

GENERALIZATIONS OF BERNOULLI'S INEQUALITY WITH APPLICATIONS

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ABSTRACT. By using methods on the theory of majorization, some new generalizations of Bernoulli's inequality are established and some applications of the generalizations are given.

1. INTRODUCTION

Let $x > -1$ and n is a positive integer. Then

$$(1.1) \quad (1+x)^n \geq 1+nx.$$

(1.1) is known as the Bernoulli's inequality which play an important role in analysis and its applications. So, during the past few years, many researchers obtained various generalizations, extensions of inequality (1.1). For example, the following generalizations and variants of (1.1) were recorded in [1, pp.127-128]:

Theorem A. *Let $x > -1$. If $\alpha > 1$ or $\alpha < 0$, then*

$$(1.2) \quad (1+x)^\alpha \geq 1+\alpha x,$$

if $0 < \alpha < 1$, then

$$(1.3) \quad (1+x)^\alpha \leq 1+\alpha x.$$

In (1.2) and (1.3), equalities holding if and only if $x = 0$.

Theorem B. *Let $a_i \geq 0$, $x_i > -1$, $i = 1, \dots, n$, and $\sum_{i=1}^n a_i \leq 1$. Then*

$$(1.4) \quad \prod_{i=1}^n (1+x_i)^{a_i} \leq 1 + \sum_{i=1}^n a_i x_i,$$

if $a_i \geq 1$ or $a_i \leq 0$, and if $x_i > 0$, or $-1 < x_i < 0$, $i = 1, \dots, n$, then

$$(1.5) \quad \prod_{i=1}^n (1+x_i)^{a_i} \geq 1 + \sum_{i=1}^n a_i x_i.$$

For more information on the Bernoulli's inequality, please refer to [4, 6, 7, 8, 9, 10] and the references therein.

In this paper, some new generalizations of Bernoulli's inequality are established by the Schur-concavity of the elementary symmetric function and the dual form of the elementary symmetric functions, and some applications of the generalizations are given. We obtain the following results.

Theorem 1. *Let m, n is a positive integer, $k = 1, \dots, n$.*

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(i) If $m \geq n$ and $x > -1$, then

$$(1.6) \quad C_m^k \left(1 + \frac{n}{m}x\right)^k \geq \sum_{i=0}^k C_n^i C_{m-n}^{k-i} (1+x)^i,$$

and

$$(1.7) \quad k C_m^k \left(1 + \frac{n}{m}x\right)^{C_m^k} \geq \prod_{i=0}^k (ix+k)^{C_n^i C_{m-n}^{k-i}}.$$

(ii) If $m < n$ and $x > -\frac{m}{n}$, then

$$(1.8) \quad C_m^k (1+x)^k \geq \sum_{i=0}^k C_m^i C_{n-m}^{k-i} \left(1 + \frac{m}{n}x\right)^i,$$

and

$$(1.9) \quad k C_m^k (1+x)^{C_m^k} \geq \prod_{i=0}^k \left(\frac{m}{n}ix+k\right)^{C_m^i C_{n-m}^{k-i}},$$

where $C_n^k = \frac{n!}{k!(n-k)!}$ is the number of combinations of n elements taken k at a time, defined $C_n^0 = 1$ and $C_n^k = 0$ for $k > n$. In (1.6) (1.7) (1.8) and (1.9), equalities holding if and only if $x = 0$.

Remark 1. When $x = 0$, (1.6) (1.7) (1.8) and (1.9) are deduce to Vandermonde identity.

$$(1.10) \quad C_m^k = \sum_{i=0}^k C_n^i C_{m-n}^{k-i}.$$

Theorem 2. If $a_i \geq 1$ or $a_i \leq 0$, and if $x_i > 0$ or $0 \geq x_i \geq -1$. Then

$$(1.11) \quad C_n^k \left(\frac{1}{n} \sum_{i=1}^n (1+x_i)^{a_i}\right)^k \geq \sum_{1 \leq i_1 < \dots < i_k \leq n} \prod_{j=1}^k (1+x_{i_j})^{a_{i_j}}$$

$$\geq \sum_{1 \leq i_1 < \dots < i_k \leq n} \prod_{j=1}^k (1+a_{i_j} x_{i_j}) \geq C_{n-1}^k + C_{n-1}^{k-1} \left(1 + \sum_{i=1}^n a_i x_i\right).$$

$$(1.12) \quad \left(\frac{k}{n} \sum_{i=1}^n (1+x_i)^{a_i}\right)^{C_n^k} \geq \prod_{1 \leq i_1 < \dots < i_k \leq n} \sum_{j=1}^k (1+x_{i_j})^{a_{i_j}}$$

$$\geq \prod_{1 \leq i_1 < \dots < i_k \leq n} \sum_{j=1}^k (1+a_{i_j} x_{i_j}) \geq k^{C_{n-1}^k} \left(k + \sum_{i=1}^n a_i x_i\right)^{C_{n-1}^{k-1}}.$$

Remark 2. When $k = n$, (1.11) is deduce to (1.5), and when $k = 1$, (1.12) is deduce to (1.5) too.

2. PROOF OF THEOREM

For our own convenience, we introduce the following notations. we assume that the set of n -dimensional row vector on real number field by \mathbb{R}^n .

$$\mathbb{R}_+^n = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_i \geq 0, i = 1, \dots, n\},$$

$$\mathbb{R}_{++}^n = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_i > 0, i = 1, \dots, n\}.$$

Let $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$. Its elementary symmetric functions are

$$E_k(\mathbf{x}) = E_k(x_1, \dots, x_n) = \sum_{1 \leq i_1 < \dots < i_k \leq n} \prod_{j=1}^k x_{i_j}, \quad k = 1, \dots, n.$$

In particular, $E_n(\mathbf{x}) = \prod_{i=1}^n x_i$, $E_1(\mathbf{x}) = \sum_{i=1}^n x_i$, and defined $E_0(\mathbf{x}) = 1$ and $E_k(\mathbf{x}) = 0$ for $k < 0$ or $k > n$.

The dual form of the elementary symmetric functions are

$$E_k^*(\mathbf{x}) = E_k^*(x_1, \dots, x_n) = \prod_{1 \leq i_1 < \dots < i_k \leq n} \sum_{j=1}^k x_{i_j}, \quad k = 1, \dots, n,$$

and defined $E_0^*(\mathbf{x}) = 1$, and $E_k^*(\mathbf{x}) = 0$ for $k < 0$ or $k > n$.

We need the following definitions and lemmas.

Definition 1 ([2, 3]). Let $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$.

- (i) \mathbf{x} is said to be majorized by \mathbf{y} (in symbols $\mathbf{x} \prec \mathbf{y}$) if $\sum_{i=1}^k x_{[i]} \leq \sum_{i=1}^k y_{[i]}$ for $k = 1, 2, \dots, n-1$ and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$, where $x_{[1]} \geq \dots \geq x_{[n]}$ and $y_{[1]} \geq \dots \geq y_{[n]}$ are rearrangements of \mathbf{x} and \mathbf{y} in a descending order, and \mathbf{x} is said to be strictly majorized by \mathbf{y} (in symbols $\mathbf{x} \prec\prec \mathbf{y}$) if \mathbf{x} is not permutation of \mathbf{y} .
- (ii) $\mathbf{x} \geq \mathbf{y}$ means $x_i \geq y_i$ for all $i = 1, 2, \dots, n$. let $\Omega \subset \mathbb{R}^n$, $\varphi: \Omega \rightarrow \mathbb{R}$ is said to be increasing if $\mathbf{x} \geq \mathbf{y}$ implies $\varphi(\mathbf{x}) \geq \varphi(\mathbf{y})$. φ is said to be decreasing if and only if $-\varphi$ is increasing.
- (iii) $\Omega \subset \mathbb{R}^n$ is called a convex set if $(\alpha x_1 + \beta y_1, \dots, \alpha x_n + \beta y_n) \in \Omega$ for any \mathbf{x} and $\mathbf{y} \in \Omega$, where α and $\beta \in [0, 1]$ with $\alpha + \beta = 1$.
- (iv) let $\Omega \subset \mathbb{R}^n$, $\varphi: \Omega \rightarrow \mathbb{R}$ is said to be a Schur-convex function on Ω if $\mathbf{x} \prec \mathbf{y}$ on Ω implies $\varphi(\mathbf{x}) \leq \varphi(\mathbf{y})$. φ is said to be a Schur-concave function on Ω if and only if $-\varphi$ is Schur-convex function. φ is said to be a strictly Schur-convex function on Ω if $\mathbf{x} \prec\prec \mathbf{y}$ on Ω implies $\varphi(\mathbf{x}) < \varphi(\mathbf{y})$, φ is said to be a strictly Schur-concave function on Ω if and only $-\varphi$ is strictly Schur-convex on Ω .

Lemma 1. [2, p. 5] Let $\mathbf{x} \in \mathbb{R}^n$ and $\bar{\mathbf{x}} = \frac{1}{n} \sum_{i=1}^n x_i$. Then $(\bar{\mathbf{x}}, \dots, \bar{\mathbf{x}}) \prec \mathbf{x}$.

Proof of Theorem 1.

Proof. From Lemma 1, we have

$$\mathbf{p} := \left(\underbrace{1 + \frac{n}{m}x, \dots, 1 + \frac{n}{m}x}_m \right) \prec \left(\underbrace{1 + x, \dots, 1 + x}_n, \underbrace{1, \dots, 1}_{m-n} \right) := \mathbf{q}$$

and $\mathbf{p} \prec\prec \mathbf{q}$ for $x \neq 0$. If $m \geq n$, from $x > -1$, we have $x + 1 > 0$ and $1 + \frac{n}{m}x > 1 - \frac{n}{m} > 0$, i.e. $\mathbf{p}, \mathbf{q} \in \mathbb{R}_{++}^n$. Since $E_k(\mathbf{x})$ be increasing and Schur-concave on \mathbb{R}_+^n and it be increasing and strictly Schur-concave on \mathbb{R}_{++}^n for $k > 1$ (see Proposition 6.7 in [2]), we have $E_k(\mathbf{p}) \geq E_k(\mathbf{q})$, i.e. (1.6) holds, and equality holding if and only if $x = 0$.

Since $E_k^*(\mathbf{x})$ be increasing and Schur-concave on \mathbb{R}_+^n and it be increasing and strictly Schur-concave on \mathbb{R}_{++}^n for $k > 1$ (see [3, p.86], [5]), we have $E_k^*(\mathbf{p}) \geq E_k^*(\mathbf{q})$, i.e. (1.7) holds, and equality holding if and only if $x = 0$.

If $m < n$, from Lemma 1, we have

$$\mathbf{p}' := \left(\underbrace{1 + x, \dots, 1 + x}_n \right) \prec \left(\underbrace{1 + \frac{n}{m}x, \dots, 1 + \frac{n}{m}x}_m, \underbrace{1, \dots, 1}_{n-m} \right) := \mathbf{q}'$$

and $\mathbf{p}' \prec \prec \mathbf{q}'$ for $x \neq 0$. From $x > -\frac{m}{n}$, we have $1 + \frac{n}{m}x > 1 - \frac{n}{m} \cdot \frac{n}{m} > 0$, i.e. $\mathbf{p}', \mathbf{q}' \in \mathbb{R}_{++}^n$. Thus we have $E_k(\mathbf{p}') \geq E_k(\mathbf{q}')$ and $E_k^*(\mathbf{p}') \geq E_k^*(\mathbf{q}')$, i.e. (1.8) and (1.9) hold, and equality holding if and only if $x = 0$.

The proof of Theorem 1 is completed. \square

Proof of Theorem 2.

Proof. Set $y = \frac{1}{n} \sum_{i=1}^n (1 + x_i)^{a_i}$, from (1.2), we have $(1 + x_i)^{a_i} \geq 1 + a_i x_i$, $i = 1, \dots, n$, and by lemma 1, it follows that

$$\begin{aligned} \underbrace{(y, \dots, y)}_n &\prec ((1 + x_1)^{a_1}, \dots, (1 + x_n)^{a_n}) \\ &\geq (1 + a_1 x_1, \dots, 1 + a_n x_n) \prec \left(1 + \sum_{i=1}^n a_i x_i, \underbrace{1, \dots, 1}_{n-1} \right). \end{aligned}$$

And then, since $E_k(\mathbf{x})$ and $E_k^*(\mathbf{x})$ are increasing and Schur-concave on \mathbb{R}_+^n and are increasing and strictly Schur-concave on \mathbb{R}_{++}^n for $k > 1$, we have

$$\begin{aligned} E_k(\underbrace{y, \dots, y}_n) &\geq E_k((1 + x_1)^{a_1}, \dots, (1 + x_n)^{a_n}) \\ &\geq E_k(1 + a_1 x_1, \dots, 1 + a_n x_n) \geq E_k \left(1 + \sum_{i=1}^n a_i x_i, \underbrace{1, \dots, 1}_{n-1} \right), \end{aligned}$$

i.e. (1.11) is holds. And

$$\begin{aligned} E_k^*(\underbrace{y, \dots, y}_n) &\geq E_k^*((1 + x_1)^{a_1}, \dots, (1 + x_n)^{a_n}) \\ &\geq E_k^*(1 + a_1 x_1, \dots, 1 + a_n x_n) \geq E_k^* \left(1 + \sum_{i=1}^n a_i x_i, \underbrace{1, \dots, 1}_{n-1} \right), \end{aligned}$$

i.e. (1.12) is holds.

The proof of Theorem 2 is completed. \square

3. APPLICATIONS

Theorem 3. Let $a_i \geq 1$ and $x_i \geq 1$, $i = 1, \dots, n$, $n \in \mathbb{N}$, $n \geq 2$. Then for $k = 1, \dots, n$, we have

$$(3.1) \quad \sum_{1 \leq i_1 < \dots < i_k \leq n} \prod_{j=1}^k (1 + x_{i_j})^{a_{i_j}} \geq \frac{2^{A_k} C_{n-1}^{k-1}}{1 + A_k} \left(\frac{n}{k} (1 + A_k) - A_n + \sum_{i=1}^n a_i x_i \right),$$

where $A_k = \min_{1 \leq i_1 < \dots < i_k \leq n} \sum_{i=j}^k a_{i_j}$.

Proof. Firstly, since $x_i \geq 1$ implies $\frac{1+x}{2} \geq 0$, from Theorem 2 we have

$$\begin{aligned} (3.2) \quad &\sum_{1 \leq i_1 < \dots < i_k \leq n} \prod_{j=1}^k \left(1 + \frac{x_{i_j} - 1}{2} \right)^{a_{i_j}} \geq C_{n-1}^k + C_{n-1}^{k-1} \left(1 + \sum_{i=1}^n \frac{a_i (x_i - 1)}{2} \right) \\ &= C_{n-1}^{k-1} \left(\frac{n}{k} + \sum_{i=1}^n \frac{a_i (x_i - 1)}{2} \right) \geq C_{n-1}^{k-1} \left(\frac{n}{k} + \sum_{i=1}^n \frac{a_i (x_i - 1)}{1 + A_k} \right) \\ &= \frac{C_{n-1}^{k-1}}{1 + A_k} \left(\frac{n}{k} (1 + A_k) - A_n + \sum_{i=1}^n a_i x_i \right). \end{aligned}$$

On the other hand, we have

$$(3.3) \quad \sum_{1 \leq i_1 < \dots < i_k \leq n} \prod_{j=1}^k (1 + x_{i_j})^{a_{i_j}} = \sum_{1 \leq i_1 < \dots < i_k \leq n} 2^{\sum_{j=1}^k a_{i_j}} \prod_{j=1}^k \left(1 + \frac{x_{i_j} - 1}{2}\right)^{a_{i_j}} \\ \geq 2^{A_k} \sum_{1 \leq i_1 < \dots < i_k \leq n} \prod_{j=1}^k \left(1 + \frac{x_{i_j} - 1}{2}\right)^{a_{i_j}}.$$

□

Combining (3.2) with (3.3), we get (3.1). The proof of Theorem 3 is completed.

Remark 3. When $k = n$, (3.1) is deduce to (7.5) in [4, p. 69]:

$$(3.4) \quad \prod_{i=1}^n (1 + x_i)^{a_i} \geq \frac{2^{A_n}}{1 + A_n} \left(1 + \sum_{i=1}^n a_i x_i\right).$$

Theorem 4. Let $a_i \geq 1$ or $a_i \leq 0$ and $0 > x_i > -1$ or $x_i > 0$, $i = 1, \dots, n, n \in \mathbb{N}, n \geq 2$. Then for $k = 1, \dots, n$, we have

$$(3.5) \quad \sum_{1 \leq i_1 < \dots < i_k \leq n} \prod_{j=1}^k (1 + x_{i_j})^{-a_{i_j}} \geq \sum_{1 \leq i_1 < \dots < i_k \leq n} \prod_{j=1}^k (1 - a_{i_j} x_{i_j} (1 + x_{i_j})^{-1}) \\ \geq C_{n-1}^k + C_{n-1}^{k-1} (1 - a_{i_j} x_{i_j} (1 + x_{i_j})^{-1})$$

Proof. Since $0 > x_i > -1$ or $x_i > 0$ implies $-x_i (1 + x_i)^{-1}$ or $0 > -x_i (1 + x_i)^{-1} > -1$, from Theorem 2 we have

$$\sum_{1 \leq i_1 < \dots < i_k \leq n} \prod_{j=1}^k (1 + x_{i_j})^{-a_{i_j}} = \sum_{1 \leq i_1 < \dots < i_k \leq n} \prod_{j=1}^k \left(1 - x_{i_j} (1 + x_{i_j})^{-1}\right)^{a_{i_j}} \\ \geq \sum_{1 \leq i_1 < \dots < i_k \leq n} \prod_{j=1}^k (1 + a_{i_j} x_{i_j} (1 + x_{i_j})^{-1}) \geq C_{n-1}^k + C_{n-1}^{k-1} (1 + a_{i_j} x_{i_j} (1 + x_{i_j})^{-1}).$$

□

The proof of Theorem 4 is completed.

Remark 4. When $k = n$, from (3.5) we have

$$\prod_{i=1}^n (1 + x_i)^{-a_i} \geq 1 - \sum_{i=1}^n a_i x_i (1 + x_i)^{-1},$$

i.e.

$$(3.6) \quad \prod_{i=1}^n (1 + x_i)^{a_i} \leq \left(1 - \sum_{i=1}^n a_i x_i (1 + x_i)^{-1}\right)^{-1}.$$

(3.6) is (7.3) in [4, p. 69].

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