mail address: mmhassany@member.ams.org

Hossein Moshtagh,
Department of Mathematics, Islamic Azad University Garmsar Branch, Garmsar, Iran
E-mail address: hs_moshtagh@yahoo.com