SOME NEW REFINEMENTS OF CAUCHY-SCHWARTZ INEQUALITY

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ABSTRACT. In this paper, we introduce some new refinements of the famous Cauchy-Schwarz inequality $|\langle x, y\rangle| \leq \|x\|\|y\|$.

1. INTRODUCTION

The famous Cauchy-Schwarz inequality states that

$$|\langle x, y\rangle| \leq \|x\|\|y\|, \tag{1.1}$$

where x and y are two linearly independent vectors in $\mathfrak{X}$, with $(\mathfrak{X}, \langle ., .\rangle)$ a real inner product space, and $\|x\| = \sqrt{\langle x, x\rangle}$. In [3] the authors, use the following inequality

$$\sigma_{a,b}.\langle x, y\rangle \leq \frac{\|y\|}{b-a} \int_a^b \delta_{x,y}(t)dt \leq \|x\|\|y\|, \tag{1.2}$$

with

$$\sigma_{a,b} = \operatorname{sgn}\left(\frac{a+b}{2}\right),$$

and

$$\delta_{x,y}(t) = 2\|x + ty\| - \|x + 2ty\|, \tag{1.3}$$

to introduce some refinements of (1.1) by approximating $\int_a^b \delta_{x,y}(t)dt$ via Hermite-Hadamard inequality. In this paper, we find some new upper bounds for $\int_a^b \delta_{x,y}(t)dt$; first, we calculate this integral explicitly, including some terms concerning Logarithmic mean [6], defined for $\alpha, \beta > 0$ as follows

$$L(\alpha, \beta) = \begin{cases} \frac{\alpha-\beta}{\log\alpha - \log\beta} & \alpha \neq \beta, \\ \alpha & \alpha = \beta, \end{cases}$$

where the base of logarithm here and all places of this work are e. Then, we use the following bound [7] for $L(\alpha, \beta)$, which holds true for $\alpha > \beta$ to make above mentioned upper bounds.

$$\frac{L(\alpha, \beta)}{\beta} > 1 + \frac{1}{2}\log\frac{\alpha}{\beta} > \frac{2\alpha}{\alpha + \beta}.$$

1991 *Mathematics Subject Classification.* 41A65.
Key words and phrases. Inequalities.

2. Explicit Calculation of $\int_a^b \delta_{x,y}(t)dt$

In this section we calculate $\int_a^b \delta_{x,y}(t)dt$ explicitly. To do this we need the following integral formula [6]

$$\int \sqrt{at^2+2bt+c}dt = \frac{ac-b^2}{2a^{\frac{3}{2}}} \log\Big(at+b+a^{\frac{1}{2}}\sqrt{at^2+2bt+c}\Big) + \frac{at+b}{2a}\sqrt{at^2+2bt+c}, \tag{2.1}$$

where $a>0$ and $\Delta' = b^2 - ac \leq 0$. Considering (1.3), we have

$$\begin{aligned} \int_a^b \delta_{x,y}(t)dt &= \int_a^b \big(2\|x+ty\| - \|x+2ty\|\big)dt \\ &= 2\int_a^b \|x+ty\|dt - \int_a^b \|x+2ty\|dt. \end{aligned} \tag{2.2}$$

Now, substituting (2.1) in (2.2), we get

$$\begin{aligned} 2\int_a^b \|x+ty\|dt &= 2\int_a^b \sqrt{\langle x+ty, x+ty\rangle}dt \\ &= 2\int_a^b \sqrt{\|y\|^2t^2 + 2 < x,y > t + \|x\|^2}dt \\ &= \frac{\|x\|^2\|y\|^2 - \langle x,y\rangle^2}{\|y\|^3} \log\left(\frac{\langle x+by, y\rangle + \|y\|\|x+by\|}{\langle x+ay, y\rangle + \|y\|\|x+ay\|}\right) \\ &\quad + \frac{1}{\|y\|^2}\Big((b\|y\|^2 + \langle x,y\rangle)\|x+by\| - (a\|y\|^2 + \langle x,y\rangle)\|x+ay\|\Big), \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} \int_a^b \|x+2ty\|dt &= \int_a^b \sqrt{\langle x+2ty, x+2ty\rangle}dt \\ &= \int_a^b \sqrt{4\|y\|^2t^2 + 4\langle x,y\rangle t + \|x\|^2}dt \\ &= \frac{\|x\|^2\|y\|^2 - \langle x,y\rangle^2}{4\|y\|^3} \log\left(\frac{\langle x+2by, y\rangle + \|y\|\|x+2by\|}{\langle x+2ay, y\rangle + \|y\|\|x+2ay\|}\right) \\ &\quad + \frac{1}{4\|y\|^2}\Big((2b\|y\|^2 + \langle x,y\rangle)\|x+2by\| - (2a\|y\|^2 + \langle x,y\rangle)\|x+2ay\|\Big), \end{aligned} \tag{2.4}$$

and considering above relations with (2.2), we obtain the following proposition.

Proposition 2.1. *With above notations, we have*

$$\int_a^b \delta_{x,y}(t)dt = \mathcal{P}_1 + \mathcal{P}_2, \tag{2.5}$$

in which

$$\mathcal{P}_1 = \gamma(\log\alpha - \log\beta),$$

with

$$\begin{aligned}\gamma &= \frac{\|x\|^2\|y\|^2-\langle x,y\rangle^2}{\|y\|^3},\\ \alpha &= \frac{\langle x+by,y\rangle+\|y\|\|x+by\|}{\langle x+ay,y\rangle+\|y\|\|x+ay\|},\\ \beta &= \left(\frac{\langle x+2by,y\rangle+\|y\|\|x+2by\|}{\langle x+2ay,y\rangle+\|y\|\|x+2ay\|}\right)^{1/4},\end{aligned}$$

and

$$\begin{aligned}\mathcal{P}_2 &= \Big(\frac{b}{2}+\frac{\langle x,y\rangle}{4\|y\|^2}\Big)\delta_{x,y}(b)-\Big(\frac{a}{2}+\frac{\langle x,y\rangle}{4\|y\|^2}\Big)\delta_{x,y}(a)\\ &\quad+\frac{\langle x,y\rangle}{2\|y\|^2}\left(\frac{\delta_{x,y}(b)-\delta_{x,y}(a)}{2}+\frac{\|x+2by\|-\|x+2ay\|}{2}\right).\end{aligned}$$

Proof. Considering (2.2), (2.3) and (2.4), we get the result for $\mathcal{P}_1$. Also, we have

$$\begin{aligned}\mathcal{P}_2 &= \frac{1}{\|y\|^2}\Big[\big(b\|y\|^2\|x+by\|+\langle x,y\rangle\|x+by\|-a\|y\|^2\|x+ay\|-\langle x,y\rangle\|x+ay\|\big)\\ &\quad-\frac{1}{4}\big(2b\|y\|^2\|x+2by\|+\langle x,y\rangle\|x+2by\|-2a\|y\|^2\|x+2ay\|-\langle x,y\rangle\|x+2ay\|\big)\Big]\\ &= \frac{1}{\|y\|^2}\Big[\|y\|^2\big(b\|x+by\|-a\|x+ay\|\big)+\langle x,y\rangle\big(\|x+by\|-\|x+ay\|\big)\\ &\quad-\frac{1}{4}\|y\|^2\big(2b\|x+2by\|-2a\|x+2ay\|\big)-\frac{\langle x,y\rangle}{4}\big(\|x+2by\|-\|x+2ay\|\big)\Big]\\ &= \big(b\|x+by\|-a\|x+ay\|\big)+\frac{\langle x,y\rangle}{\|y\|^2}\big(\|x+by\|-\|x+ay\|\big)\\ &\quad-\frac{1}{4}\big(2b\|x+2by\|-2a\|x+2ay\|\big)-\frac{\langle x,y\rangle}{4\|y\|^2}\big(\|x+2by\|-\|x+2ay\|\big)\\ &= b\Big(\frac{2\|x+by\|-\|x+2by\|}{2}\Big)-a\Big(\frac{2\|x+ay\|-\|x+2ay\|}{2}\Big)\\ &\quad+\frac{\langle x,y\rangle}{4\|y\|^2}\Big[4\|x+by\|-\|x+2by\|-4\|x+ay\|+\|x+2ay\|\Big]\\ &= \frac{b}{2}\delta_{x,y}(b)-\frac{a}{2}\delta_{x,y}(a)+\frac{\langle x,y\rangle}{4\|y\|^2}\Big[\delta_{x,y}(b)-\delta_{x,y}(a)+2\big(\|x+by\|-\|x+ay\|\big)\Big]\\ &= \frac{b}{2}\delta_{x,y}(b)-\frac{a}{2}\delta_{x,y}(a)+\frac{\langle x,y\rangle}{4\|y\|^2}\Big(\delta_{x,y}(b)-\delta_{x,y}(a)\Big)+\frac{\langle x,y\rangle}{2\|y\|^2}\Big(\|x+by\|-\|x+ay\|\Big)\\ &= \Big(\frac{b}{2}+\frac{\langle x,y\rangle}{4\|y\|^2}\Big)\delta_{x,y}(b)-\Big(\frac{a}{2}+\frac{\langle x,y\rangle}{4\|y\|^2}\Big)\delta_{x,y}(a)+\frac{\langle x,y\rangle}{2\|y\|^2}\Big(\|x+by\|-\|x+ay\|\Big)\\ &= \Big(\frac{b}{2}+\frac{\langle x,y\rangle}{4\|y\|^2}\Big)\delta_{x,y}(b)-\Big(\frac{a}{2}+\frac{\langle x,y\rangle}{4\|y\|^2}\Big)\delta_{x,y}(a)\\ &\quad+\frac{\langle x,y\rangle}{2\|y\|^2}\left(\frac{\delta_{x,y}(b)+\|x+2by\|}{2}-\frac{\delta_{x,y}(a)+\|x+2ay\|}{2}\right)\\ &= \Big(\frac{b}{2}+\frac{\langle x,y\rangle}{4\|y\|^2}\Big)\delta_{x,y}(b)-\Big(\frac{a}{2}+\frac{\langle x,y\rangle}{4\|y\|^2}\Big)\delta_{x,y}(a)\\ &\quad+\frac{\langle x,y\rangle}{2\|y\|^2}\left(\frac{\delta_{x,y}(b)-\delta_{x,y}(a)}{2}+\frac{\|x+2by\|-\|x+2ay\|}{2}\right).\end{aligned}$$

This completes the proof. □

In the next section, we use above result to approximate $\int_a^b \delta_{x,y}(t)dt$ by approximating $\mathcal{P}_1$ and $\mathcal{P}_2$, using some properties of Logarithmic mean of α and β.

3. Approximation of $\int_a^b \delta_{x,y}(t)dt$

It is known that [7] if $\alpha > \beta > 0$, then

$$\frac{L(\alpha,\beta)}{\beta} > 1 + \frac{1}{2}\log\frac{\alpha}{\beta} > \frac{2\alpha}{\alpha+\beta}.$$

Therefor, if in preceding proposition forcing $\alpha > \beta > 0$, we have

$$\mathcal{P}_1 = \gamma(\log\alpha - \log\beta) = \gamma\frac{\alpha-\beta}{L(\alpha,\beta)} < \gamma(\alpha-\beta)\frac{\alpha+\beta}{2\alpha\beta} = \gamma\frac{\alpha^2-\beta^2}{2\alpha\beta}. \tag{3.1}$$

Now we find an upper bound for $\mathcal{P}_2$. In a normed linear space $(\mathfrak{X}, \|.\|)$ we define the lower and upper semi-inner products [3] as follows

$$\langle y, x\rangle_i = \lim_{t\to 0^-}\frac{\|x+ty\|^2 - \|x\|^2}{2t},$$

and

$$\langle y, x\rangle_s = \lim_{t\to 0^+}\frac{\|x+ty\|^2 - \|x\|^2}{2t}.$$

Also, we introduce following notation

$$\Psi^p_{x,y}(t) := \frac{\langle x, x+ty\rangle_p}{\|x+ty\|} \qquad (p\in\{i,s\}).$$

It is known (respectively [3] and [5]), that

$$\delta_{x,y}(t) \le \Psi^i_{x,y}(t), \tag{3.2}$$

and

$$\Psi^i_{x,y}(t) \le \|x\|. \tag{3.3}$$

Considering (3.2) and (3.3), and using (1.1) we obtain

$$\mathcal{P}_2 \le \left(\frac{b}{2} + \frac{\|x\|}{\|y\|}\right)\|x\| - \left(\frac{a}{2} + \frac{\|x\|}{2\|y\|}\right)\|x\| = \frac{b-a}{2}\|x\|.$$

Thus, using above obtained inequality and (3.1) and if $a > 0$, we have

$$\int_a^b \delta_{x,y}(t)dt < \gamma\frac{\alpha^2-\beta^2}{2\alpha\beta} + \big(\frac{b-a}{2}\big)\|x\| < \gamma\frac{\alpha^2-\beta^2}{2\alpha\beta} + A(a,b)\|x\|. \tag{3.4}$$

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