Ky Fan's Inequality via Convexity

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Abstract

In this note, using the strict convexity and concavity of the function $f(x) = \frac{1}{1+e^x}$ on $[0, \infty)$ and $(-\infty, 0]$ respectively, we prove Ky Fan's inequality by separating the left and right hands of it by $\frac{1}{G_n+G'_n}$.

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Let $x_1, \cdots, x_n$ in $(0, 1/2]$ and $\lambda_1, \lambda_2, \cdots, \lambda_n > 0$ with $\sum_{i=1}^n \lambda_i = 1$. We denote by A_n and G_n, the arithmetic and geometric means of $x_1, \cdots, x_n$ respectively, i.e.

$$A_n = \sum_{i=1}^n \lambda_i x_i, \qquad G_n = \prod_{i=1}^n x_i^{\lambda_i}, \tag{1}$$

and also by A'_n and G'_n, the arithmetic and geometric means of $1 - x_1, \cdots, 1 - x_n$ respectively, i.e.

$$A'_n = \sum_{i=1}^n \lambda_i(1 - x_i), \qquad G'_n = \prod_{i=1}^n (1 - x_i)^{\lambda_i}. \tag{2}$$

In 1961 the following remarkable inequality, due to Ky Fan, was published for the first time in the well-known book *Inequalities* by Beckenbach and Bellman [2, p. 5]:
If $x_i \in (0, 1/2]$, *then*

$$\frac{A'_n}{G'_n} \le \frac{A_n}{G_n}, \tag{3}$$

with equality holding if and only if $x_1 = \cdots = x_n$.
Inequality (1) has evoked the interest of several mathematicians and in numerous articles new proofs, extensions, refinements and various related results have been published; see the survey paper [1],

and also for some recent results, [3-7] .
In this note, using the strict convexity and concavity of the function $f(x) = \frac{1}{1+e^x}$ on $[0, \infty)$ and $(-\infty, 0]$ respectively, we prove Ky Fan's inequality (3) by separating the left and right hands of (3) by $\frac{1}{G_n + G'_n}$:

$$\frac{A'_n}{G'_n} \leq \frac{1}{G_n + G'_n} \leq \frac{A_n}{G_n}. \tag{4}$$

Moreover, we show equality holds in each inequality in (4), if and only $x_1 = \cdots = x_n$.
It is noted that, since for $a, b, c, d > 0$ the inequality $\frac{a}{b} \leq \frac{c}{d}$ implies $\frac{a}{b} \leq \frac{a+c}{b+d} \leq \frac{c}{d}$, considering $A_n + A'_n = 1$, the inequalities (3) and (4) are equivalent.
Indeed, since $f''(x) = \frac{e^x(e^x-1)}{(1+e^x)^3}$, the function f has the foregoing convexity properties. Now, using jensen's inequality

$$f\left(\sum_{i=1}^{n} \lambda_i y_i\right) \leq \sum_{i=1}^{n} \lambda_i f(y_i),$$

for $y_i = \ln \frac{1-x_i}{x_i} \geq 0 \ \ (1 \leq i \leq n)$, we get the right hand of (4) with equality holding if and only if $\ln \frac{1-x_1}{x_1} = \cdots = \ln \frac{1-x_n}{x_n}$, or equivalently $x_1 = \cdots = x_n$. The left hand of (4) is handled by using Jensen's inequality for the convex function $-f$ on $(-\infty, 0]$ with $y_i = \ln \frac{x_i}{1-x_i} \leq 0 \ \ (1 \leq i \leq n)$.

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