SOME INEQUALITIES FOR CONVEX FUNCTIONS OF SELFADJOINT OPERATORS IN HILBERT SPACES

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Abstract. Some inequalities for convex functions of selfadjoint operators in Hilbert spaces under suitable assumptions for the involved operators are given. Applications for particular cases of interest are also provided.

1. Introduction

Let A be a selfadjoint linear operator on a complex Hilbert space $(H;\langle .,.\rangle)$. The *Gelfand map* establishes a $*$-isometrically isomorphism Φ between the set $C(Sp(A))$ of all *continuous functions* defined on the *spectrum* of A, denoted $Sp(A)$, an the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows (see for instance [6, p. 3]):

For any $f,g\in C(Sp(A))$ and any $\alpha,\beta\in\mathbb{C}$ we have

(i) $\Phi(\alpha f+\beta g)=\alpha\Phi(f)+\beta\Phi(g)$;

(ii) $\Phi(fg)=\Phi(f)\Phi(g)$ and $\Phi(\bar{f})=\Phi(f)^*$;

(iii) $\|\Phi(f)\|=\|f\|:=\sup_{t\in Sp(A)}|f(t)|$;

(iv) $\Phi(f_0)=1_H$ and $\Phi(f_1)=A$, where $f_0(t)=1$ and $f_1(t)=t$, for $t\in Sp(A)$.

With this notation we define

$$f(A):=\Phi(f)\text{ for all }f\in C(Sp(A))$$

and we call it the *continuous functional calculus* for a selfadjoint operator A.

If A is a selfadjoint operator and f is a real valued continuous function on $Sp(A)$, then $f(t)\geq 0$ for any $t\in Sp(A)$ implies that $f(A)\geq 0$, *i.e.* $f(A)$ is a positive operator on H. Moreover, if both f and g are real valued functions on $Sp(A)$ then the following important property holds:

$$\text{(P)}\qquad f(t)\geq g(t)\text{ for any }t\in Sp(A)\text{ implies that }f(A)\geq g(A)$$

in the operator order of $B(H)$.

For a recent monograph devoted to various inequalities for functions of selfadjoint operators, see [6] and the references therein. For other results, see [13], [7] and [9].

The following result that provides an operator version for the Jensen inequality is due to Mond & Pečarić [11] (see also [6, p. 5]):

Theorem 1 (Mond-Pečarić, 1993, [11])**.** *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A)\subseteq[m,M]$ for some scalars m,M with $m<M$. If f is a convex function on $[m,M]$, then*

$$\text{(MP)}\qquad f(\langle Ax,x\rangle)\leq\langle f(A)x,x\rangle$$

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for each $x \in H$ *with* $\|x\| = 1$.

The following result that provides a reverse of the Mond & Pečarić has been obtained in [3]:

Theorem 2 (Dragomir, 2008, [3])**.** *Let* I *be an interval and* $f : I \to \mathbb{R}$ *be a convex and differentiable function on* $\mathring{I}$ *(the interior of* I*) whose derivative* f' *is continuous on* $\mathring{I}$. *If* A *is a selfadjoint operators on the Hilbert space* H *with* $Sp(A) \subseteq [m, M] \subset \mathring{I}$, *then*

$$(0 \leq) \langle f(A)x, x\rangle - f(\langle Ax, x\rangle) \leq \langle f'(A)Ax, x\rangle - \langle Ax, x\rangle \cdot \langle f'(A)x, x\rangle \tag{1.1}$$

for any $x \in H$ *with* $\|x\| = 1$.

Perhaps more convenient reverses of the Mond & Pečarić result are the following inequalities that have been obtained in the same paper [3]:

Theorem 3 (Dragomir, 2008, [3])**.** *Let* I *be an interval and* $f : I \to \mathbb{R}$ *be a convex and differentiable function on* $\mathring{I}$ *(the interior of* I*) whose derivative* f' *is continuous on* $\mathring{I}$. *If* A *is a selfadjoint operators on the Hilbert space* H *with* $Sp(A) \subseteq [m, M] \subset \mathring{I}$, *then*

$$\begin{aligned}(0 \leq) \langle f(A)x, x\rangle - f(\langle Ax, x\rangle) &\leq \begin{cases} \frac{1}{2} \cdot (M - m)\left[\|f'(A)x\|^2 - \langle f'(A)x, x\rangle^2\right]^{1/2} \\ \\ \frac{1}{2} \cdot (f'(M) - f'(m))\left[\|Ax\|^2 - \langle Ax, x\rangle^2\right]^{1/2} \end{cases} \\ &\leq \frac{1}{4}(M - m)(f'(M) - f'(m))\end{aligned} \tag{1.2}$$

for any $x \in H$ *with* $\|x\| = 1$.

We also have the inequality

$$\begin{aligned}(0 \leq) \langle f(A)x, x\rangle - f(\langle Ax, x\rangle) &\leq \frac{1}{4}(M - m)(f'(M) - f'(m)) \\ &- \begin{cases} \left[\langle Mx - Ax, Ax - mx\rangle \langle f'(M)x - f'(A)x, f'(A)x - f'(m)x\rangle\right]^{\frac{1}{2}}, \\ \\ \left|\langle Ax, x\rangle - \frac{M+m}{2}\right| \left|\langle f'(A)x, x\rangle - \frac{f'(M)+f'(m)}{2}\right| \end{cases} \\ &\leq \frac{1}{4}(M - m)(f'(M) - f'(m))\end{aligned} \tag{1.3}$$

for any $x \in H$ *with* $\|x\| = 1$.

Moreover, if $m > 0$ *and* $f'(m) > 0$, *then we also have*

$$\begin{aligned}(0 \leq) \langle f(A)x, x\rangle - f(\langle Ax, x\rangle) \\ \leq \begin{cases} \frac{1}{4} \cdot \frac{(M-m)\left(f'(M)-f'(m)\right)}{\sqrt{Mmf'(M)f'(m)}} \langle Ax, x\rangle \langle f'(A)x, x\rangle, \\ \\ \left(\sqrt{M} - \sqrt{m}\right)\left(\sqrt{f'(M)} - \sqrt{f'(m)}\right)\left[\langle Ax, x\rangle \langle f'(A)x, x\rangle\right]^{\frac{1}{2}}, \end{cases}\end{aligned} \tag{1.4}$$

for any $x \in H$ *with* $\|x\| = 1$.

For generalisations to n-tuples of operators as well as for some particular cases of interest, see [3].

The main aim of the present paper is to provide more general vector inequalities for convex functions whose derivatives are continuous.

2. Some Inequalities for Two Operators

The following result holds:

Theorem 4. *Let I be an interval and $f : I \to \mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ (the interior of I) whose derivative f' is continuous on $\mathring{I}$. If A and B are selfadjoint operators on the Hilbert space H with $Sp(A), Sp(B) \subseteq [m, M] \subset \mathring{I}$, then*

$$\text{(2.1)} \quad \langle f'(A)x, x\rangle \langle By, y\rangle - \langle f'(A)Ax, x\rangle \leq \langle f(B)y, y\rangle - \langle f(A)x, x\rangle \leq \langle f'(B)By, y\rangle - \langle Ax, x\rangle \langle f'(B)y, y\rangle$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

In particular, we have

$$\text{(2.2)} \quad \langle f'(A)x, x\rangle \langle Ay, y\rangle - \langle f'(A)Ax, x\rangle \leq \langle f(A)y, y\rangle - \langle f(A)x, x\rangle \leq \langle f'(A)Ay, y\rangle - \langle Ax, x\rangle \langle f'(A)y, y\rangle$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ and

$$\text{(2.3)} \quad \langle f'(A)x, x\rangle \langle Bx, x\rangle - \langle f'(A)Ax, x\rangle \leq \langle f(B)x, x\rangle - \langle f(A)x, x\rangle \leq \langle f'(B)Bx, x\rangle - \langle Ax, x\rangle \langle f'(B)x, x\rangle$$

for any $x \in H$ with $\|x\| = 1$.

Proof. Since f is convex and differentiable on $\mathring{I}$, then we have that

$$\text{(2.4)} \quad f'(s)\cdot(t-s) \leq f(t) - f(s) \leq f'(t)\cdot(t-s)$$

for any $t, s \in [m, M]$.

Now, if we fix $t \in [m, M]$ and apply the property (P) for the operator A, then for any $x \in H$ with $\|x\| = 1$ we have

$$\text{(2.5)} \quad \langle f'(A)\cdot(t\cdot 1_H - A)x, x\rangle \leq \langle [f(t)\cdot 1_H - f(A)]x, x\rangle \leq \langle f'(t)\cdot(t\cdot 1_H - A)x, x\rangle$$

for any $t \in [m, M]$ and any $x \in H$ with $\|x\| = 1$.

The inequality (2.5) is equivalent with

$$\text{(2.6)} \quad t\langle f'(A)x, x\rangle - \langle f'(A)Ax, x\rangle \leq f(t) - \langle f(A)x, x\rangle \leq f'(t)t - f'(t)\langle Ax, x\rangle$$

for any $t \in [m, M]$ and any $x \in H$ with $\|x\| = 1$.

If we fix $x \in H$ with $\|x\| = 1$ in (2.6) and apply the property (P) for the operator B, then we get

$$\text{(2.7)} \quad \langle [\langle f'(A)x, x\rangle B - \langle f'(A)Ax, x\rangle 1_H]y, y\rangle \leq \langle [f(B) - \langle f(A)x, x\rangle 1_H]y, y\rangle \leq \langle [f'(B)B - \langle Ax, x\rangle f'(B)]y, y\rangle$$

for each $y \in H$ with $\|y\| = 1$, which is clearly equivalent to the desired inequality (2.1). ∎

Remark 1. *If we fix $x \in H$ with $\|x\| = 1$ and choose $B = \langle Ax, x\rangle \cdot 1_H$, then we obtain from the first inequality in (2.1) the reverse of the Mond-Pečarić inequality obtained by the author in* [3]. *The second inequality will provide the inequality (MP) for convex functions whose derivatives are continuous.*

The following corollary is of interest:

Corollary 1. *Let I be an interval and $f : I \to \mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ whose derivative f' is continuous on $\mathring{I}$. Also, suppose that A is a selfadjoint operator on the Hilbert space H with $Sp(A) \subseteq [m, M] \subset \mathring{I}$. If g is nonincreasing and continuous on $[m, M]$ and*

$$f'(A)[g(A) - A] \geq 0 \tag{2.8}$$

in the operator order of $B(H)$, then

$$(f \circ g)(A) \geq f(A) \tag{2.9}$$

in the operator order of $B(H)$.

Proof. If we apply the first inequality from (2.3) for $B = g(A)$ we have

$$\langle f'(A) x, x\rangle \langle g(A) x, x\rangle - \langle f'(A) Ax, x\rangle \leq \langle f(g(A)) x, x\rangle - \langle f(A) x, x\rangle \tag{2.10}$$

any $x \in H$ with $\|x\| = 1$.

We use the following Čebyšev type inequality for functions of operators established by the author in [4]:

Let A be a selfadjoint operator with $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. If $h, g : [m, M] \longrightarrow \mathbb{R}$ are continuous and *synchronous (asynchronous)* on $[m, M]$, then

$$\langle h(A) g(A) x, x\rangle \geq (\leq) \langle h(A) x, x\rangle \cdot \langle g(A) x, x\rangle \tag{2.11}$$

for any $x \in H$ with $\|x\| = 1$.

Now, since f' and g are continuous and are asynchronous on $[m, M]$, then by (2.11) we have the inequality

$$\langle f'(A) g(A) x, x\rangle \leq \langle f'(A) x, x\rangle \cdot \langle g(A) x, x\rangle \tag{2.12}$$

for any $x \in H$ with $\|x\| = 1$.

Subtracting in both sides of (2.12) the quantity $\langle f'(A) Ax, x\rangle$ and taking into account, by (2.8), that $\langle f'(A)[g(A) - A] x, x\rangle \geq 0$ for any $x \in H$ with $\|x\| = 1$, we then have

$$\begin{aligned} 0 &\leq \langle f'(A)[g(A) - A] x, x\rangle = \langle f'(A) g(A) x, x\rangle - \langle f'(A) Ax, x\rangle \\ &\leq \langle f'(A) x, x\rangle \cdot \langle g(A) x, x\rangle - \langle f'(A) Ax, x\rangle \end{aligned}$$

which together with (2.10) will produce the desired result (2.9). ∎

We provide now some particular inequalities of interest that can be derived from Theorem 4:

Example 1. ***a.*** *Let A, B two positive definite operators on H. Then we have the inequalities*

$$1 - \langle A^{-1}x, x\rangle \langle By, y\rangle \leq \langle \ln Ax, x\rangle - \langle \ln By, y\rangle \leq \langle Ax, x\rangle \langle B^{-1}y, y\rangle - 1 \tag{2.13}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

In particular, we have

$$1 - \langle A^{-1}x, x\rangle \langle Ay, y\rangle \leq \langle \ln Ax, x\rangle - \langle \ln Ay, y\rangle \leq \langle Ax, x\rangle \langle A^{-1}y, y\rangle - 1 \tag{2.14}$$

for any $x, y \in H$ *with* $\|x\| = \|y\| = 1$ *and*

$$1 - \left\langle A^{-1}x, x \right\rangle \langle Bx, x \rangle \leq \langle \ln Ax, x \rangle - \langle \ln Bx, x \rangle \leq \langle Ax, x \rangle \left\langle B^{-1}x, x \right\rangle - 1 \tag{2.15}$$

for any $x \in H$ *with* $\|x\| = 1$.

b. *With the same assumption for* A *and* B *we have the inequalities*

$$\langle By, y \rangle - \langle Ax, x \rangle \leq \langle B \ln By, y \rangle - \langle \ln Ax, x \rangle \langle By, y \rangle \tag{2.16}$$

for any $x, y \in H$ *with* $\|x\| = \|y\| = 1$.

In particular, we have

$$\langle Ay, y \rangle - \langle Ax, x \rangle \leq \langle A \ln Ay, y \rangle - \langle \ln Ax, x \rangle \langle Ay, y \rangle \tag{2.17}$$

for any $x, y \in H$ *with* $\|x\| = \|y\| = 1$ *and*

$$\langle Bx, x \rangle - \langle Ax, x \rangle \leq \langle B \ln Bx, x \rangle - \langle \ln Ax, x \rangle \langle Bx, x \rangle \tag{2.18}$$

for any $x \in H$ *with* $\|x\| = 1$.

The proof of Example **a** follows from Theorem 4 for the convex function $f(x) = -\ln x$ while the proof of the second example follows by the same theorem applied for the convex function $f(x) = x \ln x$ and performing the required calculations. The details are omitted.

The following result may be stated as well:

Theorem 5. *Let* I *be an interval and* $f : I \to \mathbb{R}$ *be a convex and differentiable function on* $\mathring{I}$ *(the interior of* I*) whose derivative* f' *is continuous on* $\mathring{I}$*. If* A *and* B *are selfadjoint operators on the Hilbert space* H *with* $Sp(A), Sp(B) \subseteq [m, M] \subset \mathring{I}$*, then*

$$\begin{aligned} f'(\langle Ax, x \rangle)(\langle By, y \rangle - \langle Ax, x \rangle) &\leq \langle f(B)y, y \rangle - f(\langle Ax, x \rangle) \\ &\leq \langle f'(B)By, y \rangle - \langle Ax, x \rangle \langle f'(B)y, y \rangle \end{aligned} \tag{2.19}$$

for any $x, y \in H$ *with* $\|x\| = \|y\| = 1$.

In particular, we have

$$\begin{aligned} f'(\langle Ax, x \rangle)(\langle Ay, y \rangle - \langle Ax, x \rangle) &\leq \langle f(A)y, y \rangle - f(\langle Ax, x \rangle) \\ &\leq \langle f'(A)Ay, y \rangle - \langle Ax, x \rangle \langle f'(A)y, y \rangle \end{aligned} \tag{2.20}$$

for any $x, y \in H$ *with* $\|x\| = \|y\| = 1$ *and*

$$\begin{aligned} f'(\langle Ax, x \rangle)(\langle Bx, x \rangle - \langle Ax, x \rangle) &\leq \langle f(B)x, x \rangle - f(\langle Ax, x \rangle) \\ &\leq \langle f'(B)Bx, x \rangle - \langle Ax, x \rangle \langle f'(B)x, x \rangle \end{aligned} \tag{2.21}$$

for any $x \in H$ *with* $\|x\| = 1$.

Proof. Since f is convex and differentiable on $\mathring{I}$, then we have that

$$f'(s) \cdot (t - s) \leq f(t) - f(s) \leq f'(t) \cdot (t - s) \tag{2.22}$$

for any $t, s \in [m, M]$.

If we choose $s = \langle Ax, x \rangle \in [m, M]$, with a fix $x \in H$ with $\|x\| = 1$, then we have

$$f'(\langle Ax, x \rangle) \cdot (t - \langle Ax, x \rangle) \leq f(t) - f(\langle Ax, x \rangle) \leq f'(t) \cdot (t - \langle Ax, x \rangle) \tag{2.23}$$

for any $t \in [m, M]$.

Now, if we apply the property (P) to the inequality (2.23) and the operator B, then we get

$$\left\langle f'\left(\langle Ax,x\rangle\right)\cdot\left(B-\langle Ax,x\rangle\cdot 1_H\right)y,y\right\rangle \\ \leq \left\langle\left[f\left(B\right)-f\left(\langle Ax,x\rangle\right)\cdot 1_H\right]y,y\right\rangle \leq \left\langle f'\left(B\right)\cdot\left(B-\langle Ax,x\rangle\cdot 1_H\right)y,y\right\rangle$$

for any $x,y\in H$ with $\|x\|=\|y\|=1$, which is equivalent with the desired result (2.19). ∎

Remark 2. *We observe that if we choose $B=A$ in (2.21) or $y=x$ in (2.20) then we recapture the Mond-Pečarić inequality and its reverse from (1.1).*

The following particular case of interest follows from Theorem 5

Corollary 2. *Assume that f,A and B are as in Theorem 5. If, either f is increasing on $[m,M]$ and $B\geq A$ in the operator order of $B(H)$ or f is decreasing and $B\leq A$, then we have the Jensen's type inequality*

$$\langle f(B)x,x\rangle \geq f\left(\langle Ax,x\rangle\right) \tag{2.24}$$

for any $x\in H$ with $\|x\|=1$.

The proof is obvious by the first inequality in (2.21) and the details are omitted.

We provide now some particular inequalities of interest that can be derived from Theorem 5:

Example 2. ***a.*** *Let A,B be two positive definite operators on H. Then we have the inequalities*

$$1-\langle Ax,x\rangle^{-1}\langle By,y\rangle \leq \ln\left(\langle Ax,x\rangle\right)-\langle \ln By,y\rangle \leq \langle Ax,x\rangle\left\langle B^{-1}y,y\right\rangle-1 \tag{2.25}$$

for any $x,y\in H$ with $\|x\|=\|y\|=1$.

In particular, we have

$$1-\langle Ax,x\rangle^{-1}\langle Ay,y\rangle \leq \ln\left(\langle Ax,x\rangle\right)-\langle \ln Ay,y\rangle \leq \langle Ax,x\rangle\left\langle A^{-1}y,y\right\rangle-1 \tag{2.26}$$

for any $x,y\in H$ with $\|x\|=\|y\|=1$ and

$$1-\langle Ax,x\rangle^{-1}\langle Bx,x\rangle \leq \ln\left(\langle Ax,x\rangle\right)-\langle \ln Bx,x\rangle \leq \langle Ax,x\rangle\left\langle B^{-1}x,x\right\rangle-1 \tag{2.27}$$

for any $x\in H$ with $\|x\|=1$.

b. *With the same assumption for A and B, we have the inequalities*

$$\langle By,y\rangle-\langle Ax,x\rangle \leq \langle B\ln By,y\rangle-\langle By,y\rangle\ln\left(\langle Ax,x\rangle\right) \tag{2.28}$$

for any $x,y\in H$ with $\|x\|=\|y\|=1$.

In particular, we have

$$\langle Ay,y\rangle-\langle Ax,x\rangle \leq \langle A\ln Ay,y\rangle-\langle Ay,y\rangle\ln\left(\langle Ax,x\rangle\right) \tag{2.29}$$

for any $x,y\in H$ with $\|x\|=\|y\|=1$ and

$$\langle Bx,x\rangle-\langle Ax,x\rangle \leq \langle B\ln Bx,x\rangle-\langle Bx,x\rangle\ln\left(\langle Ax,x\rangle\right) \tag{2.30}$$

for any $x\in H$ with $\|x\|=1$.

3. Inequalities for Two Sequences of Operators

The following result may be stated:

Theorem 6. *Let I be an interval and $f : I \to \mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ (the interior of I) whose derivative f' is continuous on $\mathring{I}$. If A_j and B_j are selfadjoint operators on the Hilbert space H with $Sp(A_j), Sp(B_j) \subseteq [m, M] \subset \mathring{I}$ for any $j \in \{1, ..., n\}$, then*

$$\begin{aligned} (3.1) \quad & \sum_{j=1}^{n} \langle f'(A_j) x_j, x_j \rangle \sum_{j=1}^{n} \langle B_j y_j, y_j \rangle - \sum_{j=1}^{n} \langle f'(A_j) A_j x_j, x_j \rangle \\ & \leq \sum_{j=1}^{n} \langle f(B_j) y_j, y_j \rangle - \sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle \\ & \leq \sum_{j=1}^{n} \langle f'(B_j) B_j y_j, y_j \rangle - \sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \sum_{j=1}^{n} \langle f'(B_j) y_j, y_j \rangle \end{aligned}$$

for any $x_j, y_j \in H$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = \sum_{j=1}^{n} \|y_j\|^2 = 1$.

In particular, we have

$$\begin{aligned} (3.2) \quad & \sum_{j=1}^{n} \langle f'(A_j) x_j, x_j \rangle \sum_{j=1}^{n} \langle A_j y_j, y_j \rangle - \sum_{j=1}^{n} \langle f'(A_j) A_j x_j, x_j \rangle \\ & \leq \sum_{j=1}^{n} \langle f(A_j) y_j, y_j \rangle - \sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle \\ & \leq \sum_{j=1}^{n} \langle f'(A_j) A_j y_j, y_j \rangle - \sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \sum_{j=1}^{n} \langle f'(A_j) y_j, y_j \rangle \end{aligned}$$

for any $x_j, y_j \in H$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = \sum_{j=1}^{n} \|y_j\|^2 = 1$ and

$$\begin{aligned} (3.3) \quad & \sum_{j=1}^{n} \langle f'(A_j) x_j, x_j \rangle \sum_{j=1}^{n} \langle B_j x_j, x_j \rangle - \sum_{j=1}^{n} \langle f'(A_j) A_j x_j, x_j \rangle \\ & \leq \sum_{j=1}^{n} \langle f(B_j) x_j, x_j \rangle - \sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle \\ & \leq \sum_{j=1}^{n} \langle f'(B_j) B_j x_j, x_j \rangle - \sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \sum_{j=1}^{n} \langle f'(B_j) x_j, x_j \rangle \end{aligned}$$

for any $x_j \in H$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

Proof. As in [6, p. 6], if we put

$$\widetilde{A} := \begin{pmatrix} A_1 & . & . & . & 0 \\ & . & & & \\ & & . & & \\ & & & . & \\ 0 & . & . & . & A_n \end{pmatrix}, \ \widetilde{B} := \begin{pmatrix} B_1 & . & . & . & 0 \\ & . & & & \\ & & . & & \\ & & & . & \\ 0 & . & . & . & B_n \end{pmatrix}$$

and

$$\widetilde{x}=\begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix},\widetilde{y}=\begin{pmatrix} y_1 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{pmatrix}$$

then we have $Sp\left(\widetilde{A}\right), Sp\left(\widetilde{B}\right) \subseteq [m, M]$, $\|\widetilde{x}\| = \|\widetilde{y}\| = 1$,

$$\left\langle f'\left(\widetilde{A}\right)\widetilde{x},\widetilde{x}\right\rangle = \sum_{j=1}^{n}\left\langle f'\left(A_j\right)x_j,x_j\right\rangle, \left\langle B\widetilde{y},\widetilde{y}\right\rangle = \sum_{j=1}^{n}\left\langle By_j,y_j\right\rangle$$

and so on.

Applying Theorem 4 for $\widetilde{A}$, $\widetilde{B}$, $\widetilde{x}$ and $\widetilde{y}$ we deduce the desired result (3.1). ∎

The following particular case may be of interest:

Corollary 3. *Let I be an interval and $f : I \to \mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ (the interior of I) whose derivative f' is continuous on $\mathring{I}$. If A_j and B_j are selfadjoint operators on the Hilbert space H with $Sp(A_j), Sp(B_j) \subseteq [m, M] \subset \mathring{I}$ for any $j \in \{1, ..., n\}$, then for any $p_j, q_j \geq 0$ with $\sum_{j=1}^{n} p_j = \sum_{j=1}^{n} q_j = 1$, we have the inequalities*

$$\begin{aligned}(3.4)\quad &\left\langle\sum_{j=1}^{n}p_jf'(A_j)x,x\right\rangle\left\langle\sum_{j=1}^{n}q_jB_jy,y\right\rangle-\left\langle\sum_{j=1}^{n}p_jf'(A_j)A_jx,x\right\rangle\\ &\leq\left\langle\sum_{j=1}^{n}q_jf(B_j)y,y\right\rangle-\left\langle\sum_{j=1}^{n}p_jf(A_j)x,x\right\rangle\\ &\leq\left\langle\sum_{j=1}^{n}q_jf'(B_j)B_jy,y\right\rangle-\left\langle\sum_{j=1}^{n}p_jA_jx,x\right\rangle\left\langle\sum_{j=1}^{n}q_jf'(B_j)y,y\right\rangle\end{aligned}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

In particular, we have

$$\begin{aligned}(3.5)\quad &\left\langle\sum_{j=1}^{n}p_jf'(A_j)x,x\right\rangle\left\langle\sum_{j=1}^{n}q_jA_jy,y\right\rangle-\left\langle\sum_{j=1}^{n}p_jf'(A_j)A_jx,x\right\rangle\\ &\leq\left\langle\sum_{j=1}^{n}q_jf(A_j)y,y\right\rangle-\left\langle\sum_{j=1}^{n}p_jf(A_j)x,x\right\rangle\\ &\leq\left\langle\sum_{j=1}^{n}q_jf'(A_j)B_jy,y\right\rangle-\left\langle\sum_{j=1}^{n}p_jA_jx,x\right\rangle\left\langle\sum_{j=1}^{n}q_jf'(A_j)y,y\right\rangle\end{aligned}$$

for any $x, y \in H$ *with* $\|x\| = \|y\| = 1$ *and*

$$\left\langle \sum_{j=1}^{n} p_j f'(A_j)\, x, x \right\rangle \left\langle \sum_{j=1}^{n} p_j B_j x, x \right\rangle - \left\langle \sum_{j=1}^{n} p_j f'(A_j)\, A_j x, x \right\rangle \tag{3.6}$$
$$\leq \left\langle \sum_{j=1}^{n} p_j f(B_j)\, x, x \right\rangle - \left\langle \sum_{j=1}^{n} p_j f(A_j)\, x, x \right\rangle$$
$$\leq \left\langle \sum_{j=1}^{n} p_j f'(B_j)\, B_j x, x \right\rangle - \left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle \left\langle \sum_{j=1}^{n} p_j f'(B_j)\, x, x \right\rangle$$

for any $x \in H$ *with* $\|x\| = 1$.

Proof. Follows from Theorem 6 on choosing $x_j = \sqrt{p_j} \cdot x$, $y_j = \sqrt{q_j} \cdot y$, $j \in \{1, ..., n\}$, where $p_j, q_j \geq 0, j \in \{1, ..., n\}$, $\sum_{j=1}^{n} p_j = \sum_{j=1}^{n} q_j = 1$ and $x, y \in H$, with $\|x\| = \|y\| = 1$. The details are omitted. ∎

Example 3. ***a.*** *Let* A_j, B_j, $j \in \{1, ..., n\}$, *be two sequences of positive definite operators on* H. *Then we have the inequalities*

$$1 - \sum_{j=1}^{n} \left\langle A_j^{-1} x_j, x_j \right\rangle \sum_{j=1}^{n} \left\langle B_j y_j, y_j \right\rangle \tag{3.7}$$
$$\leq \sum_{j=1}^{n} \left\langle \ln A_j x_j, x_j \right\rangle - \sum_{j=1}^{n} \left\langle \ln B_j y_j, y_j \right\rangle \leq \sum_{j=1}^{n} \left\langle A_j x_j, x_j \right\rangle \sum_{j=1}^{n} \left\langle B_j^{-1} y_j, y_j \right\rangle - 1$$

for any $x_j, y_j \in H$, $j \in \{1, ..., n\}$ *with* $\sum_{j=1}^{n} \|x_j\|^2 = \sum_{j=1}^{n} \|y_j\|^2 = 1$.

b. *With the same assumption for* A_j *and* B_j *we have the inequalities*

$$\sum_{j=1}^{n} \left\langle B_j y_j, y_j \right\rangle - \sum_{j=1}^{n} \left\langle A_j x_j, x_j \right\rangle \tag{3.8}$$
$$\leq \sum_{j=1}^{n} \left\langle B_j \ln B_j y_j, y_j \right\rangle - \sum_{j=1}^{n} \left\langle \ln A_j x_j, x_j \right\rangle \sum_{j=1}^{n} \left\langle B_j y_j, y_j \right\rangle$$

for any $x_j, y_j \in H$, $j \in \{1, ..., n\}$ *with* $\sum_{j=1}^{n} \|x_j\|^2 = \sum_{j=1}^{n} \|y_j\|^2 = 1$.

Finally, we have

Example 4. ***a.*** *Let* A_j, B_j, $j \in \{1, ..., n\}$, *be two sequences of positive definite operators on* H. *Then for any* $p_j, q_j \geq 0$ *with* $\sum_{j=1}^{n} p_j = \sum_{j=1}^{n} q_j = 1$, *we have the inequalities*

$$1 - \left\langle \sum_{j=1}^{n} p_j A_j^{-1} x, x \right\rangle \left\langle \sum_{j=1}^{n} q_j B_j y, y \right\rangle \tag{3.9}$$
$$\leq \left\langle \sum_{j=1}^{n} p_j \ln A_j x, x \right\rangle - \left\langle \sum_{j=1}^{n} q_j \ln B_j y, y \right\rangle$$
$$\leq \left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle \left\langle \sum_{j=1}^{n} q_j B_j^{-1} y, y \right\rangle - 1$$

for any $x, y \in H$ *with* $\|x\| = \|y\| = 1$.

b. *With the same assumption for* A_j, B_j, p_j *and* q_j, *we have the inequalities*

$$\left\langle \sum_{j=1}^{n} q_j B_j y, y \right\rangle - \left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle \leq \left\langle \sum_{j=1}^{n} q_j B_j \ln B_j y, y \right\rangle - \left\langle \sum_{j=1}^{n} p_j \ln A_j x, x \right\rangle \left\langle \sum_{j=1}^{n} q_j B_j y, y \right\rangle \tag{3.10}$$

for any $x, y \in H$ *with* $\|x\| = \|y\| = 1$.

Remark 3. *We observe that all the other inequalities for two operators obtained in Section 2 can be extended for two sequences of operators in a similar way. However, the details are left to the interested reader.*

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