SOLUTION OF AN OPEN PROBLEM PROPOSED BY FENG QI

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Abstract. By using methods on the theory of majorization, the inequalities between the sum of power and the exponential of sum of a nonnegative sequence is established, and so an open problem proposed by Feng Qi is solved.

1. Introduction

Throughout the paper we assume that the set of n-dimensional row vector on real number field by $\mathbb{R}^n$, and

$$\mathbb{R}^n_+ = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_i \geq 0, i = 1, \ldots, n\}.$$

In particular, $\mathbb{R}^1$ and $\mathbb{R}^1_+$ denoted by $\mathbb{R}$ and $\mathbb{R}_+$ respectively.

In [1], the following inequality between the sum of squares and the exponential of sum of a nonnegative sequence is obtained: For $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n_+$ and $n \geq 2$, inequality

$$\frac{e^2}{4}\sum_{i=1}^{n} x_i^2 \leq \exp\left(\sum_{i=1}^{n} x_i\right) \tag{1}$$

is valid. The equality in (1) holds if $x_i = 2$ for some given $1 \leq i \leq n$ and $x_j = 0$ for all $1 \leq j \leq n$ with $j \neq i$. The constant $\frac{e^2}{4}$ in (1) is the best possible.

At the end of the paper [1], an open problem was posed: For $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n_+$ and $n \geq 2$, determine the best possible constants $\alpha, \lambda \in \mathbb{R}$ and $0 < \beta, \mu < \infty$ such that

$$\beta\sum_{i=1}^{n} x_i^\alpha \leq \exp\left(\sum_{i=1}^{n} x_i\right) \leq \mu\sum_{i=1}^{n} x_i^\lambda. \tag{2}$$

Firstly, we show that the right inequality in (2) is generally untenable. In fact, when $n = 2$, the right inequality in (2) is deduce to

$$e^{x_1+x_2} \leq \mu\left(x_1^\lambda + x_2^\lambda\right). \tag{3}$$

Taking $x_2 = 0$, (3) is deduce to

$$\frac{e^{x_1}}{x_1^\lambda} \leq \mu. \tag{4}$$

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For fixed $\lambda > 0$, $\frac{e^{x_1}}{x_1^{\lambda}} \to \infty$ as $x_1 \to \infty$, hence (4) is does not holds if x_1 is large enough. This is, there not exist μ such that (4) is uniformly holds for any $(x_1, x_2) \in \mathbb{R}_+^2$.

For the lift inequality in (2), we shall establish following results.

Theorem 1. *Let $(x_1, x_2, \ldots, x_n) \in \mathbb{R}_+^n$, $n \geq 2$. If $\alpha \geq 1$, then inequality*

$$\frac{e^{\alpha}}{\alpha^{\alpha}} \left(\sum_{i=1}^{n} x_i^{\alpha} \right) \leq \exp\left(\sum_{i=1}^{n} x_i \right) \tag{5}$$

is valid. The equality in (5) holds if and only if $x_i = \alpha$ for some given $1 \leq i \leq n$ and $x_j = 0$ for all $1 \leq j \leq n$ with $j \neq i$.

Theorem 2. *Let $\{x_i\}_{i=1}^{\infty}$ be a nonnegative sequence such that $\sum_{i=1}^{\infty} x_i < \infty$. For given $\alpha \geq 1$, inequality*

$$\frac{e^{\alpha}}{\alpha^{\alpha}} \sum_{i=1}^{\infty} x_i^{\alpha} \leq \exp\left(\sum_{i=1}^{\infty} x_i \right) \tag{6}$$

is valid.

Remark 1. *Taking $\alpha = 2$, the left inequality of (5) deduce to (1). And taking $x_i = \frac{1}{i}$ for $i \in \mathbb{N}$ in (6) and rearranging gives*

$$\alpha - \alpha \ln \alpha + \ln \zeta(\alpha) \leq \zeta(1) \tag{7}$$

where $\zeta(\alpha) = \sum_{i=1}^{\infty} \frac{1}{i^{\alpha}}$ is Riemann Zata function.

We need the following definitions and lemmas.

Definition 1. [2, 3] *Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$.*

(1) *x is said to be majorized by y (in symbols $x \prec y$) if $\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}$ for $k = 1, 2, \ldots, n-1$ and $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$, where $x_{[1]} \geq \cdots \geq x_{[n]}$ and $y_{[1]} \geq \cdots \geq y_{[n]}$ are rearrangements of x and y in a descending order, and x is said to strictly majorized by y (in symbols $x \prec\prec y$) if x is not permutation of y.*

(2) *let $\Omega \subset \mathbb{R}^n$, $f : \Omega \to \mathbb{R}$ is said to be a strictly Schur-convex function on Ω if $x \prec\prec y$ on Ω implies $f(x) < f(y)$. f is said to be a strictly Schur-concave function on Ω if and only if $-f$ is strictly Schur-convex function on Ω.*

Definition 2. [2] *Let set $\Omega \subseteq \mathbb{R}^n$. Ω is said to be a convex set if $x, y \in \Omega, 0 \leq \alpha \leq 1$ implies $\alpha x + (1-\alpha)y = (\alpha x_1 + (1-\alpha)y_1, \ldots, \alpha x_n + (1-\alpha)y_n) \in \Omega$.*

Lemma 1 ([2, p. 5])**.** *Let $\Omega \subset \mathbb{R}^n$ is symmetric and has a nonempty interior convex set. Ω^0 is the interior of Ω. $f : \Omega \to \mathbb{R}$ is continuous on Ω and differentiable in Ω^0. Then f is the strictly Schur-convex $(Schur-concave)$ function, if and only if f is symmetric on Ω and*

$$(x_1 - x_2)\left(\frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial x_2} \right) > 0 (< 0) \tag{8}$$

holds for any $x \in \Omega^0$ and $x_1 \neq x_2$.

Lemma 2 ([2, p. 5])**.** *For any given positive real number s and α, we have*

$$\frac{e^{\alpha}}{\alpha^{\alpha}} \leq \frac{e^{s}}{s^{\alpha}}, \tag{9}$$

and the equality in (9) holds if and only if $s = \alpha$.

Proof. Let $\varphi(s) = \alpha \ln s - s$. Then $\varphi^{'}(s) = \frac{\alpha}{s} - 1 \le 0$ for $s \ge \alpha > 0$, this means that $\varphi(s)$ is increasing, and $\varphi^{'}(s) \ge 0$ for $0 < s \le \alpha$, this means that $\varphi(s)$ is decreasing. Hence, for any $s > 0$, we have

$$\varphi(s) = \alpha \ln s - s \le \varphi(\alpha) = \alpha \ln \alpha - \alpha,$$

i.e. (9) is holds, and the equality in (9) holds if and only if $s = \alpha$. □

2. Proofs of theorems

Now we are in a position to prove our theorems.

Proof of Theorem 1. Let

$$f(x) = f(x_1, \dots, x_n) = \ln\left(\sum_{i=1}^{n} x_i^{\alpha}\right) - s. \tag{10}$$

where $s = \sum_{i=1}^{n} x_i$. Simple calculation gives

$$\Delta := (x_1 - x_2)\left(\frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial x_2}\right) = \frac{\alpha(x_1 - x_2)(x_1^{\alpha-1} - x_2^{\alpha-1})}{\sum_{i=1}^{n} x_i^{\alpha}}.$$

When $\alpha > 1$, since $x^{\alpha-1}$ is strictly increasing on $\mathbb{R}$, $(x_1 - x_2)(x_1^{\alpha-1} - x_2^{\alpha-1}) > 0$ for $x_1 \ne x_2$, and then $\Delta > 0$. From Lemma 1, it follows that $f(x)$ is a strictly Schur-convex function on $\mathbb{R}_+^n$. It is easy to see that

$$x = (x_1, \cdots, x_n) \prec \left(s, \underbrace{0, \dots, 0}_{n-1}\right) = y, \tag{11}$$

and $x \prec\prec y$ unless $x_i = s$ for some given $1 \le i \le n$ and $x_j = 0$ for all $1 \le j \le n$ with $j \ne i$. Hence,

$$f(x_1, \dots, x_n) = \ln\left(\sum_{i=1}^{n} x_i^{\alpha}\right) - s \le f\left(s, \underbrace{0, \dots, 0}_{n-1}\right) = \alpha \ln s - s. \tag{12}$$

i.e.

$$\frac{e^s}{s^{\alpha}}\left(\sum_{i=1}^{n} x_i^{\alpha}\right) \le \exp\left(\sum_{i=1}^{n} x_i\right), \tag{13}$$

and equality in (13) holds if and only if $x_i = s$ for some given $1 \le i \le n$ and $x_j = 0$ for all $1 \le j \le n$ with $j \ne i$. Combining (13) with (9), we get (5), and equality in (5) holds if and only if $x_i = \alpha$ for some given $1 \le i \le n$ and $x_j = 0$ for all $1 \le j \le n$ with $j \ne i$.

The proof of Theorem 1 is complete. □

Proof of Theorem 2. Letting $n \to \infty$ in Theorem 1 yields Theorem 2 straightforwardly. □

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