SOME INEQUALITIES FOR THE ČEBYŠEV FUNCTIONAL OF TWO FUNCTIONS OF SELFADJOINT OPERATORS IN HILBERT SPACES

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Abstract. Some inequalities for the Čebyšev functional of two functions of selfadjoint linear operators in Hilbert spaces, under suitable assumptions for the involved functions and operators, are given.

1. Introduction

Let A be a selfadjoint linear operator on a complex Hilbert space $(H;\langle .,.\rangle)$. The *Gelfand map* establishes a $*$-isometrically isomorphism Φ between the set $C(Sp(A))$ of all *continuous functions* defined on the *spectrum* of A, denoted $Sp(A)$, an the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows (see for instance [5, p. 3]):

For any $f,g \in C(Sp(A))$ and any $\alpha,\beta \in \mathbb{C}$ we have

(i) $\Phi(\alpha f+\beta g)=\alpha\Phi(f)+\beta\Phi(g)$;

(ii) $\Phi(fg)=\Phi(f)\Phi(g)$ and $\Phi(\bar{f})=\Phi(f)^*$;

(iii) $\|\Phi(f)\|=\|f\|:=\sup_{t\in Sp(A)}|f(t)|$;

(iv) $\Phi(f_0)=1_H$ and $\Phi(f_1)=A$, where $f_0(t)=1$ and $f_1(t)=t$, for $t\in Sp(A)$.

With this notation we define

$$f(A):=\Phi(f) \text{ for all } f\in C(Sp(A))$$

and we call it the *continuous functional calculus* for a selfadjoint operator A.

If A is a selfadjoint operator and f is a real valued continuous function on $Sp(A)$, then $f(t)\geq 0$ for any $t\in Sp(A)$ implies that $f(A)\geq 0$, *i.e.* $f(A)$ is a *positive operator* on H. Moreover, if both f and g are real valued functions on $Sp(A)$ then the following important property holds:

$$f(t)\geq g(t) \text{ for any } t\in Sp(A) \text{ implies that } f(A)\geq g(A) \tag{P}$$

in the operator order of $B(H)$.

For a recent monograph devoted to various inequalities for functions of selfadjoint operators, see [5] and the references therein.

For other results see [7], [8], [9] and [10].

We say that the functions $f,g:[a,b]\longrightarrow\mathbb{R}$ are *synchronous (asynchronous)* on the interval $[a,b]$ if they satisfy the following condition:

$$(f(t)-f(s))(g(t)-g(s))\geq(\leq)\,0 \text{ for each } t,s\in[a,b].$$

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It is obvious that, if f, g are monotonic and have the same monotonicity on the interval $[a,b]$, then they are synchronous on $[a,b]$ while if they have opposite monotonicity, they are asynchronous.

For some extensions of the discrete *Čebyšev inequality* for *synchronous (asynchronous)* sequences of vectors in an inner product space, see [3] and [4].

For a selfadjoint operator A on the Hilbert space A with $Sp(A) \subseteq [m,M]$ for some real numbers $m < M$ and for $f, g : [m,M] \longrightarrow \mathbb{R}$ that are continuous functions *on* $[m,M]$, we can define the following *Čebyšev functional*

$$C(f,g;A;x) := \langle f(A)g(A)x,x\rangle - \langle f(A)x,x\rangle \cdot \langle g(A)x,x\rangle$$

where $x \in H$ with $\|x\| = 1$.

The following result provides an inequality of Čebyšev type for functions of selfadjoint operators, see [1]:

Theorem 1 (Dragomir, 2008, [1])**.** *Let A be a selfadjoint operator with $Sp(A) \subseteq [m,M]$ for some real numbers $m < M$. If $f, g : [m,M] \longrightarrow \mathbb{R}$ are continuous and synchronous (asynchronous) on $[m,M]$, then*

$$C(f,g;A;x) \geq (\leq)\, 0 \tag{1.1}$$

for any $x \in H$ with $\|x\| = 1$.

The following result of Grüss' Type can be stated as well, see [2]:

Theorem 2 (Dragomir, 2008, [2])**.** *Let A be a selfadjoint operator on the Hilbert space $(H;\langle .,.\rangle)$ and assume that $Sp(A) \subseteq [m,M]$ for some scalars $m < M$. If f and g are continuous on $[m,M]$ and $\gamma := \min_{t\in[m,M]} f(t)$ and $\Gamma := \max_{t\in[m,M]} f(t)$ then*

$$|C(f,g;A;x)| \leq \frac{1}{2}\cdot(\Gamma-\gamma)\left[C(g,g;A;x)\right]^{1/2}\left(\leq \frac{1}{4}(\Gamma-\gamma)(\Delta-\delta)\right) \tag{1.2}$$

for each $x \in H$ with $\|x\| = 1$, where $\delta := \min_{t\in[m,M]} g(t)$ and $\Delta := \max_{t\in[m,M]} g(t)$.

The main aim of this paper is to provide other inequalities for the Čebyšev functional. Applications for particular functions of interest are also given.

2. The Case of Lipschitzian Functions

The following result can be stated:

Theorem 3. *Let A be a selfadjoint operator with $Sp(A) \subseteq [m,M]$ for some real numbers $m < M$. If $f : [m,M] \longrightarrow \mathbb{R}$ is Lipschitzian with the constant $L > 0$ and $g : [m,M] \longrightarrow \mathbb{R}$ is continuous with $\delta := \min_{t\in[m,M]} g(t)$ and $\Delta := \max_{t\in[m,M]} g(t)$, then*

$$|C(f,g;A;x)| \leq \frac{1}{2}(\Delta-\delta)L\langle \ell_{A,x}(A)x,x\rangle \leq \frac{\sqrt{2}}{2}(\Delta-\delta)LC(e,e;A;x) \tag{2.1}$$

for any $x \in H$ with $\|x\| = 1$, where

$$\ell_{A,x}(t) := \langle |t\cdot 1_H - A|\,x,x\rangle$$

is a continuous function on $[m,M]$, $e(t) = t$ and

$$C(e,e;A;x) = \|Ax\|^2 - \langle Ax,x\rangle^2\,(\geq 0). \tag{2.2}$$

Proof. First of all, by the Jensen inequality for convex functions of selfadjoint operators (see for instance [5, p. 5]) applied for the modulus, we can state that

$$|\langle h(A)x,x\rangle| \le \langle |h(A)|x,x\rangle \tag{M}$$

for any $x \in H$ with $\|x\| = 1$, where h is a continuous function on $[m, M]$.

Since f is Lipschitzian with the constant $L > 0$, then for any $t, s \in [m, M]$ we have

$$|f(t) - f(s)| \le L|t - s|. \tag{2.3}$$

Now, if we fix $t \in [m, M]$ and apply the property (P) for the inequality (2.3) and the operator A we get

$$\langle |f(t)\cdot 1_H - f(A)|x,x\rangle \le L\langle |t\cdot 1_H - A|x,x\rangle, \tag{2.4}$$

for any $x \in H$ with $\|x\| = 1$.

Utilising the property (M) we get

$$|f(t) - \langle f(A)x,x\rangle| = |\langle f(t)\cdot 1_H - f(A)x,x\rangle| \le \langle |f(t)\cdot 1_H - f(A)|x,x\rangle$$

which together with (2.4) gives

$$|f(t) - \langle f(A)x,x\rangle| \le L\ell_{A,x}(t) \tag{2.5}$$

for any $t \in [m, M]$ and for any $x \in H$ with $\|x\| = 1$.

Since $\delta := \min_{t\in[m,M]} g(t)$ and $\Delta := \max_{t\in[m,M]} g(t)$, we also have

$$\left|g(t) - \frac{\Delta+\delta}{2}\right| \le \frac{1}{2}(\Delta - \delta) \tag{2.6}$$

for any $t \in [m, M]$ and for any $x \in H$ with $\|x\| = 1$.

If we multiply the inequality (2.5) with (2.6) we get

$$\begin{aligned}
&\left|f(t)g(t) - \langle f(A)x,x\rangle g(t) - \frac{\Delta+\delta}{2}f(t) + \frac{\Delta+\delta}{2}\langle f(A)x,x\rangle\right| \\
&\le \frac{1}{2}(\Delta-\delta)L\ell_{A,x}(t) = \frac{1}{2}(\Delta-\delta)L\langle |t\cdot 1_H - A|x,x\rangle \\
&\le \frac{1}{2}(\Delta-\delta)L\left\langle |t\cdot 1_H - A|^2 x,x\right\rangle^{1/2} \\
&= \frac{1}{2}(\Delta-\delta)L\left(\left\langle A^2x,x\right\rangle - 2\langle Ax,x\rangle t + t^2\right)^{1/2},
\end{aligned} \tag{2.7}$$

for any $t \in [m, M]$ and for any $x \in H$ with $\|x\| = 1$.

Now, if we apply the property (P) for the inequality (2.7) and a selfadjoint operator B with $Sp(B) \subset [m, M]$, then we get the following inequality of interest in itself:

$$\begin{aligned}
&|\langle f(B)g(B)y,y\rangle - \langle f(A)x,x\rangle\langle g(B)y,y\rangle \\
&-\frac{\Delta+\delta}{2}\langle f(B)y,y\rangle + \frac{\Delta+\delta}{2}\langle f(A)x,x\rangle\Big| \\
&\le \frac{1}{2}(\Delta-\delta)L\langle \ell_{A,x}(B)y,y\rangle \\
&\le \frac{1}{2}(\Delta-\delta)L\left\langle \left(\left\langle A^2x,x\right\rangle 1_H - 2\langle Ax,x\rangle B + B^2\right)^{1/2}y,y\right\rangle \\
&\le \frac{1}{2}(\Delta-\delta)L\left(\left\langle A^2x,x\right\rangle - 2\langle Ax,x\rangle\langle By,y\rangle + \left\langle B^2y,y\right\rangle\right)^{1/2},
\end{aligned} \tag{2.8}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

Finally, if we choose in (2.8) $y = x$ and $B = A$, then we deduce the desired result (2.1). □

In the case of two Lipschitzian functions, the following result may be stated as well:

Theorem 4. *Let A be a selfadjoint operator with $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. If $f, g : [m, M] \longrightarrow \mathbb{R}$ are Lipschitzian with the constants $L, K > 0$, then*

$$|C(f, g; A; x)| \leq LKC(e, e; A; x), \tag{2.9}$$

for any $x \in H$ with $\|x\| = 1$.

Proof. Since $f, g : [m, M] \longrightarrow \mathbb{R}$ are Lipschitzian, then

$$|f(t) - f(s)| \leq L|t - s| \text{ and } |g(t) - g(s)| \leq K|t - s|$$

for any $t, s \in [m, M]$, which gives the inequality

$$|f(t) g(t) - f(t) g(s) - f(s) g(t) + f(s) g(s)| \leq KL\left(t^2 - 2ts + s^2\right)$$

for any $t, s \in [m, M]$.

Now, fix $t \in [m, M]$ and if we apply the properties (P) and (M) for the operator A we get successively

$$\begin{aligned} &|f(t) g(t) - \langle g(A) x, x\rangle f(t) - \langle f(A) x, x\rangle g(t) + \langle f(A) g(A) x, x\rangle| \\ &= |\langle [f(t) g(t) \cdot 1_H - f(t) g(A) - f(A) g(t) + f(A) g(A)] x, x\rangle| \\ &\leq \langle |f(t) g(t) \cdot 1_H - f(t) g(A) - f(A) g(t) + f(A) g(A)| x, x\rangle \\ &\leq KL\left\langle \left(t^2 \cdot 1_H - 2tA + A^2\right) x, x\right\rangle = KL\left(t^2 - 2t\langle Ax, x\rangle + \left\langle A^2 x, x\right\rangle\right) \end{aligned} \tag{2.10}$$

for any $x \in H$ with $\|x\| = 1$.

Further, fix $x \in H$ with $\|x\| = 1$. On applying the same properties for the inequality (2.10) and another selfadjoint operator B with $Sp(B) \subset [m, M]$, we have

$$\begin{aligned} &|\langle f(B) g(B) y, y\rangle - \langle g(A) x, x\rangle \langle f(B) y, y\rangle \\ &\qquad - \langle f(A) x, x\rangle \langle g(B) y, y\rangle + \langle f(A) g(A) x, x\rangle| \\ &= |\langle [f(B) g(B) - \langle g(A) x, x\rangle f(B) - \langle f(A) x, x\rangle g(B) + \langle f(A) g(A) x, x\rangle 1_H] y, y\rangle| \\ &\leq \langle |f(B) g(B) - \langle g(A) x, x\rangle f(B) - \langle f(A) x, x\rangle g(B) + \langle f(A) g(A) x, x\rangle 1_H| y, y\rangle \\ &\leq KL\left\langle \left(B^2 - 2\langle Ax, x\rangle B + \left\langle A^2 x, x\right\rangle 1_H\right) y, y\right\rangle \\ &= KL\left(\left\langle B^2 y, y\right\rangle - 2\langle Ax, x\rangle \langle By, y\rangle + \left\langle A^2 x, x\right\rangle\right) \end{aligned} \tag{2.11}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$, which is an inequality of interest in its own right.

Finally, on making $B = A$ and $y = x$ in (2.11) we deduce the desired result (2.9). □

3. Some Inequalities for Sequences of Operators

Consider the sequence of selfadjoint operators $\mathbf{A}=(A_1,...,A_n)$ with $Sp(A_j)\subseteq[m,M]$ for $j\in\{1,...,n\}$ and for some scalars $m<M$. If $\mathbf{x}=(x_1,...,x_n)\in H^n$ are such that $\sum_{j=1}^n\|x_j\|^2=1$, then we can consider the following Čebyšev type functional

$$C(f,g;\mathbf{A},\mathbf{x}):=\sum_{j=1}^n\langle f(A_j)g(A_j)x_j,x_j\rangle-\sum_{j=1}^n\langle f(A_j)x_j,x_j\rangle\cdot\sum_{j=1}^n\langle g(A_j)x_j,x_j\rangle.$$

As a particular case of the above functional and for a probability sequence $\mathbf{p}=(p_1,...,p_n)$, i.e., $p_j\geq 0$ for $j\in\{1,...,n\}$ and $\sum_{j=1}^n p_j=1$, we can also consider the functional

$$C(f,g;\mathbf{A},\mathbf{p},x):=\left\langle\sum_{j=1}^n p_jf(A_j)g(A_j)x,x\right\rangle-\left\langle\sum_{j=1}^n p_jf(A_j)x,x\right\rangle\cdot\left\langle\sum_{j=1}^n p_jg(A_j)x,x\right\rangle$$

where $x\in H$, $\|x\|=1$.

We know, from [1] that for the sequence of selfadjoint operators $\mathbf{A}=(A_1,...,A_n)$ with $Sp(A_j)\subseteq[m,M]$ for $j\in\{1,...,n\}$ and for the synchronous (asynchronous) functions $f,g:[m,M]\longrightarrow\mathbb{R}$ we have the inequality

$$C(f,g;\mathbf{A},\mathbf{x})\geq(\leq)\,0 \tag{3.1}$$

for any $\mathbf{x}=(x_1,...,x_n)\in H^n$ with $\sum_{j=1}^n\|x_j\|^2=1$. Also, for any probability distribution $\mathbf{p}=(p_1,...,p_n)$ and any $x\in H$, $\|x\|=1$ we have

$$C(f,g;\mathbf{A},\mathbf{p},x)\geq(\leq)\,0. \tag{3.2}$$

On the other hand, the following Grüss' type inequality is valid as well [2]:

$$|C(f,g;\mathbf{A},\mathbf{x})|\leq\frac{1}{2}\cdot(\Gamma-\gamma)[C(g,g;\mathbf{A},\mathbf{x})]^{1/2}\left(\leq\frac{1}{4}(\Gamma-\gamma)(\Delta-\delta)\right) \tag{3.3}$$

for any $\mathbf{x}=(x_1,...,x_n)\in H^n$ with $\sum_{j=1}^n\|x_j\|^2=1$, where f and g are continuous on $[m,M]$ and $\gamma:=\min_{t\in[m,M]}f(t)$, $\Gamma:=\max_{t\in[m,M]}f(t)$, $\delta:=\min_{t\in[m,M]}g(t)$ and $\Delta:=\max_{t\in[m,M]}g(t)$.

Similarly, for any probability distribution $\mathbf{p}=(p_1,...,p_n)$ and any $x\in H$, $\|x\|=1$ we also have the inequality:

$$|C(f,g;\mathbf{A},\mathbf{p},x)|\leq\frac{1}{2}\cdot(\Gamma-\gamma)[C(g,g;\mathbf{A},\mathbf{p},x)]^{1/2}\left(\leq\frac{1}{4}(\Gamma-\gamma)(\Delta-\delta)\right). \tag{3.4}$$

We can state now the following new result:

Theorem 5. *Let $\mathbf{A}=(A_1,...,A_n)$ be a sequence of selfadjoint operators with $Sp(A_j)\subseteq[m,M]$ for $j\in\{1,...,n\}$ and for some scalars $m<M$. If $f:[m,M]\longrightarrow\mathbb{R}$ is Lipschitzian with the constant $L>0$ and $g:[m,M]\longrightarrow\mathbb{R}$ is continuous with*

$\delta := \min_{t\in[m,M]} g(t)$ and $\Delta := \max_{t\in[m,M]} g(t)$, then

$$|C(f,g;\mathbf{A},\mathbf{x})| \leq \frac{1}{2}(\Delta-\delta) L \sum_{k=1}^{n} \langle \ell_{\mathbf{A},\mathbf{x}}(A_k) x_k, x_k\rangle \leq \frac{\sqrt{2}}{2}(\Delta-\delta) LC(e,e;\mathbf{A};\mathbf{x}) \tag{3.5}$$

for any $\mathbf{x} = (x_1,...,x_n) \in H^n$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$, where

$$\ell_{\mathbf{A},\mathbf{x}}(t) := \sum_{j=1}^{n} \langle |t\cdot 1_H - A_j| x_j, x_j\rangle$$

is a continuous function on $[m,M]$, $e(t) = t$ and

$$C(e,e;\mathbf{A};\mathbf{x}) = \sum_{j=1}^{n} \|Ax_j\|^2 - \left(\sum_{j=1}^{n} \langle A_j x_j, x_j\rangle\right)^2 (\geq 0).$$

Proof. As in [5, p. 6], if we put

$$\widetilde{A} := \begin{pmatrix} A_1 & . & . & . & 0 \\ & . & & & \\ & & . & & \\ & & & . & \\ 0 & . & . & . & A_n \end{pmatrix} \text{ and } \widetilde{x} = \begin{pmatrix} x_1 \\ . \\ . \\ . \\ x_n \end{pmatrix}$$

then we have $Sp\left(\widetilde{A}\right) \subseteq [m,M]$, $\|\widetilde{x}\| = 1$,

$$\left\langle f\left(\widetilde{A}\right) g\left(\widetilde{A}\right)\widetilde{x},\widetilde{x}\right\rangle = \sum_{j=1}^{n} \langle f(A_j) g(A_j) x_j, x_j\rangle,$$

$$\left\langle f\left(\widetilde{A}\right)\widetilde{x},\widetilde{x}\right\rangle = \sum_{j=1}^{n} \langle f(A_j) x_j, x_j\rangle, \left\langle g\left(\widetilde{A}\right)\widetilde{x},\widetilde{x}\right\rangle = \sum_{j=1}^{n} \langle g(A_j) x_j, x_j\rangle,$$

and so on.

Applying Theorem 3 for $\widetilde{A}$ and $\widetilde{x}$ we deduce the desired result (3.5). □

As a particular case we have:

Corollary 1. *Let $\mathbf{A} = (A_1,...,A_n)$ be a sequence of selfadjoint operators with $Sp(A_j) \subseteq [m,M]$ for $j \in \{1,...,n\}$ and for some scalars $m < M$. If $f : [m,M] \longrightarrow \mathbb{R}$ is Lipschitzian with the constant $L > 0$ and $g : [m,M] \longrightarrow \mathbb{R}$ is continuous with $\delta := \min_{t\in[m,M]} g(t)$ and $\Delta := \max_{t\in[m,M]} g(t)$, then for any $p_j \geq 0, j \in \{1,...,n\}$ with $\sum_{j=1}^{n} p_j = 1$ and $x \in H$ with $\|x\| = 1$ we have*

$$|C(f,g;\mathbf{A},\mathbf{p},x)| \leq \frac{1}{2}(\Delta-\delta) L \left\langle \sum_{k=1}^{n} p_k \ell_{\mathbf{A},\mathbf{p},x}(A_k) x, x\right\rangle \leq \frac{\sqrt{2}}{2}(\Delta-\delta) LC(e,e;\mathbf{A},\mathbf{p},x) \tag{3.6}$$

where

$$\ell_{\mathbf{A},\mathbf{p},x}(t) := \left\langle \sum_{j=1}^{n} p_j |t\cdot 1_H - A_j| x, x\right\rangle$$

is a continuous function on $[m,M]$ *and*

$$C(e,e;\mathbf{A},\mathbf{p},x)=\sum_{j=1}^{n}p_j\left\|Ax_j\right\|^2-\left\langle\sum_{j=1}^{n}p_jA_jx,x\right\rangle^2\ (\geq 0).$$

Proof. In we choose in Theorem 5 $x_j=\sqrt{p_j}\cdot x,\ j\in\{1,...,n\}$, where $p_j\geq 0, j\in\{1,...,n\}$, $\sum_{j=1}^{n}p_j=1$ and $x\in H$, with $\|x\|=1$ then a simple calculation shows that the inequality (3.5) becomes (3.6). The details are omitted. □

In a similar way we obtain the following results as well:

Theorem 6. *Let* $\mathbf{A}=(A_1,...,A_n)$ *be a sequence of selfadjoint operators with* $Sp(A_j)\subseteq[m,M]$ *for* $j\in\{1,...,n\}$ *and for some scalars* $m<M$. *If* $f,g:[m,M]\longrightarrow\mathbb{R}$ *are Lipschitzian with the constants* $L,K>0$, *then*

$$|C(f,g;\mathbf{A},\mathbf{x})|\leq LKC(e,e;\mathbf{A},\mathbf{x}), \tag{3.7}$$

for any $\mathbf{x}=(x_1,...,x_n)\in H^n$ *with* $\sum_{j=1}^{n}\|x_j\|^2=1$.

Corollary 2. *Let* $\mathbf{A}=(A_1,...,A_n)$ *be a sequence of selfadjoint operators with* $Sp(A_j)\subseteq[m,M]$ *for* $j\in\{1,...,n\}$ *and for some scalars* $m<M$. *If* $f,g:[m,M]\longrightarrow\mathbb{R}$ *are Lipschitzian with the constants* $L,K>0$, *then for any* $p_j\geq 0, j\in\{1,...,n\}$ *with* $\sum_{j=1}^{n}p_j=1$ *we have*

$$|C(f,g;\mathbf{A},\mathbf{p},x)|\leq LKC(e,e;\mathbf{A},\mathbf{p},x), \tag{3.8}$$

for any $x\in H$ *with* $\|x\|=1$.

4. The Case of (φ,Φ) −Lipschitzian Functions

The following lemma may be stated.

Lemma 1. *Let* $u:[a,b]\to\mathbb{R}$ *and* $\varphi,\Phi\in\mathbb{R}$ *with* $\Phi>\varphi$. *The following statements are equivalent:*

(i) *The function* $u-\frac{\varphi+\Phi}{2}\cdot e$, *where* $e(t)=t, t\in[a,b]$, *is* $\frac{1}{2}(\Phi-\varphi)$ −*Lipschitzian;*

(ii) *We have the inequality:*

$$\varphi\leq\frac{u(t)-u(s)}{t-s}\leq\Phi\quad\text{for each}\quad t,s\in[a,b]\quad\text{with } t\neq s; \tag{4.1}$$

(iii) *We have the inequality:*

$$\varphi(t-s)\leq u(t)-u(s)\leq\Phi(t-s)\quad\text{for each}\quad t,s\in[a,b]\quad\text{with } t>s. \tag{4.2}$$

Following [6], we can introduce the concept:

Definition 1. *The function* $u:[a,b]\to\mathbb{R}$ *which satisfies one of the equivalent conditions (i) – (iii) is said to be* (φ,Φ) −*Lipschitzian on* $[a,b]$.

Notice that in [6], the definition was introduced on utilising the statement (iii) and only the equivalence (i) ⇔ (iii) was considered.

Utilising *Lagrange's mean value theorem*, we can state the following result that provides practical examples of (φ,Φ) −Lipschitzian functions.

Proposition 1. *Let* $u:[a,b]\to\mathbb{R}$ *be continuous on* $[a,b]$ *and differentiable on* (a,b). *If*

$$-\infty<\gamma:=\inf_{t\in(a,b)}u'(t),\qquad\sup_{t\in(a,b)}u'(t)=:\Gamma<\infty \tag{4.3}$$

then u *is* (γ,Γ) −*Lipschitzian on* $[a,b]$.

The following result can be stated:

Theorem 7. *Let A be a selfadjoint operator with $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. If $f : [m, M] \longrightarrow \mathbb{R}$ is $(\varphi, \Phi)-$Lipschitzian on $[a, b]$ and $g : [m, M] \longrightarrow \mathbb{R}$ is continuous with $\delta := \min_{t\in[m,M]} g(t)$ and $\Delta := \max_{t\in[m,M]} g(t)$, then*

$$\left|C(f, g; A; x) - \frac{\varphi + \Phi}{2} C(e, g; A; x)\right| \leq \frac{1}{4}(\Delta - \delta)(\Phi - \varphi)\langle \ell_{A,x}(A)x, x\rangle \leq \frac{\sqrt{2}}{4}(\Delta - \delta)(\Phi - \varphi) C(e, e; A; x) \tag{4.4}$$

for any $x \in H$ with $\|x\| = 1$.

The proof follows by Theorem 3 applied for the $\frac{1}{2}(\Phi - \varphi)-$Lipschitzian function $f - \frac{\varphi+\Phi}{2} \cdot e$ (see Lemma 1) and the details are omitted.

Theorem 8. *Let A be a selfadjoint operator with $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and $f, g : [m, M] \longrightarrow \mathbb{R}$. If f is $(\varphi, \Phi)-$Lipschitzian and g is $(\psi, \Psi)-$Lipschitzian on $[a, b]$, then*

$$\left|C(f, g; A; x) - \frac{\Phi + \varphi}{2} C(e, g; A; x) - \frac{\Psi + \psi}{2} C(f, e; A; x) + \frac{\Phi + \varphi}{2} \cdot \frac{\Psi + \psi}{2} C(e, e; A; x)\right| \leq \frac{1}{4}(\Phi - \varphi)(\Psi - \psi) C(e, e; A; x), \tag{4.5}$$

for any $x \in H$ with $\|x\| = 1$.

The proof follows by Theorem 4 applied for the $\frac{1}{2}(\Phi - \varphi)-$Lipschitzian function $f - \frac{\varphi+\Phi}{2} \cdot e$ and the $\frac{1}{2}(\Psi - \psi)-$Lipschitzian function $g - \frac{\Psi+\psi}{2} \cdot e$. The details are omitted.

Similar results can be derived for sequences of operators, however they will not be presented here.

5. Some Applications

It is clear that all the inequalities obtained in the previous sections can be applied to obtain particular inequalities of interest for different selections of the functions f and g involved. However we will present here only some particular results that can be derived from the inequality

$$|C(f, g; A; x)| \leq LKC(e, e; A; x), \tag{5.1}$$

that holds for the Lipschitzian functions f and g, the first with the constant $L > 0$ and the second with the constant $K > 0$.

1. Now, if we consider the functions $f, g : [m, M] \subset (0, \infty) \to \mathbb{R}$ with $f(t) = t^p, g(t) = t^q$ and $p, q \in (-\infty, 0) \cup (0, \infty)$ then they are Lipschitzian with the constants $L = \|f'\|_\infty$ and $K = \|g'\|_\infty$. Since $f'(t) = pt^{p-1}, g(t) = qt^{q-1}$, hence

$$\|f'\|_\infty = \begin{cases} pM^{p-1} & \text{for } p \in [1, \infty), \\ \\ |p|\, m^{p-1} & \text{for } p \in (-\infty, 0) \cup (0, 1) \end{cases}$$

and

$$\left\|g'\right\|_{\infty}=\begin{cases} qM^{q-1} & \text{for } q\in[1,\infty), \\ \\ |q|\, m^{q-1} & \text{for } q\in(-\infty,0)\cup(0,1) \end{cases}.$$

Therefore we can state the following inequalities for the powers of a positive definite operator A with $Sp(A)\subset[m,M]\subset(0,\infty)$.

If $p,q\geq 1$, then

$$(0\leq)\left\langle A^{p+q}x,x\right\rangle-\langle A^{p}x,x\rangle\cdot\langle A^{q}x,x\rangle\leq pqM^{p+q-2}\left(\|Ax\|^{2}-\langle Ax,x\rangle^{2}\right) \tag{5.2}$$

for each $x\in H$ with $\|x\|=1$.

If $p\geq 1$ and $q\in(-\infty,0)\cup(0,1)$, then

$$\left|\left\langle A^{p+q}x,x\right\rangle-\langle A^{p}x,x\rangle\cdot\langle A^{q}x,x\rangle\right|\leq p\,|q|\,M^{p-1}m^{q-1}\left(\|Ax\|^{2}-\langle Ax,x\rangle^{2}\right) \tag{5.3}$$

for each $x\in H$ with $\|x\|=1$.

If $p\in(-\infty,0)\cup(0,1)$ and $q\geq 1$, then

$$\left|\left\langle A^{p+q}x,x\right\rangle-\langle A^{p}x,x\rangle\cdot\langle A^{q}x,x\rangle\right|\leq |p|\,qM^{q-1}m^{p-1}\left(\|Ax\|^{2}-\langle Ax,x\rangle^{2}\right) \tag{5.4}$$

for each $x\in H$ with $\|x\|=1$.

If $p,q\in(-\infty,0)\cup(0,1)$, then

$$\left|\left\langle A^{p+q}x,x\right\rangle-\langle A^{p}x,x\rangle\cdot\langle A^{q}x,x\rangle\right|\leq |pq|\,m^{p+q-2}\left(\|Ax\|^{2}-\langle Ax,x\rangle^{2}\right) \tag{5.5}$$

for each $x\in H$ with $\|x\|=1$.

Moreover, if we take $p=1$ and $q=-1$ in (5.3), then we get the following result

$$(0\leq)\langle Ax,x\rangle\cdot\left\langle A^{-1}x,x\right\rangle-1\leq m^{-2}\left(\|Ax\|^{2}-\langle Ax,x\rangle^{2}\right) \tag{5.6}$$

for each $x\in H$ with $\|x\|=1$.

2. Consider now the functions $f,g:[m,M]\subset(0,\infty)\to\mathbb{R}$ with $f(t)=t^{p}, p\in(-\infty,0)\cup(0,\infty)$ and $g(t)=\ln t$. Then g is also Lipschitzian with the constant $K=\|g'\|_{\infty}=m^{-1}$. Applying the inequality (5.1) we then have for any $x\in H$ with $\|x\|=1$ that

$$(0\leq)\langle A^{p}\ln Ax,x\rangle-\langle A^{p}x,x\rangle\cdot\langle\ln Ax,x\rangle\leq pM^{p-1}m^{-1}\left(\|Ax\|^{2}-\langle Ax,x\rangle^{2}\right) \tag{5.7}$$

if $p\geq 1$,

$$(0\leq)\langle A^{p}\ln Ax,x\rangle-\langle A^{p}x,x\rangle\cdot\langle\ln Ax,x\rangle\leq pm^{p-2}\left(\|Ax\|^{2}-\langle Ax,x\rangle^{2}\right) \tag{5.8}$$

if $p\in(0,1)$ and

$$(0\leq)\langle A^{p}x,x\rangle\cdot\langle\ln Ax,x\rangle-\langle A^{p}\ln Ax,x\rangle\leq(-p)\,m^{p-2}\left(\|Ax\|^{2}-\langle Ax,x\rangle^{2}\right) \tag{5.9}$$

if $p\in(-\infty,0)$.

3. Now consider the functions $f,g:[m,M]\subset\mathbb{R}\to\mathbb{R}$ given by $f(t)=\exp(\alpha t)$ and $g(t)=\exp(\beta t)$ with α,β nonzero real numbers. It is obvious that

$$\left\|f'\right\|_{\infty}=|\alpha|\times\begin{cases} \exp(\alpha M) & \text{for } \alpha>0, \\ \\ \exp(\alpha m) & \text{for } \alpha<0 \end{cases}$$

and

$$\|g'\|_\infty = |\beta| \times \begin{cases} \exp(\beta M) & \text{for } \beta > 0, \\ \\ \exp(\beta m) & \text{for } \beta < 0 \end{cases}.$$

Finally, on applying the inequality (5.1) we get

$$(5.10) \qquad (0 \leq) \langle \exp[(\alpha+\beta) A] x, x\rangle - \langle \exp(\alpha A) x, x\rangle \cdot \langle \exp(\beta A) x, x\rangle$$
$$\leq \quad |\alpha\beta| \left(\|Ax\|^2 - \langle Ax, x\rangle^2\right) \times \begin{cases} \exp[(\alpha+\beta) M] & \text{for } \alpha, \beta > 0, \\ \\ \exp[(\alpha+\beta) m] & \text{for } \alpha, \beta < 0 \end{cases}$$

and

$$(5.11) \qquad (0 \leq) \langle \exp(\alpha A) x, x\rangle \cdot \langle \exp(\beta A) x, x\rangle - \langle \exp[(\alpha+\beta) A] x, x\rangle$$
$$\leq \quad |\alpha\beta| \left(\|Ax\|^2 - \langle Ax, x\rangle^2\right) \times \begin{cases} \exp(\alpha M + \beta m) & \text{for } \alpha > 0, \beta < 0 \\ \\ \exp(\alpha m + \beta M) & \text{for } \alpha < 0, \beta > 0 \end{cases}$$

for each $x \in H$ with $\|x\| = 1$.

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