On a new nonlinear Volterra type sum-difference equation *

B.G PACHPATTE

November 22, 2007

Abstract

The aim of the present paper is to study the properties of solutions of a new nonlinear Volterra type sum-difference equation. The finite difference inequality with explicit estimate is used to establish the results.

1 Introduction

Consider the nonlinear Volterra type sum-difference equation

$$x(n) = g(n) + \sum_{s=0}^{n-1} f(n, s, x(s), \Delta x(s)), \tag{1.1}$$

where x, g, f are real-valued functions. Let R denotes the set of real numbers and $R_+ = [0, \infty), N_0 = \{0, 1, 2, ...\}$ be the given subsets of R . Let $D(A, B)$ denotes the class of functions from the set A to the set B . We assume that $x, g \in D(N_0, R)$ and for $0 \leq s \leq n; s, n \in N_0$, $f \in D(N_0^2 \times R^2, R)$. For the functions $w(m), z(m, n)$, $m, n \in N_0$ we define the operators Δ and Δ_1 by $\Delta w(m) = w(m+1) - w(m)$ and $\Delta_1 z(m, n) = z(m+1, n) - z(m, n)$. We use the usual conventions that empty sums and products are taken to be 0 and 1 respectively.

In the past few years a number of papers have been devoted to the study of equations of the form (1.1), particularly when the function f is of the form $f(t, s, x)$, see [1-4,6] and the references cited therein. In [2] Kwapisz has studied the existence of solutions of certain boundary value problems for difference equations by using the techniques employed in the theory of ordinary differential equations. One can formulate existence and uniqueness result for the solutions

* *1991 Mathematics Subject Classifications:* 26D10, 3910.
Key words and Phrases: Volterra type, sum-difference equation, finite difference inequality, explicit estimate, uniqueness, boundedness, continuous dependence.

of equation (1.1) by following the method used in [5]. The main objective of this paper is to study the uniqueness, boundedness and continuous dependence of solutions of equation (1.1) under some suitable conditions on the functions involved therein. The special version of the finite difference inequality established by Pachpatte (see [3, p. 21]) is used to establish the results.

2 Uniqueness and boundedness

By a solution of equation (1.1) we mean a function $x(n)$, $n \in N_0$ which satisfies the equation (1.1). It is easy to observe that the solution $x(n)$ of equation (1.1) satisfies (see [3, p. 22]) the following sum-difference equation

$$\Delta x(n) = \Delta g(n) + f(n+1, n, x(n), \Delta x(n)) + \sum_{s=0}^{n-1} \Delta_1 f(n, s, x(s), \Delta x(s)), \tag{2.1}$$

for $n \in N_0$. The existence and uniqueness result for the solutions of equation (1.1) can be formulated by closely looking at the proof of Theorem 1 given in [5] (see also [2]). However, since there is a close parallelism, here we are not intend to discuss the details of such a result.

We need the following special version of the finite difference inequality established by Pachpatte (see [3, Theorem 1.3.4, p.21]). We shall state it in the following lemma for completeness.

Lemma . Let $u(n), a(n) \in D(N_0, R_+)$ and for $0 \le s \le n; s, n \in N_0$, $k(n,s), \Delta_1 k(n,s) \in D\left(N_0^2, R_+\right)$. If

$$u(n) \le a(n) + \sum_{s=0}^{n-1} k(n,s)\, u(s),$$

for $n \in N_0$, then

$$u(n) \le a(n) + \sum_{s=0}^{n-1} B(s) \prod_{\sigma=s+1}^{n-1} [1 + A(\sigma)],$$

for $n \in N_0$, where

$$A(n) = k(n+1, n) + \sum_{s=0}^{n-1} \Delta_1 k(n,s), \tag{2.2}$$

$$B(n) = k(n+1, n)\, a(n) + \sum_{s=0}^{n-1} \Delta_1 k(n,s)\, a(s), \tag{2.3}$$

for $n \in N_0$.

First, we formulate the following theorem concerning the uniqueness of solutions of equation (1.1) without existence part.

Theorem 1. Assume that

$$|f(n,s,u,v) - f(n,s,\bar{u},\bar{v})| \le h_1(n,s)\left[|u-\bar{u}| + |v-\bar{v}|\right], \tag{2.4}$$

$$|\Delta_1 f(n,s,u,v) - \Delta_1 f(n,s,\bar{u},\bar{v})| \le h_2(n,s)\left[|u-\bar{u}| + |v-\bar{v}|\right], \tag{2.5}$$

where $h_i(n,s), \Delta_1 h_i(n,s) \in D\left(N_0^2, R_+\right)$ for $i = 1, 2$ and $0 \le s \le n; s, n \in N_0$. Let $h(n,s) = h_1(n,s) + h_2(n,s)$ and assume that $h_1(n+1,n) \le c$, where $c < 1$ is a constant. Then the equation (1.1) has at most one solution on N_0

Proof. Let $x(n)$ and $y(n)$ be two solutions of equation (1.1) on N_0 . Using the facts that $x(n), y(n)$ are the solutions of equation (1.1) and the hypotheses (2.4), (2.5) we have

$$
\begin{aligned}
&|x(n) - y(n)| + |\Delta x(n) - \Delta y(n)| \\
&\le \sum_{s=0}^{n-1} |f(n,s,x(s),\Delta x(s)) - f(n,s,y(s),\Delta y(s))| \\
&+ |f(n+1,n,x(n),\Delta x(n)) - f(n+1,s,y(n),\Delta y(n))| \\
&+ \sum_{s=0}^{n-1} |\Delta_1 f(n,s,x(s),\Delta x(s)) - \Delta_1 f(n,s,y(s),\Delta y(s))| \\
&\le \sum_{s=0}^{n-1} h_1(n,s)\left[|x(s) - y(s)| + |\Delta x(s) - \Delta y(s)|\right] \\
&+ h_1(n+1,n)\left[|x(n) - y(n)| + |\Delta x(n) - \Delta y(n)|\right] \\
&+ \sum_{s=0}^{n-1} h_2(n,s)\left[|x(s) - y(s)| + |\Delta x(s) - \Delta y(s)|\right].
\end{aligned} \tag{2.6}
$$

From (2.6) and using the assumption $h_1(n+1,n) \le c$, we observe that

$$
\begin{aligned}
&|x(n) - y(n)| + |\Delta x(n) - \Delta y(n)| \\
&\le \frac{1}{1-c} \sum_{s=0}^{n-1} h(n,s)\left[|x(s) - y(s)| + |\Delta x(s) - \Delta y(s)|\right].
\end{aligned} \tag{2.7}
$$

Now an application of Lemma (with $a(n) = 0$) to (2.7) yields

$$|x(n) - y(n)| + |\Delta x(n) - \Delta y(n)| \le 0,$$

which implies $x(n) = y(n)$ for $n \in N_0$. Thus there is at most one solution to equation (1.1) on N_0 .

The following theorem concerning the estimate on the solution of equation (1.1) holds.

Theorem 2. Assume that

$$|g(n)| + |\Delta g(n)| \leq p(n), \tag{2.8}$$

$$|f(n,s,u,v)| \leq r_1(n,s)[|u| + |v|], \tag{2.9}$$

$$|\Delta_1 f(n,s,u,v)| \leq r_2(n,s)[|u| + |v|], \tag{2.10}$$

where $p(n) \in D(N_0, R_+)$, $r_i(n,s), \Delta_1 r_i(n,s) \in D\left(N_0^2, R_+\right)$ for $i = 1,2$ and $0 \leq s \leq n; s, n \in N_0$. Let $r(n,s) = r_1(n,s) + r_2(n,s)$ and assume that $r_1(n+1,n) \leq d$, where $d < 1$ is a constant. If $x(n)$, $n \in N_0$ is any solution of equation (1.1), then

$$|x(n)| + |\Delta x(n)| \leq \frac{p(n)}{1-d} + \sum_{s=0}^{n-1} B_1(s) \prod_{\sigma=s+1}^{n-1} [1 + A_1(\sigma)], \tag{2.11}$$

for $n \in N_0$, where $A_1(n)$ and $B_1(n)$ are defined respectively by the right hand sides of (2.2) and (2.3), replacing $k(n,s)$ by $\frac{r(n,s)}{1-d}$ and $a(n)$ by $\frac{p(n)}{1-d}$.

Proof. Using the fact that $x(n)$, $n \in N_0$ is a solution of equation (1.1) and the hypotheses (2.8)-(2.10) we have

$$\begin{aligned} |x(n)| + |\Delta x(n)| &\leq |g(n)| + |\Delta g(n)| + \sum_{s=0}^{n-1} |f(n,s,x(s),\Delta x(s))| \\ &+ |f(n+1,n,x(n),\Delta x(n))| + \sum_{s=0}^{n-1} |\Delta_1 f(n,s,x(s),\Delta x(s))| \\ &\leq p(n) + \sum_{s=0}^{n-1} r_1(n,s)[|x(s)| + |\Delta x(s)|] \\ &+ r_1(n+1,n)[|x(n)| + |\Delta x(n)|] + \sum_{s=0}^{n-1} r_2(n,s)[|x(s)| + |\Delta x(s)|]. \end{aligned} \tag{2.12}$$

From (2.12) and using the assumption $r_1(n+1,n) \leq d$, we observe that

$$|x(n)| + |\Delta x(n)| \leq \frac{p(n)}{1-d} + \frac{1}{1-d} \sum_{s=0}^{n-1} r(n,s)[|x(s)| + |\Delta x(s)|]. \tag{2.13}$$

Now an application of Lemma to (2.13) yields (2.11).

Remark 1 We note that the estimate obtained in (2.11) yields not only the bound on the solution $x(n)$ of equation (1.1) but also the bound on $\Delta x(n)$. If the estimate on the right hand side in (2.11) is bounded, then the solution $x(n)$ and also $\Delta x(n)$ are bounded for $n \in N_0$.

Now, we shall obtain the estimate on the solution of equation (1.1), assuming that f and $\Delta_1 f$ satisfy Lipschitz type conditions.

Theorem 3. Assume that the hypotheses of Theorem 1 hold and

$$q(n) = |f(n+1, n, g(n), \Delta g(n))| + \sum_{s=0}^{n-1} |f(n, s, g(s), \Delta g(s))|$$

$$+ \sum_{s=0}^{n-1} |\Delta_1 f(n, s, g(s), \Delta g(s))|, \tag{2.14}$$

for $n \in N_0$. If $x(n)$,$n \in N_0$ is any solution of equation (1.1), then

$$|x(n) - g(n)| + |\Delta x(n) - \Delta g(n)|$$

$$\leq \frac{q(n)}{1-c} + \sum_{s=0}^{n-1} B_2(s) \prod_{\sigma=s+1}^{n-1} [1 + A_2(\sigma)], \tag{2.15}$$

for $n \in N_0$, where $A_2(n)$ and $B_2(n)$ are defined respectively by the right hand sides of (2.2) and (2.3), replacing $k(n,s)$ by $\frac{h(n,s)}{1-c}$ and $a(n)$ by $\frac{q(n)}{1-c}$.

Proof. Using the fact that $x(n)$, $n \in N_0$ is a solution of equation (1.1) and the hypotheses (2.4),(2.5) we have

$$|x(n) - g(n)| + |\Delta x(n) - \Delta g(n)|$$

$$\leq \sum_{s=0}^{n-1} |f(n, s, x(s), \Delta x(s)) - f(n, s, g(s), \Delta g(s))|$$

$$+ \sum_{s=0}^{n-1} |f(n, s, g(s), \Delta g(s))|$$

$$+ |f(n+1, n, x(n), \Delta x(n)) - f(n+1, n, g(n), \Delta g(n))|$$

$$+ |f(n+1, n, g(n), \Delta g(n))|$$

$$+ \sum_{s=0}^{n-1} |\Delta_1 f(n, s, x(s), \Delta x(s)) - \Delta_1 f(n, s, g(s), \Delta g(s))|$$

$$+ \sum_{s=0}^{n-1} |\Delta_1 f(n, s, g(s), \Delta g(s))|$$

$$\leq q(n) + \sum_{s=0}^{n-1} h_1(n,s) [|x(s) - g(s)| + |\Delta x(s) - \Delta g(s)|]$$

$$+ h_1(n+1, n) [|x(n) - g(n)| + |\Delta x(n) - \Delta g(n)|]$$

$$+\sum_{s=0}^{n-1} h_2(n,s)\left[|x(s)-g(s)|+|\Delta x(s)-\Delta g(s)|\right]. \tag{2.16}$$

From (2.16) and using the assumption $h_1(n+1,n)\le c$, we observe that

$$|x(n)-g(n)|+|\Delta x(n)-\Delta g(n)|$$

$$\le \frac{q(n)}{1-c}+\frac{1}{1-c}\sum_{s=0}^{n-1} h(n,s)\left[|x(s)-g(s)|+|\Delta x(s)-\Delta g(s)|\right]. \tag{2.17}$$

Now an application of Lemma to (2.17) yields (2.15).

3 Continuous dependence

In this section we shall deal with the continuous dependence of solutions of equation (1.1) on the functions involved therein and also the continuous dependence of solutions of equations of the form (1.1) on parameters.

Consider the equation (1.1) and the corresponding Volterra type sum-difference equation

$$y(n)=G(n)+\sum_{s=0}^{n-1} F(n,s,y(s),\Delta y(s)), \tag{3.1}$$

for $n\in N_0$, where $y,G\in D(N_0,R)$, and for $0\le s\le n; s,n\in N_0$, $F\in D\left(N_0^2\times R^2,R\right)$.

The following theorem deals with the continuous dependence of solutions of equation (1.1) on the functions involved therein.

Theorem 4. Assume that the hypotheses of Theorem 1 hold. Suppose that

$$|g(n)-G(n)|+|\Delta g(n)-\Delta G(n)|$$

$$+|f(n+1,n,y(n),\Delta y(n))-F(n+1,n,y(n),\Delta y(n))|$$

$$+\sum_{s=0}^{n-1}|f(n,s,y(s),\Delta y(s))-F(n,s,y(s),\Delta y(s))|$$

$$+\sum_{s=0}^{n-1}|\Delta_1 f(n,s,y(s),\Delta y(s))-\Delta_1 F(n,s,y(s),\Delta y(s))|\le\beta(n), \tag{3.2}$$

where g,f and G,F are the functions involved in (1.1) and (3.1) and $\beta(n)\in D(N_0,R)$. Let $x(n)$ and $y(n)$, $n\in N_0$ be the solutions of equations (1.1) and (3.1) respectively. Then the solution $x(n)$ of equation (1.1) depends continuously on the functions involved on the right hand side of equation (1.1).

Proof. Using the facts that $x(n)$ and $y(n)$ are the solutions of equations (1.1) and (3.1) and hypotheses (2.4),(2.5),(3.2) we have

$$
\begin{aligned}
&|x(n)-y(n)|+|\Delta x(n)-\Delta y(n)| \\
&\leq |g(n)-G(n)|+|\Delta g(n)-\Delta G(n)| \\
&+\sum_{s=0}^{n-1}|f(n,s,x(s),\Delta x(s))-f(n,s,y(s),\Delta y(s))| \\
&+\sum_{s=0}^{n-1}|f(n,s,y(s),\Delta y(s))-F(n,s,y(s),\Delta y(s))| \\
&+|f(n+1,n,x(n),\Delta x(n))-f(n+1,n,y(n),\Delta y(n))| \\
&+|f(n+1,n,y(n),\Delta y(n))-F(n+1,n,y(n),\Delta y(n))| \\
&+\sum_{s=0}^{n-1}|\Delta_1 f(n,s,x(s),\Delta x(s))-\Delta_1 f(n,s,y(s),\Delta y(s))| \\
&+\sum_{s=0}^{n-1}|\Delta_1 f(n,s,y(s),\Delta y(s))-\Delta_1 F(n,s,y(s),\Delta y(s))| \\
&\leq \beta(n)+\sum_{s=0}^{n-1}h_1(n,s)\left[|x(s)-y(s)|+|\Delta x(s)-\Delta y(s)|\right] \\
&+h_1(n+1,n)\left[|x(n)-y(n)|+|\Delta x(n)-\Delta y(n)|\right] \\
&+\sum_{s=0}^{n-1}h_2(n,s)\left[|x(s)-y(s)|+|\Delta x(s)-\Delta y(s)|\right]. \qquad (3.3)
\end{aligned}
$$

From (3.3) and using the assumption $h_1(n+1,n)\leq c$, we observe that

$$
\begin{aligned}
&|x(n)-y(n)|+|\Delta x(n)-\Delta y(n)| \\
&\leq \frac{\beta(n)}{1-c}+\frac{1}{1-c}\sum_{s=0}^{n-1}h(n,s)\left[|x(s)-y(s)|+|\Delta x(s)-\Delta y(s)|\right]. \qquad (3.4)
\end{aligned}
$$

Now an application of Lemma to (3.4) yields

$$
\begin{aligned}
&|x(n)-y(n)|+|\Delta x(n)-\Delta y(n)| \\
&\leq \frac{\beta(n)}{1-c}+\sum_{s=0}^{n-1}B_3(s)\prod_{\sigma=s+1}^{n-1}[1+A_3(\sigma)], \qquad (3.5)
\end{aligned}
$$

for $n\in N_0$,where $A_3(n)$ and $B_3(n)$ are defined respectively by (2.2) and (2.3), replacing $k(n,s)$ by $\frac{h(n,s)}{1-c}$ and $a(n)$ by $\frac{\beta(n)}{1-c}$. From (3.5) it follows that the solutions of equation (1.1) depends continuously on the functions involved on the right hand side of equation (1.1).

Next, we consider the following Volterra type sum-difference equations

$$z(n) = g(n) + \sum_{s=0}^{n-1} f(n, s, z(s), \Delta z(s), \mu), \tag{3.6}$$

and

$$z(n) = g(n) + \sum_{s=0}^{n-1} f(n, s, z(s), \Delta z(s), \mu_0), \tag{3.7}$$

for $n \in N_0$, where $z, g \in D(N_0, R)$ and for $0 \le s \le n; s, n \in N_0$, $f \in D\left(N_0^2 \times R^2 \times R, R\right)$.

Finally, we present the following theorem which deals with the continuous dependency of solutions of equations (3.6) and (3.7) on parameters.

Theorem 5. Assume that the function f in (3.6) and (3.7) satisfy the conditions

$$|f(n, s, u, v, \mu) - f(n, s, \bar{u}, \bar{v}, \mu)| \le \gamma_1(n, s)\left[|u - \bar{u}| + |v - \bar{v}|\right], \tag{3.8}$$

$$|f(n, s, u, v, \mu) - f(n, s, u, v, \mu_0)| \le e_1(n, s)|\mu - \mu_0|, \tag{3.9}$$

$$|\Delta_1 f(n, s, u, v, \mu) - \Delta_1 f(n, s, \bar{u}, \bar{v}, \mu)| \le \gamma_2(n, s)\left[|u - \bar{u}| + |v - \bar{v}|\right], \tag{3.10}$$

$$|\Delta_1 f(n, s, u, v, \mu) - \Delta_1 f(n, s, u, v, \mu_0)| \le e_2(n, s)|\mu - \mu_0|, \tag{3.11}$$

where $\gamma_i(n, s), \Delta_1\gamma_i(n, s), e_i(n, s) \in D\left(N_0^2, R_+\right)$ for $i = 1, 2$ and $0 \le s \le n; s, n \in N_0$. Let $\gamma(n, s) = \gamma_1(n, s) + \gamma_2(n, s)$, $e(n, s) = e_1(n, s) + e_2(n, s)$, $\delta(n) = e_1(n+1, n) + \sum_{s=0}^{n-1} e(n, s)$ and assume that $\gamma_1(n+1, n) \le m$, where $m < 1$ is a constant. Let $z_1(n)$ and $z_2(n)$ be the solutions of equations (3.6) and (3.7) respectively. Then

$$|z_1(n) - z_2(n)| + |\Delta z_1(n) - \Delta z_2(n)|$$
$$\le \frac{|\mu - \mu_0|}{1 - m}\delta(n) + \sum_{s=0}^{n-1} B_4(s) \prod_{\sigma=s+1}^{n-1} [1 + A_4(\sigma)], \tag{3.12}$$

where $A_4(n)$ and $B_4(n)$ are defined respectivey by the right hand sides of (2.2) and (2.3), replacing $k(n, s)$ by $\frac{\gamma(n,s)}{1-m}$ and $a(n)$ by $\frac{|\mu-\mu_0|}{1-m}\delta(n)$.

Proof. Using the facts that $z_1(n)$ and $z_2(n)$ are the solutions of equations (3.6) and (3.7) and the hypotheses (3.9)-(3.11) we have

$$|z_1(n) - z_2(n)| + |\Delta z_1(n) - \Delta z_2(n)|$$
$$\le \sum_{s=0}^{n-1} |f(n, s, z_1(s), \Delta z_1(s), \mu) - f(n, s, z_2(s), \Delta z_2(s), \mu)|$$

$$+\sum_{s=0}^{n-1}\left|f\left(n,s,z_2\left(s\right),\Delta z_2\left(s\right),\mu\right)-f\left(n,s,z_2\left(s\right),\Delta z_2\left(s\right),\mu_0\right)\right|$$

$$+\left|f\left(n+1,n,z_1\left(n\right),\Delta z_1\left(n\right),\mu\right)-f\left(n+1,n,z_2\left(n\right),\Delta z_2\left(n\right),\mu\right)\right|$$

$$+\left|f\left(n+1,n,z_2\left(n\right),\Delta z_2\left(n\right),\mu\right)-f\left(n+1,n,z_2\left(n\right),\Delta z_2\left(n\right),\mu_0\right)\right|$$

$$+\sum_{s=0}^{n-1}\left|\Delta_1 f\left(n,s,z_1\left(s\right),\Delta z_1\left(s\right),\mu\right)-\Delta_1 f\left(n,s,z_2\left(s\right),\Delta z_2\left(s\right),\mu\right)\right|$$

$$+\sum_{s=0}^{n-1}\left|\Delta_1 f\left(n,s,z_2\left(s\right),\Delta z_2\left(s\right),\mu\right)-\Delta_1 f\left(n,s,z_2\left(s\right),\Delta z_2\left(s\right),\mu_0\right)\right|$$

$$\leq\sum_{s=0}^{n-1}\gamma_1\left(n,s\right)\left[\left|z_1\left(s\right)-z_2\left(s\right)\right|+\left|\Delta z_1\left(s\right)-\Delta z_2\left(s\right)\right|\right]$$

$$+\sum_{s=0}^{n-1}e_1\left(n,s\right)\left|\mu-\mu_0\right|$$

$$+\gamma_1\left(n+1,n\right)\left[\left|z_1\left(n\right)-z_2\left(n\right)\right|+\left|\Delta z_1\left(n\right)-\Delta z_2\left(n\right)\right|\right]$$

$$+e_1\left(n+1,n\right)\left|\mu-\mu_0\right|$$

$$+\sum_{s=0}^{n-1}\gamma_2\left(n,s\right)\left[\left|z_1\left(s\right)-z_2\left(s\right)\right|+\left|\Delta z_1\left(s\right)-\Delta z_2\left(s\right)\right|\right]$$

$$+\sum_{s=0}^{n-1}e_2\left(n,s\right)\left|\mu-\mu_0\right|. \tag{2.13}$$

From (2.13) and using the assumption $\gamma_1\left(n+1,n\right)\leq m$, we observe that

$$\left|z_1\left(n\right)-z_2\left(n\right)\right|+\left|\Delta z_1\left(n\right)-\Delta z_2\left(n\right)\right|$$

$$\leq\frac{\left|\mu-\mu_0\right|}{1-m}\delta\left(n\right)+\frac{1}{1-m}\sum_{s=0}^{n-1}\gamma\left(n,s\right)\left[\left|z_1\left(s\right)-z_2\left(s\right)\right|+\left|\Delta z_1\left(s\right)-\Delta z_2\left(s\right)\right|\right]. \tag{3.14}$$

Now an application of Lemma to (3.14) yields (3.12), which shows the dependency of solutions of equations (3.6) and (3.7) on parameters.

Remark 2. We note that in [3, p.269] (see also [1, p.221] and [4, p.237]) the explicit bounds on the solutions of equation (1.1) when f is of the form $f(n,s,x(s))$ are given. Here our approach to the study of equation (1.1) is somewhat different and we believe that the results obtained here are of independent interest.

References

1. D. Bainov and P. Simeonov, Integral Inequalities and Applications, Kluwer Academic Publishers, Dordrecht, 1992.

2. M. Kwapisz, Some existence and uniqueness results for boundary value problems for difference equations, Applicable Analysis 37 (1990), 169-182.

3. B.G. Pachpatte, Inequalities for Finite Difference Equations, Marcel Dekker, Inc.,New York, 2002.

4. B.G. Pachpatte, Integral and Finite Difference Inequalities and Applications, North-Holland Mathematics studies 205, Elsevier Science B.V., 2006.

5. B.G. Pachpatte, On higher order Volterra-Fredholm integrodifferential equation, Fasciculi Mathematici, Nr. 37 (2006), 35-48.

6. S. Sugiyama, On the stability of difference equations, Bull.Sci.Engr.Research Lab. Waseda Univ. 45 (1969), 140-144.

57 Shri Niketan Colony, Near Abhinay Talkies, Aurangabad (Maharashtra) India

E-mail: bgpachpatte@gmail.com