Some basic theorems on difference-differential equations *

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November 22, 2007

Abstract

In this paper some basic theorems on the existence, uniqueness and continuous dependence of solutions of a certain difference-differential equation are established. The well known Banach fixed point theorem and the Gronwall-Bellman integral inequality are used to establish the results.

1 Introduction

Let R^n denotes the real n-dimensional Euclidean space with appropriate norm denoted by $|.|$. Let $R_+ = [0, \infty)$ be the given subset of R, the set of real numbers. Consider the difference-differential equation

$$x'(t) = f(t, x(t), x(t-1)), \tag{1.1}$$

for $t \in R_+$ under the initial conditions

$$x(t-1) = \phi(t) \ \ (0 \leq t < 1), x(0) = x_0, \tag{1.2}$$

where $f \in C(R_+ \times R^n \times R^n, R^n)$ and $\phi(t)$ is a continuous function which has the $\lim_{t \to 1-0} \phi(t)$. If we consider the solutions of equation (1.1) for $t \in R_+$, it happens that we obtain a function $x(t-1)$ unable to define as a solution for $0 \leq t < 1$. Hence, we have to impose some condition, for example the first condition in (1.2). In this case it is sufficient to consider the ordinary differential equation

$$x'(t) = f(t, x(t), \phi(t)), \tag{1.3}$$

for $0 \leq t < 1$, with the second condition in (1.2). We note that, here it is essential to obtain the solutions of equation (1.1) for $0 \leq t < \infty$. It is easy to

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* *1991 Mathematics Subject Classifications:* 34K05, 34K10, 34K40

Key words and Phrases: Difference-differential equations, Banach fixed point theorem, Gronwall-Bellman integral inequality, existence and uniqueness, continuous dependence.

observe that the integral equations which are equivalent to the equation (1.1) with (1.2) are given by

$$x(t) = x_0 + \int_0^t f(s, x(s), \phi(s))\, ds, \tag{1.4}$$

for $0 \leq t < 1$ and

$$x(t) = x_0 + \int_0^1 f(s, x(s), \phi(s))\, ds + \int_1^t f(s, x(s), x(s-1))\, ds, \tag{1.5}$$

for $0 \leq t < \infty$.

In a paper published in 1960, S. Sugiyama [7] (see also [8]) studied the existence and uniqueness of solutions of equation (1.1) under the initial conditions (1.2) by using Tychonov's fixed point theorem, the method of successive approximations and the comparison method. The main objective of the present paper is to study the existence, uniqueness and continuous dependence of solutions of equation (1.1) with (1.2) (IVP (1.1)-(1.2) for short). The main tools employed in the analysis are based on the applications of the Banach fixed point theorem (see [4, p. 37]) and the Gronwall-Bellman integral inequality (see [6, p.11]).

2 Existence and uniqueness

Let S be the space of those functions $z(t) \in R^n$ which are continuous for $t \in R_+$ and fulfill the condition

$$|z(t)| = o(\exp(\lambda t)), \tag{2.1}$$

where $\lambda > 0$ is a constant. In the space S we define the norm (see [2, 5])

$$|z|_S = \sup_{t \in R_+} [|z(t)| \exp(-\lambda t)]. \tag{2.2}$$

It is easy to see that S with norm defined in (2.2) is a Banach space. We note that the condition (2.1) implies that there exists a constant $N \geq 0$ such that $|z(t)| \leq N \exp(\lambda t)$ for $t \in R_+$. Using this fact in (2.2) we observe that

$$|z|_S \leq N. \tag{2.3}$$

Now we shall prove the following main result of this section.

Theorem 1. Assume that
(i) the function f in equation (1.1) satisfies the condition

$$|f(t, x, y) - f(t, \bar{x}, \bar{y})| \leq h(t)[|x - \bar{x}| + |y - \bar{y}|], \tag{2.4}$$

where $h \in C(R_+, R_+)$,
(ii) for λ as in (2.1)
(a) there exists a nonnegative constant α such that $\alpha < 1$ and

$$\int_0^t [h(s) + h(s+1)] \exp(\lambda s)\, ds \le \alpha \exp(\lambda t), \tag{2.5}$$

(b) there exists a nonnegative constant β such that

$$|x_0| + \int_0^1 h(s) |\phi(s)|\, ds + \int_0^t |f(s,0,0)|\, ds \le \beta \exp(\lambda t), \tag{2.6}$$

for $t \in R_+$. Then the IVP (1.1)-(1.2) has a unique solution on R_+.

Proof. Let $x(t) \in S$ and define the operator T by (see [7])

$$Tx(t) = x_0 + \int_0^t f(s, x(s), \phi(s))\, ds, \tag{2.7}$$

for $0 \le t < 1$ and

$$Tx(t) = x_0 + \int_0^1 f(s, x(s), \phi(s))\, ds + \int_1^t f(s, x(s), x(s-1))\, ds, \tag{2.8}$$

for $1 \le t < \infty$. First we shall show that Tx maps S into itself. Evidently, Tx is continuous on R_+ and $Tx \in R^n$. We verify that (2.1) is fulfilled. We consider the following two cases.
(i) The case $0 \le t < 1$. From (2.7), using the hypotheses and (2.3) we have

$$\begin{aligned} |Tx(t)| &\le |x_0| + \int_0^t |f(s, x(s), \phi(s)) - f(s,0,0)|\, ds + \int_0^t |f(s,0,0)|\, ds \\ &\le |x_0| + \int_0^t h(s) [|x(s)| + |\phi(s)|]\, ds + \int_0^t |f(s,0,0)|\, ds \\ &\le |x_0| + \int_0^1 h(s) |\phi(s)|\, ds + \int_0^t |f(s,0,0)|\, ds + |x|_S \int_0^t h(s) \exp(\lambda s)\, ds \\ &\le [\beta + N\alpha] \exp(\lambda t). \end{aligned} \tag{2.9}$$

(ii) The case $1 \le t < \infty$. From (2.8), using the hypotheses and (2.3) we have

$$|Tx(t)| \le |x_0| + \int_0^1 |f(s, x(s), \phi(s)) - f(s,0,0)|\, ds + \int_0^1 |f(s,0,0)|\, ds$$

$$+ \int_1^t |f(s, x(s), x(s-1)) - f(s,0,0)|\, ds + \int_1^t |f(s,0,0)|\, ds$$

$$\le |x_0| + \int_0^1 h(s)\,[|x(s)| + |\phi(s)|]\, ds + \int_0^t |f(s,0,0)| ds$$

$$+ \int_1^t h(s)\,[|x(s)| + |x(s-1)|]\, ds$$

$$= |x_0| + \int_0^1 h(s)\,|\phi(s)|\, ds + \int_0^t |f(s,0,0)|\, ds$$

$$+ \int_0^1 h(s)\,|x(s)|\, ds + \int_1^t h(s)\,|x(s)|\, ds + \int_1^t h(s)\,|x(s-1)|\, ds$$

$$\le \beta \exp(\lambda t) + \int_0^t h(s)\,|x(s)|\, ds + I_1, \tag{2.10}$$

where

$$I_1 = \int_1^t h(s)\,|x(s-1)|\, ds. \tag{2.11}$$

By making the change of variable in the integral in (2.11) we obtain

$$I_1 = \int_0^{t-1} h(\sigma+1)\,|x(\sigma)|\, d\sigma \le \int_0^t h(\sigma+1)\,|x(\sigma)|\, d\sigma. \tag{2.12}$$

Using (2.12) in (2.10) we get

$$|Tx(t)| \le \beta \exp(\lambda t) + |x|_S \int_0^t [h(s) + h(s+1)] \exp(\lambda s)\, ds$$

$$\le [\beta + N\alpha] \exp(\lambda t). \tag{2.13}$$

From (2.9) and (2.13) it follows that $Tx \in S$. This proves that T maps S into itself.

Next, we verify that the operator T is a contraction map. Let $x(t), y(t) \in S$. We consider the following two cases.
(i) The case $0 \leq t < 1$. From (2.7) and using the hypotheses we have

$$|Tx(t) - Ty(t)| \leq \int_0^t |f(s, x(s), \phi(s)) - f(s, y(s), \phi(s))| ds$$

$$\leq \int_0^t h(s) |x(s) - y(s)| ds$$

$$\leq |x - y|_S \int_0^t h(s) \exp(\lambda s) ds$$

$$\leq |x - y|_S \alpha \exp(\lambda t). \tag{2.14}$$

(ii) The case $1 \leq t < \infty$. From (2.8) and using the hypotheses we have

$$|Tx(t) - Ty(t)| \leq \int_0^1 |f(s, x(s), \phi(s)) - f(s, y(s), \phi(s))| ds$$

$$+ \int_1^t |f(s, x(s), x(s-1)) - f(s, y(s), y(s-1))| ds$$

$$\leq \int_0^1 h(s) |x(s) - y(s)| ds + \int_1^t h(s) [|x(s) - y(s)| + |x(s-1) - y(s-1)|] ds$$

$$= \int_0^t h(s) |x(s) - y(s)| ds + I_2, \tag{2.15}$$

where

$$I_2 = \int_1^t h(s)|x(s-1) - y(s-1)| ds. \tag{2.16}$$

By making the change of variable in the integral in (2.16) we obtain

$$I_2 \leq \int_0^t h(s+1) |x(s) - y(s)| ds. \tag{2.17}$$

Using (2.17) in (2.15) we get

$$|Tx(t) - Ty(t)| \le |x-y|_S \int_0^t [h(s) + h(s+1)] \exp(\lambda s)\, ds$$

$$\le |x-y|_S\, \alpha \exp(\lambda t)\,. \tag{2.18}$$

From (2.14) and (2.18) we observe that

$$|Tx - Ty|_S \le \alpha\, |x-y|_S\,.$$

Since $\alpha < 1$, it follows from Banach fixed point theorem (see [4, p. 37]) that T has a unique fixed in S. The fixed point of T is however a solution of IVP(1.1)-(1.2). The proof is complete.

Remark 1. We note that the norm defined in (2.2) was first used by Bielecki [2] for proving global existence and uniqueness of solutions of ordinary differential equations. For the developments related to the Bielecki's method, see [3].

The following theorem deals with the uniqueness of solutions of IVP (1.1)-(1.2) without existence part.

Theorem 2. Assume that the function f in equation (1.1) satisfies the condition (2.4). Then the IVP (1.1)-(1.2) has at most one solution on R_+.

Proof. Let $x_1(t)$ and $x_2(t)$ be two solutions of IVP (1.1)-(1.2) and $u(t) = |x_1(t) - x_2(t)|\,,\ t \in R_+$. We consider the following two cases.
(i) The case $0 \le t < 1$. From the hypotheses we have

$$u(t) \le \int_0^t |f(s, x_1(s), \phi(s)) - f(s, x_2(s), \phi(s))| ds$$

$$\le \int_0^t h(s)\, |x_1(s) - x_2(s)| ds$$

$$= \int_0^t h(s)\, u(s) ds. \tag{2.19}$$

Now a suitable application of Gronwall-Bellman inequality (see [6, p. 11]) yields

$$|x_1(t) - x_2(t)| \le 0. \tag{2.20}$$

(ii) The case $1 \le t < \infty$. From the hypotheses we have

$$u(t) \le \int_0^1 |f(s, x_1(s), \phi(s)) - f(s, x_2(s), \phi(s))| ds$$

$$+ \int_1^t |f(s, x_1(s), x_1(s-1)) - f(s, x_2(s), x_2(s-1))| ds$$

$$\le \int_0^1 h(s) |x_1(s) - x_2(s)| ds$$

$$+ \int_1^t h(s) [|x_1(s) - x_2(s)| + |x_1(s-1) - x_2(s-1)|] ds$$

$$= \int_0^t h(s) |x_1(s) - x_2(s)| ds + I_3, \tag{2.21}$$

where

$$I_3 = \int_1^t h(s) |x_1(s-1) - x_2(s-1)| ds. \tag{2.22}$$

By making the change of variable in the integral in (2.22) we observe that

$$I_3 \le \int_0^t h(s+1) |x_1(s) - x_2(s)| ds. \tag{2.23}$$

Using (2.23) in (2.21) we obtain

$$u(t) \le \int_0^t [h(s) + h(s+1)] u(s) ds. \tag{2.24}$$

Now a suitable application of Gronwall-Bellman inequality (see [6, p. 37]) to (2.24) yields

$$|x_1(t) - x_2(t)| \le 0. \tag{2.25}$$

From (2.20) and (2.25) we get $x_1(t) = x_2(t)$ for $t \in R_+$. Thus there is at most one solution to the IVP (1.1)-(1.2) on R_+.

3 Continuous dependence

In this section we study the continuous dependence of solutions of equation (1.1) on the given initial data, the function f involved in equation (1.1) and also the continuous dependence of solutions of equations of the form (1.1) on parameters.

First, we shall give the following theorem concerning the continuous dependence of solutions of equation (1.1) on given initial data.

Theorem 3. Assume that the function f in equation (1.1) satisfies the condition (2.4). Let $x_1(t)$ and $x_2(t)$ be the solutions of equation (1.1) with the given initial conditions

$$x_1(t-1) = \phi_1(t) \ \ (0 \le t < 1), \quad x_1(0) = c_1, \tag{3.1}$$

and

$$x_2(t-1) = \phi_2(t) \ \ (0 \le t < 1), \quad x_2(0) = c_2, \tag{3.2}$$

respectively, where c_1, c_2 are constants. Then

$$|x_1(t) - x_2(t)| \le c \exp\left(\int_0^t h(s)\, ds\right), \tag{3.3}$$

for $0 \le t < 1$ and

$$|x_1(t) - x_2(t)| \le c \exp\left(\int_0^t [h(s) + h(s+1)]\, ds\right), \tag{3.4}$$

for $1 \le t < \infty$, where

$$c = |c_1 - c_2| + \int_0^1 h(s)\, |\phi_1(s) - \phi_2(s)| ds. \tag{3.5}$$

Proof. Let $u(t) = |x_1(t) - x_1(t)|$ for $t \in R_+$. We consider the following two cases.
(i) The case $0 \le t < 1$. From the hypotheses it follows that

$$u(t) \le |c_1 - c_2| + \int_0^t h(s)\, |f(s, x_1(s), \phi_1(s)) - f(s, x_2(s), \phi_2(s))| ds$$

$$\le |c_1 - c_2| + \int_0^t h(s)\, [|x_1(s) - x_2(s)| + |\phi_1(s) - \phi_2(s)|] ds$$

$$\leq |c_1 - c_2| + \int_0^1 h(s)\,|\phi_1(s) - \phi_2(s)|ds + \int_0^t h(s)\,|x_1(s) - x_2(s)|ds$$

$$= c + \int_0^t h(s)\,u(s)\,ds. \tag{3.6}$$

Now applying Gronwall-Bellman inequality (see [6, p. 11]) to (3.6) yields (3.3). (ii) The case $1 \leq t < \infty$. By following the similar arguments as in the proof of Theorem 2 in case (ii), from the hypotheses it follows that

$$u(t) \leq |c_1 - c_2| + \int_0^1 |f(s, x_1(s), \phi_1(s)) - f(s, x_2(s), \phi_2(s))|ds$$

$$+ \int_1^t |f(s, x_1(s), x_1(s-1)) - f(s, x_2(s), x_2(s-1))|ds$$

$$\leq |c_1 - c_2| + \int_0^1 h(s)\,[|x_1(s) - x_2(s)| + |\phi_1(s) - \phi_2(s)|]ds$$

$$+ \int_1^t h(s)\,[|x_1(s) - x_2(s)| + |x_1(s-1) - x_2(s-1)|]ds$$

$$= |c_1 - c_2| + \int_0^1 h(s)\,|\phi_1(s) - \phi_2(s)|ds + \int_0^t h(s)\,|x_1(s) - x_2(s)|ds$$

$$+ \int_1^t h(s)\,|x_1(s-1) - x_2(s-1)|ds$$

$$\leq c + \int_0^t [h(s) + h(s+1)]u(s)\,ds. \tag{3.7}$$

Now applying Gronwall-Bellman inequality (see [6, p. 11]) to (3.7) yields (3.4). From (3.3) and (3.4) it follows that the solutions of equation (1.1) depends on the given initial data.

Now, we consider the IVP (1.1)-(1.2) and the corresponding IVP

$$y'(t) = F(t, y(t), y(t-1)), \tag{3.8}$$

for $t \in R_+$ under the initial conditions

$$y(t-1) = \psi(t), \;\; y(0) = y_0, \tag{3.9}$$

where $F \in C(R_+ \times R^n \times R^n, R^n)$ and $\psi(t)$ is a continuous function which has the $\lim_{t \to 1-0} \psi(t)$.

The following theorem shows the continuous dependence of solutions of IVP (1.1)-(1.2) on the functions involved therein.

Theorem 4. Assume that the function f in equation (1.1) satifies the condition (2.4) and

$$|x_0 - y_0| + \int_0^1 h(s)\,|\phi(s) - \psi(s)|\,ds$$

$$+ \int_0^1 |f(s, y(s), \phi(s)) - F(s, y(s), \psi(s))|ds \le \varepsilon_1, \tag{3.10}$$

$$\int_1^t |f(s, y(s), y(s-1)) - F(s, y(s), y(s-1))|ds \le \varepsilon_2, \tag{3.11}$$

where x_0, ϕ, f and y_0, ψ, F are as in IVP (1.1)-(1.2) and IVP (3.8)-(3.9), $\varepsilon_1, \varepsilon_2$ are nonnegative constants and $y(t)$ is a solution of IVP (3.8)-(3.9). Then the solution $x(t)$ of IVP (1.1)-(1.2) depends continuously on the functions involved therein.

Proof. Let $u(t) = |x(t) - y(t)|$ for $t \in R_+$. We consider the following two cases.
(i) The case $0 \le t < 1$. From the hypotheses we have

$$u(t) \le |x_0 - y_0| + \int_0^t |f(s, y(s), \phi(s)) - F(s, y(s), \psi(s))|ds$$

$$\le |x_0 - y_0| + \int_0^t |f(s, y(s), \phi(s)) - f(s, y(s), \psi(s))|ds$$

$$+ \int_0^t |f(s, y(s), \psi(s)) - F(s, y(s), \psi(s))|ds$$

$$\le |x_0 - y_0| + \int_0^t h(s)\,[|x(s) - y(s)| + |\phi(s) - \psi(s)|]\,ds$$

$$+ \int_0^1 |f(s, y(s), \psi(s)) - F(s, y(s), \psi(s))|ds$$

$$\leq |x_0 - y_0| + \int_0^t h(s) |x(s) - y(s)| \, ds + \int_0^1 h(s) |\phi(s) - \psi(s)| \, ds$$

$$+ \int_0^1 |f(s, y(s), \psi(s)) - F(s, y(s), \psi(s))| ds$$

$$\leq \varepsilon_1 + \int_0^t h(s) \, u(s) \, ds. \tag{3.12}$$

Now an application of Gronwall-Bellman inequality (see [6, p. 11]) to (3.12) yields

$$|x(t) - y(t)| \leq \varepsilon_1 \exp\left(\int_0^t h(s) \, ds\right). \tag{3.13}$$

(ii) The case $1 \leq t < \infty$. Following the arguments as in the proof of Theorem 2 in case (ii), from the hypotheses we have

$$u(t) \leq |x_0 - y_0| + \int_0^1 |f(s, x(s), \phi(s)) - F(s, y(s), \psi(s))| \, ds$$

$$+ \int_1^t |f(s, x(s), x(s-1)) - F(s, y(s), y(s-1))| \, ds$$

$$\leq |x_0 - y_0| + \int_0^1 |f(s, x(s), \phi(s)) - f(s, y(s), \psi(s))| \, ds$$

$$+ \int_0^1 |f(s, y(s), \psi(s)) - F(s, y(s), \psi(s))| \, ds$$

$$+ \int_1^t |f(s, x(s), x(s-1)) - f(s, y(s), y(s-1))| \, ds$$

$$+ \int_1^t |f(s, y(s), y(s-1)) - F(s, y(s), y(s-1))| \, ds$$

$$\leq |x_0 - y_0| + \int_0^1 h(s) \left[|x(s) - y(s)| + |\phi(s) - \psi(s)|\right] ds$$

$$+ \int_0^1 |f(s, y(s), \psi(s)) - F(s, y(s), \psi(s))| \, ds$$

$$+\int_1^t h(s)\left[|x(s)-y(s)|+|x(s-1)-y(s-1)|\right]ds$$

$$+\int_1^t |f(s,y(s),y(s-1))-F(s,y(s),y(s-1))|\,ds$$

$$\leq |x_0-y_0|+\int_0^1 h(s)\,|\phi(s)-\psi(s)|\,ds$$

$$+\int_0^1 |f(s,y(s),\psi(s))-F(s,y(s),\psi(s))|\,ds$$

$$+\int_0^t h(s)\,|x(s)-y(s)|\,ds+\int_0^t h(s+1)\,|x(s)-y(s)|\,ds$$

$$+\int_1^t |f(s,y(s),y(s-1))-F(s,y(s),y(s-1))|\,ds$$

$$\leq (\varepsilon_1+\varepsilon_2)+\int_0^t [h(s)+h(s+1)]\,u(s)\,ds. \tag{3.14}$$

Now an application of Gronwall-Bellman inequality (see [6, p. 11]) to (3.14) yields

$$|x(t)-y(t)|\leq (\varepsilon_1+\varepsilon_2)\exp\left(\int_0^t [h(s)+h(s+1)]\,ds\right). \tag{3.15}$$

From (3.13) and (3.15) it follows that the IVP (1.1)-(1.2) depends continuously on the functions involved therein.

Next, we consider the following difference-differential equations

$$x'(t)=g(t,x(t),x(t-1),\mu), \tag{3.16}$$

and

$$x'(t)=g(t,x(t),x(t-1),\mu_0), \tag{3.17}$$

for $t\in R_+$ with the given initial conditions in (1.2), where $g\in C(R_+\times R^n\times R^n\times R,R^n)$.

Finally, we present the following theorem which deals with the dependency of solutions of IVP (3.16)-(1.2) and IVP (3.17)-(1.2) on parameters.

Theorem 5. Assume that the function g satisfy the conditions

$$|g(t,x,y,\mu) - g(t,\bar{x},\bar{y},\mu)| \le p(t)\left[|x-\bar{x}| + |y-\bar{y}|\right], \tag{3.18}$$

$$|g(t,x,y,\mu) - g(t,x,y,\mu_0)| \le q(t)\,|\mu-\mu_0|, \tag{3.19}$$

where $p, q \in C(R_+, R_+)$. Let $x_1(t)$ and $x_2(t)$ be the solutions of IVP (3.16)-(1.2) and IVP (3.17)-(1.2) respectively. Then

$$|x_1(t) - x_2(t)| \le \left(|\mu-\mu_0| \int_0^1 q(s)\right) \exp\left(\int_0^t p(s)\,ds\right), \tag{3.20}$$

for $0 \le t < 1$ and

$$|x_1(t) - x_2(t)| \le \left(|\mu-\mu_0| \int_0^t q(s)\right) \exp\left(\int_0^t [p(s) + p(s+1)]\,ds\right), \tag{3.21}$$

for $1 \le t < \infty$.

Proof. Let $u(t) = |x_1(t) - x_2(t)|$ for $t \in R_+$. We consider the following two cases.
(i) The case $0 \le t < 1$. From the hypotheses we have

$$\begin{aligned}
u(t) &\le \int_0^t |g(s,x_1(s),\phi(s),\mu) - g(s,x_2(s),\phi(s),\mu_0)|ds \\
&\le \int_0^t |g(s,x_1(s),\phi(s),\mu) - g(s,x_2(s),\phi(s),\mu)|ds \\
&+ \int_0^t |g(s,x_2(s),\phi(s),\mu) - g(s,x_2(s),\phi(s),\mu_0)|ds \\
&\le \int_0^t p(s)\,|x_1(s) - x_2(s)|ds + \int_0^t q(s)\,|\mu-\mu_0|ds \\
&\le |\mu-\mu_0| \int_0^1 q(s)\,ds + \int_0^t p(s)\,u(s)\,ds.
\end{aligned} \tag{3.22}$$

Now an application of the Gronwall-Bellman inequality (see [6, p. 11]) to (3.22) yields (3.20).

(ii) The case $1 \leq t < \infty$. By following the arguments as in the proof of Theorem 2 in case (ii), from the hypotheses we have

$$u(t) \leq \int_0^1 |g(s, x_1(s), \phi(s), \mu) - g(s, x_2(s), \phi(s), \mu_0)| ds$$

$$+ \int_1^t |g(s, x_1(s), x_1(s-1), \mu) - g(s, x_2(s), x_2(s-1), \mu_0)| ds$$

$$\leq \int_0^1 |g(s, x_1(s), \phi(s), \mu) - g(s, x_2(s), \phi(s), \mu)| ds$$

$$+ \int_0^1 |g(s, x_2(s), \phi(s), \mu) - g(s, x_2(s), \phi(s), \mu_0)| ds$$

$$+ \int_1^t |g(s, x_1(s), x_1(s-1), \mu) - g(s, x_2(s), x_2(s-1), \mu)| ds$$

$$+ \int_1^t |g(s, x_2(s), x_2(s-1), \mu) - g(s, x_2(s), x_2(s-1), \mu_0)| ds$$

$$\leq \int_0^1 p(s) |x_1(s) - x_2(s)| ds + \int_0^1 q(s) |\mu - \mu_0| ds$$

$$+ \int_1^t p(s) [|x_1(s) - x_2(s)| + |x_1(s-1) - x_2(s-1)|] ds + \int_1^t q(s) |\mu - \mu_0| ds$$

$$= \int_0^t q(s) |\mu - \mu_0| \, ds + \int_0^t p(s) |x_1(s) - x_2(s)| \, ds$$

$$+ \int_1^t p(s) |x_1(s-1) - x_2(s-1)| ds$$

$$\leq |\mu - \mu_0| \int_0^t q(s) \, ds + \int_0^t [p(s) + p(s+1)] \, u(s) \, ds. \tag{3.23}$$

Now an application of a suitable version of the Gronwall-Bellman inequality (see [6, p.12]) to (3.23) yields (3.21). From (3.20) and (3.21) it follows that the solutions of IVP (3.16)-(1.2) and IVP (3.17)-(1.2) depends on parameters.

Remark 2. We note that in the literature there are many papers dealing with existence, uniqueness and other properties of solutions of equations of the form (1.1), see [1] and the references cited therein. We believe that the results given above are of independent interest and will be useful for future reference..

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