GRÜSS' TYPE INEQUALITIES FOR FUNCTIONS OF SELFADJOINT OPERATORS IN HILBERT SPACES

S.S. DRAGOMIR

Abstract. Some inequalities of Grüss' type for functions of selfadjoint operators in Hilbert spaces, under suitable assumptions for the involved operators, are given.

1. Introduction

In 1935, G. Grüss [19] proved the following integral inequality which gives an approximation of the integral of the product in terms of the product of the integrals as follows:

$$\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx\right| \leq \frac{1}{4}(\Phi-\phi)(\Gamma-\gamma), \tag{1.1}$$

where $f,\ g:[a,b]\to\mathbb{R}$ are integrable on $[a,b]$ and satisfy the condition

$$\phi\leq f(x)\leq\Phi,\ \gamma\leq g(x)\leq\Gamma \tag{1.2}$$

for each $x\in[a,b]$, where ϕ,Φ,γ,Γ are given real constants.

Moreover, the constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller one.

In 1950, M. Biernacki, H. Pidek and C. Ryll-Nardjewski [22, Chapter X] established the following discrete version of Grüss' inequality:

Let $a=(a_1,...,a_n)$, $b=(b_1,...,b_n)$ be two $n-$tuples of real numbers such that $r\leq a_i\leq R$ and $s\leq b_i\leq S$ for $i=1,...,n$. Then one has

$$\left|\frac{1}{n}\sum_{i=1}^n a_ib_i-\frac{1}{n}\sum_{i=1}^n a_i\cdot\frac{1}{n}\sum_{i=1}^n b_i\right|\leq\frac{1}{n}\left[\frac{n}{2}\right]\left(1-\frac{1}{n}\left[\frac{n}{2}\right]\right)(R-r)(S-s), \tag{1.3}$$

where $[x]$ denotes the integer part of $x,\ x\in\mathbb{R}$.

For a simple proof of (1.1) as well as for some other integral inequalities of Grüss type, see Chapter X of the recent book [22]. For other related results see the papers [1]-[3], [4]-[6], [7]-[9], [10]-[16], [18], [25], [27] and the references therein.

Date: August 05, 2008.

1991 *Mathematics Subject Classification.* 47A63; 47A99.

Key words and phrases. Selfadjoint operators, Grüss inequality, Functions of Selfadjoint operators.

2. Operator Inequalities

Let A be a selfadjoint linear operator on a complex Hilbert space $(H;\langle .,.\rangle)$. The *Gelfand map* establishes a $*$-isometrically isomorphism Φ between the set $C(Sp(A))$ of all *continuous functions* defined on the *spectrum* of A, denoted $Sp(A)$, an the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows (see for instance [20, p. 3]):

For any $f,g\in C(Sp(A))$ and any $\alpha,\beta\in\mathbb{C}$ we have

(i) $\Phi(\alpha f+\beta g)=\alpha\Phi(f)+\beta\Phi(g)$;

(ii) $\Phi(fg)=\Phi(f)\Phi(g)$ and $\Phi(\bar{f})=\Phi(f)^*$;

(iii) $\|\Phi(f)\|=\|f\|:=\sup_{t\in Sp(A)}|f(t)|$;

(iv) $\Phi(f_0)=1_H$ and $\Phi(f_1)=A$, where $f_0(t)=1$ and $f_1(t)=t$, for $t\in Sp(A)$.

With this notation we define

$$f(A):=\Phi(f) \text{ for all } f\in C(Sp(A))$$

and we call it the *continuous functional calculus* for a selfadjoint operator A.

If A is a selfadjoint operator and f is a real valued continuous function on $Sp(A)$, then $f(t)\geq 0$ for any $t\in Sp(A)$ implies that $f(A)\geq 0$, *i.e.* $f(A)$ is a positive operator on H. Moreover, if both f and g are real valued functions on $Sp(A)$ then the following important property holds:

$$f(t)\geq g(t) \text{ for any } t\in Sp(A) \text{ implies that } f(A)\geq g(A) \tag{P}$$

in the operator order of $B(H)$.

For a recent monograph devoted to various inequalities for functions of selfadjoint operators, see [20] and the references therein. For other results, see [24], [21] and [26].

The following operator version of the Grüss inequality was obtained by Mond & Pečarić in [23]:

Theorem 1 (Mond-Pečarić, 1993, [23])**.** *Let C_j, $j\in\{1,...,n\}$ be selfadjoint operators on the Hilbert space $(H,\langle .,.\rangle)$ and such that $m_j\cdot 1_H\leq C_j\leq M_j\cdot 1_H$ for $j\in\{1,...,n\}$, where 1_H is the identity operator on H. Further, let $g_j,h_j:[m_j,Mj]\to\mathbb{R}$, $j\in\{1,...,n\}$ be functions such that*

$$\varphi\cdot 1_H\leq g_j(C_j)\leq\Phi\cdot 1_H \text{ and } \gamma\cdot 1_H\leq h_j(C_j)\leq\Gamma\cdot 1_H \tag{2.1}$$

for each $j\in\{1,...,n\}$.

If $x_j\in H$, $j\in\{1,...,n\}$ are such that $\sum_{j=1}^n\|x_j\|^2=1$, then

$$\left|\sum_{j=1}^n\langle g_j(C_j)h_j(C_j)x_j,x_j\rangle-\sum_{j=1}^n\langle g_j(C_j)x_j,x_j\rangle\cdot\sum_{j=1}^n\langle h_j(C_j)x_j,x_j\rangle\right| \leq\frac{1}{4}(\Phi-\varphi)(\Gamma-\gamma). \tag{2.2}$$

If $C_j, j\in\{1,...,n\}$ are selfadjoint operators such that $Sp(C_j)\subseteq[m,M]$ for $j\in\{1,...,n\}$ and for some scalars $m<M$ and if $g,h:[m,M]\longrightarrow\mathbb{R}$ are continuous then by the Mond-Pečarić inequality we deduce the following version of the Grüss

inequality for operators

$$\left|\sum_{j=1}^{n}\left\langle g\left(C_{j}\right)h\left(C_{j}\right)x_{j},x_{j}\right\rangle-\sum_{j=1}^{n}\left\langle g\left(C_{j}\right)x_{j},x_{j}\right\rangle\cdot\sum_{j=1}^{n}\left\langle h\left(C_{j}\right)x_{j},x_{j}\right\rangle\right| \leq\frac{1}{4}\left(\Phi-\varphi\right)\left(\Gamma-\gamma\right), \tag{2.3}$$

where $x_j \in H, j \in \{1, ..., n\}$ are such that $\sum_{j=1}^{n} \|x_j\|^2 = 1$ and $\varphi = \min_{t\in[m,M]} g(t)$, $\Phi = \max_{t\in[m,M]} g(t)$, $\gamma = \min_{t\in[m,M]} h(t)$ and $\Gamma = \max_{t\in[m,M]} h(t)$.

In particular, if the selfadjoint operator C satisfy the condition $Sp(C) \subseteq [m, M]$ for some scalars $m < M$, then

$$\left|\langle g(C)h(C)x,x\rangle-\langle g(C)x,x\rangle\cdot\langle h(C)x,x\rangle\right|\leq\frac{1}{4}\left(\Phi-\varphi\right)\left(\Gamma-\gamma\right), \tag{2.4}$$

for any $x \in H$ with $\|x\| = 1$.

We say that the functions $f, g : [a, b] \longrightarrow \mathbb{R}$ are *synchronous (asynchronous)* on the interval $[a, b]$ if they satisfy the following condition:

$$(f(t)-f(s))(g(t)-g(s))\geq(\leq)\,0 \text{ for each } t,s\in[a,b].$$

It is obvious that, if f, g are monotonic and have the same monotonicity on the interval $[a, b]$, then they are synchronous on $[a, b]$ while if they have opposite monotonicity, they are asynchronous.

In the recent paper [17] the following *Čebyšev type inequality* for operators has been obtained:

Theorem 2 (Dragomir, 2008, [17]). *Let A be a selfadjoint operator on the Hilbert space $(H, \langle ., .\rangle)$ with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. If $f, g : [m, M] \longrightarrow \mathbb{R}$ are continuous and synchronous (asynchronous) on $[m, M]$, then*

$$\langle f(A)g(A)x,x\rangle\geq(\leq)\langle f(A)x,x\rangle\cdot\langle g(A)x,x\rangle \tag{2.5}$$

for any $x \in H$ with $\|x\| = 1$.

This can be generalised for n operators as follows:

Theorem 3 (Dragomir, 2008, [17]). *Let A_j be selfadjoint operators with $Sp(A_j) \subseteq [m, M]$ for $j \in \{1, ..., n\}$ and for some scalars $m < M$. If $f, g : [m, M] \longrightarrow \mathbb{R}$ are continuous and synchronous (asynchronous) on $[m, M]$, then*

$$\sum_{j=1}^{n}\left\langle f\left(A_{j}\right)g\left(A_{j}\right)x_{j},x_{j}\right\rangle\geq(\leq)\sum_{j=1}^{n}\left\langle f\left(A_{j}\right)x_{j},x_{j}\right\rangle\cdot\sum_{j=1}^{n}\left\langle g\left(A_{j}\right)x_{j},x_{j}\right\rangle, \tag{2.6}$$

for each $x_j \in H, j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

Another version for n operators is incorporated in:

Theorem 4 (Dragomir, 2008, [17]). *Let A_j be selfadjoint operators with $Sp(A_j) \subseteq [m, M]$ for $j \in \{1, ..., n\}$ and for some scalars $m < M$. If $f, g : [m, M] \longrightarrow \mathbb{R}$ are continuous and synchronous (asynchronous) on $[m, M]$, then*

$$\left\langle\sum_{j=1}^{n}p_{j}f\left(A_{j}\right)g\left(A_{j}\right)x,x\right\rangle\geq(\leq)\left\langle\sum_{j=1}^{n}p_{j}f\left(A_{j}\right)x,x\right\rangle\cdot\left\langle\sum_{j=1}^{n}p_{j}g\left(A_{j}\right)x,x\right\rangle, \tag{2.7}$$

for any $p_j \geq 0, j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} p_j = 1$ and $x \in H$ with $\|x\| = 1$.

Motivated by the above results we investigate in this paper other Grüss' type inequalities for selfadjoint operators in Hilbert spaces. Some of the obtained results improve the inequalities (2.3) and (2.4) derived from the Mond-Pečarić inequality. Others provide different operator versions for the celebrated Grüss' inequality mentioned above. Examples for power functions and the logarithmic function are given as well.

3. An Inequality of Grüss' Type for One Operator

The following result may be stated:

Theorem 5. *Let A be a selfadjoint operator on the Hilbert space $(H;\langle .,.\rangle)$ and assume that $Sp(A)\subseteq[m,M]$ for some scalars $m<M$. If f and g are continuous on $[m,M]$ and $\gamma:=\min_{t\in[m,M]}f(t)$ and $\Gamma:=\max_{t\in[m,M]}f(t)$ then*

$$\begin{aligned}(3.1)\quad &\Big|\langle f(A)g(A)y,y\rangle-\langle f(A)y,y\rangle\cdot\langle g(A)x,x\rangle\\ &\qquad-\frac{\gamma+\Gamma}{2}\left[\langle g(A)y,y\rangle-\langle g(A)x,x\rangle\right]\Big|\\ &\leq\frac{1}{2}\cdot(\Gamma-\gamma)\left[\|g(A)y\|^2+\langle g(A)x,x\rangle^2-2\langle g(A)x,x\rangle\langle g(A)y,y\rangle\right]^{1/2}\end{aligned}$$

for any $x,y\in H$ with $\|x\|=\|y\|=1$.

Proof. First of all, observe that, for each $\lambda\in\mathbb{R}$ and $x,y\in H$, $\|x\|=\|y\|=1$ we have the identity

$$\begin{aligned}(3.2)\quad &\langle(f(A)-\lambda\cdot 1_H)(g(A)-\langle g(A)x,x\rangle\cdot 1_H)y,y\rangle\\ &=\langle f(A)g(A)y,y\rangle-\lambda\cdot[\langle g(A)y,y\rangle-\langle g(A)x,x\rangle]-\langle g(A)x,x\rangle\langle f(A)y,y\rangle.\end{aligned}$$

Taking the modulus in (3.2) we have

$$\begin{aligned}(3.3)\quad &|\langle f(A)g(A)y,y\rangle-\lambda\cdot[\langle g(A)y,y\rangle-\langle g(A)x,x\rangle]\\ &-\langle g(A)x,x\rangle\langle f(A)y,y\rangle|\\ &=|\langle(g(A)-\langle g(A)x,x\rangle\cdot 1_H)y,(f(A)-\lambda\cdot 1_H)y\rangle|\\ &\leq\|g(A)y-\langle g(A)x,x\rangle y\|\,\|f(A)y-\lambda y\|\\ &=\left[\|g(A)y\|^2+\langle g(A)x,x\rangle^2-2\langle g(A)x,x\rangle\langle g(A)y,y\rangle\right]^{1/2}\\ &\times\|f(A)y-\lambda y\|\\ &\leq\left[\|g(A)y\|^2+\langle g(A)x,x\rangle^2-2\langle g(A)x,x\rangle\langle g(A)y,y\rangle\right]^{1/2}\\ &\times\|f(A)-\lambda\cdot 1_H\|\end{aligned}$$

for any $x,y\in H$, $\|x\|=\|y\|=1$.

Now, since $\gamma=\min_{t\in[m,M]}f(t)$ and $\Gamma=\max_{t\in[m,M]}f(t)$, then by the property (P) we have that $\gamma\leq\langle f(A)y,y\rangle\leq\Gamma$ for each $y\in H$ with $\|y\|=1$ which is clearly equivalent with

$$\left|\langle f(A)y,y\rangle-\frac{\gamma+\Gamma}{2}\|y\|^2\right|\leq\frac{1}{2}(\Gamma-\gamma)$$

or with

$$\left|\left\langle\left(f(A)-\frac{\gamma+\Gamma}{2}1_H\right)y,y\right\rangle\right|\leq\frac{1}{2}(\Gamma-\gamma)$$

for each $y \in H$ with $\|y\| = 1$.

Taking the supremum in this inequality we get

$$\left\| f(A) - \frac{\gamma + \Gamma}{2} \cdot 1_H \right\| \le \frac{1}{2} (\Gamma - \gamma),$$

which together with the inequality (3.3) applied for $\lambda = \frac{\gamma+\Gamma}{2}$ produces the desired result (3.1). ∎

As a particular case of interest we can derive from the above theorem the following result of Grüss' type that improves (2.4):

Corollary 1. *With the assumptions in Theorem 5 we have*

$$\text{(3.4)} \quad |\langle f(A) g(A) x, x\rangle - \langle f(A) x, x\rangle \cdot \langle g(A) x, x\rangle| \le \frac{1}{2} \cdot (\Gamma - \gamma) \left[\|g(A) x\|^2 - \langle g(A) x, x\rangle^2 \right]^{1/2} \left(\le \frac{1}{4} (\Gamma - \gamma)(\Delta - \delta) \right)$$

for each $x \in H$ *with* $\|x\| = 1$, *where* $\delta := \min_{t \in [m,M]} g(t)$ *and* $\Delta := \max_{t \in [m,M]} g(t)$.

Proof. The first inequality follows from (3.1) by putting $y = x$.

Now, if we write the first inequality in (3.4) for $f = g$ we get

$$0 \le \|g(A) x\|^2 - \langle g(A) x, x\rangle^2 = \langle g^2(A) x, x\rangle - \langle g(A) x, x\rangle^2 \le \frac{1}{2} (\Delta - \delta) \left[\|g(A) x\|^2 - \langle g(A) x, x\rangle^2 \right]^{1/2}$$

which implies that

$$\left[\|g(A) x\|^2 - \langle g(A) x, x\rangle^2 \right]^{1/2} \le \frac{1}{2} (\Delta - \delta)$$

for each $x \in H$ with $\|x\| = 1$.

This together with the first part of (3.4) proves the desired bound. ∎

The following particular cases that hold for power function are of interest:

Example 1. *Let* A *be a selfadjoint operator with* $Sp(A) \subseteq [m, M]$ *for some scalars* $m < M$.

If A *is positive* $(m \ge 0)$ *and* $p, q > 0$, *then*

$$\text{(3.5)} \quad (0 \le) \langle A^{p+q} x, x\rangle - \langle A^p x, x\rangle \cdot \langle A^q x, x\rangle \le \frac{1}{2} \cdot (M^p - m^p) \left[\|A^q x\|^2 - \langle A^q x, x\rangle^2 \right]^{1/2} \left[\le \frac{1}{4} \cdot (M^p - m^p)(M^q - m^q) \right]$$

for each $x \in H$ *with* $\|x\| = 1$.

If A *is positive definite* $(m > 0)$ *and* $p, q < 0$, *then*

$$\text{(3.6)} \quad (0 \le) \langle A^{p+q} x, x\rangle - \langle A^p x, x\rangle \cdot \langle A^q x, x\rangle \le \frac{1}{2} \cdot \frac{M^{-p} - m^{-p}}{M^{-p} m^{-p}} \left[\|A^q x\|^2 - \langle A^q x, x\rangle^2 \right]^{1/2} \left[\le \frac{1}{4} \cdot \frac{M^{-p} - m^{-p}}{M^{-p} m^{-p}} \frac{M^{-q} - m^{-q}}{M^{-q} m^{-q}} \right]$$

for each $x \in H$ *with* $\|x\| = 1$.

If A is positive definite $(m>0)$ and $p<0$, $q>0$ then

$$(0\leq)\left\langle A^{p}x,x\right\rangle \cdot\left\langle A^{q}x,x\right\rangle -\left\langle A^{p+q}x,x\right\rangle \leq\frac{1}{2}\cdot\frac{M^{-p}-m^{-p}}{M^{-p}m^{-p}}\left[\left\Vert A^{q}x\right\Vert ^{2}-\left\langle A^{q}x,x\right\rangle ^{2}\right]^{1/2}\left[\leq\frac{1}{4}\cdot\frac{M^{-p}-m^{-p}}{M^{-p}m^{-p}}\left(M^{q}-m^{q}\right)\right] \tag{3.7}$$

for each $x\in H$ with $\|x\|=1$.

If A is positive definite $(m>0)$ and $p>0$, $q<0$ then

$$(0\leq)\left\langle A^{p}x,x\right\rangle \cdot\left\langle A^{q}x,x\right\rangle -\left\langle A^{p+q}x,x\right\rangle \leq\frac{1}{2}\cdot\left(M^{p}-m^{p}\right)\left[\left\Vert A^{q}x\right\Vert ^{2}-\left\langle A^{q}x,x\right\rangle ^{2}\right]^{1/2}\left[\leq\frac{1}{4}\cdot\left(M^{p}-m^{p}\right)\frac{M^{-q}-m^{-q}}{M^{-q}m^{-q}}\right] \tag{3.8}$$

for each $x\in H$ with $\|x\|=1$.

We notice that the positivity of the quantities in the left hand side of the above inequalities (3.5)-(3.8) follows from the Theorem 2.

The following particular cases when one function is a power while the second is the logarithm are of interest as well:

Example 2. *Let A be a positive definite operator with $Sp(A)\subseteq[m,M]$ for some scalars $0<m<M$.*

If $p>0$ then

$$(0\leq)\left\langle A^{p}\ln Ax,x\right\rangle -\left\langle A^{p}x,x\right\rangle \cdot\left\langle \ln Ax,x\right\rangle \leq\begin{cases}\frac{1}{2}\cdot\left(M^{p}-m^{p}\right)\left[\left\Vert \ln Ax\right\Vert ^{2}-\left\langle \ln Ax,x\right\rangle ^{2}\right]^{1/2}\\ \\ \ln\sqrt{\frac{M}{m}}\cdot\left[\left\Vert A^{p}x\right\Vert ^{2}-\left\langle A^{p}x,x\right\rangle ^{2}\right]^{1/2}\end{cases}\left[\leq\frac{1}{2}\cdot\left(M^{p}-m^{p}\right)\ln\sqrt{\frac{M}{m}}\right] \tag{3.9}$$

for each $x\in H$ with $\|x\|=1$.

If $p<0$ then

$$(0\leq)\left\langle A^{p}x,x\right\rangle \cdot\left\langle \ln Ax,x\right\rangle -\left\langle A^{p}\ln Ax,x\right\rangle \leq\begin{cases}\frac{1}{2}\cdot\frac{M^{-p}-m^{-p}}{M^{-p}m^{-p}}\left[\left\Vert \ln Ax\right\Vert ^{2}-\left\langle \ln Ax,x\right\rangle ^{2}\right]^{1/2}\\ \\ \ln\sqrt{\frac{M}{m}}\cdot\left[\left\Vert A^{p}x\right\Vert ^{2}-\left\langle A^{p}x,x\right\rangle ^{2}\right]^{1/2}\end{cases}\left[\leq\frac{1}{2}\cdot\frac{M^{-p}-m^{-p}}{M^{-p}m^{-p}}\ln\sqrt{\frac{M}{m}}\right] \tag{3.10}$$

for each $x\in H$ with $\|x\|=1$.

4. An Inequality of Grüss' Type for n Operators

The following multiple operator version of Theorem 5 holds:

Theorem 6. *Let A_j be selfadjoint operators with $Sp(A_j)\subseteq[m,M]$ for $j\in\{1,...,n\}$ and for some scalars $m<M$. If $f,g:[m,M]\longrightarrow\mathbb{R}$ are continuous*

and $\gamma := \min_{t\in[m,M]} f(t)$ *and* $\Gamma := \max_{t\in[m,M]} f(t)$ *then*

$$\left|\sum_{j=1}^{n}\langle f(A_j)\,g(A_j)\,y_j,y_j\rangle-\sum_{j=1}^{n}\langle f(A_j)\,y_j,y_j\rangle\cdot\sum_{j=1}^{n}\langle g(A_j)\,x_j,x_j\rangle -\frac{\gamma+\Gamma}{2}\left[\sum_{j=1}^{n}\langle g(A_j)\,y_j,y_j\rangle-\sum_{j=1}^{n}\langle g(A_j)\,x_j,x_j\rangle\right]\right| \leq\frac{1}{2}(\Gamma-\gamma) \times\left[\sum_{j=1}^{n}\|g(A_j)\,y_j\|^2+\left(\sum_{j=1}^{n}\langle g(A_j)\,x_j,x_j\rangle\right)^2-2\sum_{j=1}^{n}\langle g(A_j)\,x_j,x_j\rangle\sum_{j=1}^{n}\langle g(A_j)\,y_j,y_j\rangle\right]^{1/2} \tag{4.1}$$

for each $x_j,y_j\in H, j\in\{1,...,n\}$ *with* $\sum_{j=1}^{n}\|x_j\|^2=\sum_{j=1}^{n}\|y_j\|^2=1.$

Proof. As in [20, p. 6], if we put

$$\widetilde{A}:=\begin{pmatrix} A_1 & . & . & . & 0\\ & . & & & \\ & & . & & \\ & & & . & \\ 0 & . & . & . & A_n\end{pmatrix}\text{ and }\widetilde{x}=\begin{pmatrix}x_1\\ .\\ .\\ .\\ x_n\end{pmatrix},\widetilde{y}=\begin{pmatrix}y_1\\ .\\ .\\ .\\ y_n\end{pmatrix}$$

then we have $Sp\left(\widetilde{A}\right)\subseteq[m,M]$, $\|\widetilde{x}\|=\|\widetilde{y}\|=1$

$$\left\langle f\left(\widetilde{A}\right)g\left(\widetilde{A}\right)\widetilde{y},\widetilde{y}\right\rangle=\sum_{j=1}^{n}\langle f(A_j)\,g(A_j)\,y_j,y_j\rangle,\left\langle g\left(\widetilde{A}\right)\widetilde{x},\widetilde{x}\right\rangle=\sum_{j=1}^{n}\langle g(A_j)\,x_j,x_j\rangle,$$

$$\left\langle f\left(\widetilde{A}\right)\widetilde{y},\widetilde{y}\right\rangle=\sum_{j=1}^{n}\langle f(A_j)\,y_j,y_j\rangle,\ \left\langle g\left(\widetilde{A}\right)\widetilde{y},\widetilde{y}\right\rangle=\sum_{j=1}^{n}\langle g(A_j)\,y_j,y_j\rangle,$$

and

$$\left\|g\left(\widetilde{A}\right)\widetilde{y}\right\|^2=\sum_{j=1}^{n}\|g(A_j)\,y_j\|^2,\left\langle g\left(\widetilde{A}\right)\widetilde{x},\widetilde{x}\right\rangle^2=\left(\sum_{j=1}^{n}\langle g(A_j)\,x_j,x_j\rangle\right)^2.$$

Applying Theorem 5 for $\widetilde{A}$, $\widetilde{x}$ and $\widetilde{y}$ we deduce the desired result (4.1). ∎

The following particular case provides a refinement of the Mond- Pečarić result from (2.3).

Corollary 2. *With the assumptions of Theorem 6 we have*

$$\left|\sum_{j=1}^{n}\langle f(A_j)\,g(A_j)\,x_j,x_j\rangle-\sum_{j=1}^{n}\langle f(A_j)\,x_j,x_j\rangle\cdot\sum_{j=1}^{n}\langle g(A_j)\,x_j,x_j\rangle\right| \leq\frac{1}{2}\cdot(\Gamma-\gamma)\left[\sum_{j=1}^{n}\|g(A_j)\,x_j\|^2-\left(\sum_{j=1}^{n}\langle g(A_j)\,x_j,x_j\rangle\right)^2\right]^{1/2}\left(\leq\frac{1}{4}(\Gamma-\gamma)(\Delta-\delta)\right) \tag{4.2}$$

for each $x_j \in H, j \in \{1,...,n\}$ *with* $\sum_{j=1}^n \|x_j\|^2 = 1$ *where* $\delta := \min_{t\in[m,M]} g(t)$ *and* $\Delta := \max_{t\in[m,M]} g(t)$.

Example 3. *Let* A_j, $j \in \{1,...,n\}$ *be a selfadjoint operators with* $Sp(A_j) \subseteq [m,M]$, $j \in \{1,...,n\}$ *for some scalars* $m < M$.

If A_j *are positive* $(m \geq 0)$ *and* $p,q > 0$, *then*

$$(4.3)\quad (0 \leq) \sum_{j=1}^n \left\langle A_j^{p+q} x_j, x_j \right\rangle - \sum_{j=1}^n \left\langle A_j^p x_j, x_j \right\rangle \cdot \sum_{j=1}^n \left\langle A_j^q x_j, x_j \right\rangle \\ \leq \frac{1}{2} \cdot (M^p - m^p) \left[\sum_{j=1}^n \left\| A_j^q x_j \right\|^2 - \left(\sum_{j=1}^n \left\langle A_j^q x_j, x_j \right\rangle \right)^2 \right]^{1/2} \\ \left[\leq \frac{1}{4} \cdot (M^p - m^p)(M^q - m^q) \right]$$

for each $x_j \in H, j \in \{1,...,n\}$ *with* $\sum_{j=1}^n \|x_j\|^2 = 1$.

If A_j *are positive definite* $(m > 0)$ *and* $p,q < 0$, *then*

$$(4.4)\quad (0 \leq) \sum_{j=1}^n \left\langle A_j^{p+q} x_j, x_j \right\rangle - \sum_{j=1}^n \left\langle A_j^p x_j, x_j \right\rangle \cdot \sum_{j=1}^n \left\langle A_j^q x_j, x_j \right\rangle \\ \leq \frac{1}{2} \cdot \frac{M^{-p} - m^{-p}}{M^{-p} m^{-p}} \left[\sum_{j=1}^n \left\| A_j^q x_j \right\|^2 - \left(\sum_{j=1}^n \left\langle A_j^q x_j, x_j \right\rangle \right)^2 \right]^{1/2} \\ \left[\leq \frac{1}{4} \cdot \frac{M^{-p} - m^{-p}}{M^{-p} m^{-p}} \frac{M^{-q} - m^{-q}}{M^{-q} m^{-q}} \right]$$

for each $x_j \in H, j \in \{1,...,n\}$ *with* $\sum_{j=1}^n \|x_j\|^2 = 1$.

If A_j *are positive definite* $(m > 0)$ *and* $p < 0$, $q > 0$ *then*

$$(4.5)\quad (0 \leq) \sum_{j=1}^n \left\langle A_j^p x_j, x_j \right\rangle \cdot \sum_{j=1}^n \left\langle A_j^q x_j, x_j \right\rangle - \sum_{j=1}^n \left\langle A_j^{p+q} x_j, x_j \right\rangle \\ \leq \frac{1}{2} \cdot \frac{M^{-p} - m^{-p}}{M^{-p} m^{-p}} \left[\sum_{j=1}^n \left\| A_j^q x_j \right\|^2 - \left(\sum_{j=1}^n \left\langle A_j^q x_j, x_j \right\rangle \right)^2 \right]^{1/2} \\ \left[\leq \frac{1}{4} \cdot \frac{M^{-p} - m^{-p}}{M^{-p} m^{-p}} (M^q - m^q) \right]$$

for each $x_j \in H, j \in \{1,...,n\}$ *with* $\sum_{j=1}^n \|x_j\|^2 = 1$.

If A_j *are positive definite* $(m>0)$ *and* $p>0$, $q<0$ *then*

$$(4.6)\quad (0\leq)\sum_{j=1}^{n}\left\langle A_j^{p}x_j,x_j\right\rangle\cdot\sum_{j=1}^{n}\left\langle A_j^{q}x_j,x_j\right\rangle-\sum_{j=1}^{n}\left\langle A_j^{p+q}x_j,x_j\right\rangle$$
$$\leq\frac{1}{2}\cdot(M^p-m^p)\left[\sum_{j=1}^{n}\left\|A_j^{q}x_j\right\|^2-\left(\sum_{j=1}^{n}\left\langle A_j^{q}x_j,x_j\right\rangle\right)^2\right]^{1/2}$$
$$\left[\leq\frac{1}{4}\cdot(M^p-m^p)\,\frac{M^{-q}-m^{-q}}{M^{-q}m^{-q}}\right]$$

for each $x_j\in H, j\in\{1,...,n\}$ *with* $\sum_{j=1}^{n}\|x_j\|^2=1$.

We notice that the positivity of the quantities in the left hand side of the above inequalities (4.3)-(4.6) follows from the Theorem 3.

The following particular cases when one function is a power while the second is the logarithm are of interest as well:

Example 4. *Let* A_j *be positive definite operators with* $Sp(A_j)\subseteq[m,M]$, $j\in\{1,...,n\}$ *for some scalars* $0<m<M$.

If $p>0$ *then*

$$(4.7)\quad (0\leq)\sum_{j=1}^{n}\left\langle A_j^{p}\ln A_jx_j,x_j\right\rangle-\sum_{j=1}^{n}\left\langle A_j^{p}x_j,x_j\right\rangle\cdot\sum_{j=1}^{n}\left\langle \ln A_jx_j,x_j\right\rangle$$
$$\leq\begin{cases}\frac{1}{2}\cdot(M^p-m^p)\left[\sum_{j=1}^{n}\|\ln A_jx_j\|^2-\left(\sum_{j=1}^{n}\langle\ln A_jx_j,x_j\rangle\right)^2\right]^{1/2}\\[2ex] \ln\sqrt{\frac{M}{m}}\cdot\left[\sum_{j=1}^{n}\left\|A_j^{p}x_j\right\|^2-\left(\sum_{j=1}^{n}\left\langle A_j^{p}x_j,x_j\right\rangle\right)^2\right]^{1/2}\end{cases}$$
$$\left[\leq\frac{1}{2}\cdot(M^p-m^p)\ln\sqrt{\frac{M}{m}}\right]$$

for each $x_j\in H, j\in\{1,...,n\}$ *with* $\sum_{j=1}^{n}\|x_j\|^2=1$.

If $p<0$ *then*

$$(4.8)\quad (0\leq)\sum_{j=1}^{n}\left\langle A_j^{p}x_j,x_j\right\rangle\cdot\sum_{j=1}^{n}\langle\ln A_jx_j,x_j\rangle-\sum_{j=1}^{n}\left\langle A_j^{p}\ln A_jx_j,x_j\right\rangle$$
$$\leq\begin{cases}\frac{1}{2}\frac{M^{-p}-m^{-p}}{M^{-p}m^{-p}}\left[\sum_{j=1}^{n}\|\ln A_jx_j\|^2-\left(\sum_{j=1}^{n}\langle\ln A_jx_j,x_j\rangle\right)^2\right]^{1/2}\\[2ex] \ln\sqrt{\frac{M}{m}}\cdot\left[\sum_{j=1}^{n}\left\|A_j^{p}x_j\right\|^2-\left(\sum_{j=1}^{n}\left\langle A_j^{p}x_j,x_j\right\rangle\right)^2\right]^{1/2}\end{cases}$$
$$\left[\leq\frac{1}{2}\cdot\frac{M^{-p}-m^{-p}}{M^{-p}m^{-p}}\ln\sqrt{\frac{M}{m}}\right]$$

for each $x_j\in H, j\in\{1,...,n\}$ *with* $\sum_{j=1}^{n}\|x_j\|^2=1$.

5. Another Inequality of Grüss' Type for n Operators

The following different result for n operators can be stated as well:

Theorem 7. *Let A_j be selfadjoint operators with $Sp(A_j) \subseteq [m, M]$ for $j \in \{1, ..., n\}$ and for some scalars $m < M$. If f and g are continuous on $[m, M]$ and $\gamma := \min_{t\in[m,M]} f(t)$ and $\Gamma := \max_{t\in[m,M]} f(t)$ then for any $p_j \geq 0, j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} p_j = 1$ we have*

$$\begin{aligned} (5.1)\quad & \Bigg| \left\langle \sum_{k=1}^{n} p_k f(A_k) g(A_k) y, y \right\rangle \\ & \quad - \frac{\gamma+\Gamma}{2} \cdot \left[\left\langle \sum_{k=1}^{n} p_k g(A_k) y, y \right\rangle - \left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle \right] \\ & \quad - \left\langle \sum_{k=1}^{n} p_k f(A_k) y, y \right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle \Bigg| \\ & \leq \frac{\Gamma-\gamma}{2} \left[\sum_{k=1}^{n} p_k \|g(A_k) y\|^2 - 2 \left\langle \sum_{k=1}^{n} p_k g(A_k) y, y \right\rangle \left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle \right. \\ & \qquad \left. + \left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle^2 \right]^{1/2}, \end{aligned}$$

for each $x, y \in H$ with $\|x\| = \|y\| = 1$.

Proof. Follows from Theorem 6 on choosing $x_j = \sqrt{p_j} \cdot x$, $y_j = \sqrt{p_j} \cdot y$, $j \in \{1, ..., n\}$, where $p_j \geq 0, j \in \{1, ..., n\}$, $\sum_{j=1}^{n} p_j = 1$ and $x, y \in H$, with $\|x\| = \|y\| = 1$. The details are omitted. ∎

Remark 1. *The case $n = 1$ (therefore $p = 1$) in (5.1) provides the result from Theorem 5.*

As a particular case of interest we can derive from the above theorem the following result of Grüss' type:

Corollary 3. *With the assumptions of Theorem 7 we have*

$$\begin{aligned} (5.2)\quad & \left| \left\langle \sum_{k=1}^{n} p_k f(A_k) g(A_k) x, x \right\rangle - \left\langle \sum_{k=1}^{n} p_k f(A_k) x, x \right\rangle \cdot \left\langle \sum_{k=1}^{n} p_k g(A_k) x, x \right\rangle \right| \\ & \leq \frac{\Gamma-\gamma}{2} \left(\sum_{k=1}^{n} p_k \|g(A_k) x\|^2 - \left\langle \sum_{k=1}^{n} p_k g(A_k) x, x \right\rangle^2 \right)^{1/2} \\ & \qquad \left[\leq \frac{1}{4} \cdot (\Gamma - \gamma)(\Delta - \delta) \right] \end{aligned}$$

for each $x \in H$ with $\|x\| = 1$, where $\delta := \min_{t\in[m,M]} g(t)$ and $\Delta := \max_{t\in[m,M]} g(t)$.

Proof. It is similar with the proof from Corollary 1 and the details are omitted. ∎

The following particular cases that hold for power function are of interest:

Example 5. *Let* A_j, $j \in \{1,...,n\}$ *be a selfadjoint operators with* $Sp(A_j) \subseteq [m, M]$, $j \in \{1,...,n\}$ *for some scalars* $m < M$ *and* $p_j \geq 0, j \in \{1,...,n\}$ *with* $\sum_{j=1}^{n} p_j = 1$.

If A_j, $j \in \{1,...,n\}$ *are positive* $(m \geq 0)$ *and* $p, q > 0$, *then*

$$(0 \leq) \left\langle \sum_{k=1}^{n} p_k A_k^{p+q} x, x \right\rangle - \left\langle \sum_{k=1}^{n} p_k A_k^{p} x, x \right\rangle \cdot \left\langle \sum_{k=1}^{n} p_k A_k^{q} x, x \right\rangle \leq \frac{1}{2} \cdot (M^p - m^p) \left[\sum_{k=1}^{n} p_k \left\| A_k^q x \right\|^2 - \left\langle \sum_{k=1}^{n} p_k A_k^{q} x, x \right\rangle^2 \right]^{1/2} \left[\leq \frac{1}{4} \cdot (M^p - m^p)(M^q - m^q) \right] \tag{5.3}$$

for each $x \in H$ *with* $\|x\| = 1$.

If A_j, $j \in \{1,...,n\}$ *are positive definite* $(m > 0)$ *and* $p, q < 0$, *then*

$$(0 \leq) \left\langle \sum_{k=1}^{n} p_k A_k^{p+q} x, x \right\rangle - \left\langle \sum_{k=1}^{n} p_k A_k^{p} x, x \right\rangle \cdot \left\langle \sum_{k=1}^{n} p_k A_k^{q} x, x \right\rangle \leq \frac{1}{2} \cdot \frac{M^{-p} - m^{-p}}{M^{-p} m^{-p}} \left[\sum_{k=1}^{n} p_k \left\| A_k^q x \right\|^2 - \left\langle \sum_{k=1}^{n} p_k A_k^{q} x, x \right\rangle^2 \right]^{1/2} \left[\leq \frac{1}{4} \cdot \frac{M^{-p} - m^{-p}}{M^{-p} m^{-p}} \frac{M^{-q} - m^{-q}}{M^{-q} m^{-q}} \right] \tag{5.4}$$

for each $x \in H$ *with* $\|x\| = 1$.

If A_j, $j \in \{1,...,n\}$ *are positive definite* $(m > 0)$ *and* $p < 0$, $q > 0$ *then*

$$(0 \leq) \left\langle \sum_{k=1}^{n} p_k A_k^{p} x, x \right\rangle \cdot \left\langle \sum_{k=1}^{n} p_k A_k^{q} x, x \right\rangle - \left\langle \sum_{k=1}^{n} p_k A_k^{p+q} x, x \right\rangle \leq \frac{1}{2} \cdot \frac{M^{-p} - m^{-p}}{M^{-p} m^{-p}} \left[\sum_{k=1}^{n} p_k \left\| A_k^q x \right\|^2 - \left\langle \sum_{k=1}^{n} p_k A_k^{q} x, x \right\rangle^2 \right]^{1/2} \left[\leq \frac{1}{4} \cdot \frac{M^{-p} - m^{-p}}{M^{-p} m^{-p}} (M^q - m^q) \right] \tag{5.5}$$

for each $x \in H$ *with* $\|x\| = 1$.

If A_j, $j\in\{1,...,n\}$ *are positive definite* $(m>0)$ *and* $p>0$, $q<0$ *then*

$$(0\leq)\left\langle\sum_{k=1}^{n}p_kA_k^px,x\right\rangle\cdot\left\langle\sum_{k=1}^{n}p_kA_k^qx,x\right\rangle-\left\langle\sum_{k=1}^{n}p_kA_k^{p+q}x,x\right\rangle \tag{5.6}$$
$$\leq\frac{1}{2}\cdot(M^p-m^p)\left[\sum_{k=1}^{n}p_k\left\|A_k^qx\right\|^2-\left\langle\sum_{k=1}^{n}p_kA_k^qx,x\right\rangle^2\right]^{1/2}$$
$$\left[\leq\frac{1}{4}\cdot(M^p-m^p)\frac{M^{-q}-m^{-q}}{M^{-q}m^{-q}}\right]$$

for each $x\in H$ *with* $\|x\|=1$.

We notice that the positivity of the quantities in the left hand side of the above inequalities (5.3)-(5.6) follows from the Theorem 4.

The following particular cases when one function is a power while the second is the logarithm are of interest as well:

Example 6. *Let* A_j, $j\in\{1,...,n\}$ *be positive definite operators with* $Sp(A_j)\subseteq[m,M]$, $j\in\{1,...,n\}$ *for some scalars* $0<m<M$ *and* $p_j\geq0, j\in\{1,...,n\}$ *with* $\sum_{j=1}^{n}p_j=1$.

If $p>0$ *then*

$$(0\leq)\left\langle\sum_{k=1}^{n}p_kA_k^p\ln A_kx,x\right\rangle-\left\langle\sum_{k=1}^{n}p_kA_k^px,x\right\rangle\cdot\left\langle\sum_{k=1}^{n}p_k\ln A_kx,x\right\rangle \tag{5.7}$$
$$\leq\begin{cases}\frac{1}{2}\cdot(M^p-m^p)\cdot\left[\sum_{k=1}^{n}p_k\left\|\ln A_kx\right\|^2-\left\langle\sum_{k=1}^{n}p_k\ln A_kx,x\right\rangle^2\right]^{1/2}\\ \\ \ln\sqrt{\frac{M}{m}}\cdot\left[\sum_{k=1}^{n}p_k\left\|A_k^px\right\|^2-\left\langle\sum_{k=1}^{n}p_kA_k^px,x\right\rangle^2\right]^{1/2}\end{cases}$$
$$\left[\leq\frac{1}{2}\cdot(M^p-m^p)\ln\sqrt{\frac{M}{m}}\right]$$

for each $x\in H$ *with* $\|x\|=1$.

If $p<0$ *then*

$$(0\leq)\left\langle\sum_{k=1}^{n}p_kA_k^px,x\right\rangle\cdot\left\langle\sum_{k=1}^{n}p_k\ln A_kx,x\right\rangle-\left\langle\sum_{k=1}^{n}p_kA_k^p\ln A_kx,x\right\rangle \tag{5.8}$$
$$\leq\begin{cases}\frac{1}{2}\cdot\frac{M^{-p}-m^{-p}}{M^{-p}m^{-p}}\left[\sum_{k=1}^{n}p_k\left\|\ln A_kx\right\|^2-\left\langle\sum_{k=1}^{n}p_k\ln A_kx,x\right\rangle^2\right]^{1/2}\\ \\ \ln\sqrt{\frac{M}{m}}\cdot\left[\sum_{k=1}^{n}p_k\left\|A_k^px\right\|^2-\left\langle\sum_{k=1}^{n}p_kA_k^px,x\right\rangle^2\right]^{1/2}\end{cases}$$
$$\left[\leq\frac{1}{2}\cdot\frac{M^{-p}-m^{-p}}{M^{-p}m^{-p}}\ln\sqrt{\frac{M}{m}}\right]$$

for each $x\in H$ *with* $\|x\|=1$.

The following norm inequalities may be stated as well:

Corollary 4. *Let A_j be selfadjoint operators with $Sp(A_j) \subseteq [m, M]$ for $j \in \{1, ..., n\}$ and for some scalars $m < M$. If $f, g : [m, M] \longrightarrow \mathbb{R}$ are continuous, then for each $p_j \geq 0, j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} p_j = 1$ we have the norm inequality:*

$$\left\| \sum_{j=1}^{n} p_j f(A_j) g(A_j) \right\| \leq \left\| \sum_{j=1}^{n} p_j f(A_j) \right\| \cdot \left\| \sum_{j=1}^{n} p_j g(A_j) \right\| + \frac{1}{4} (\Gamma - \gamma)(\Delta - \delta), \tag{5.9}$$

where $\gamma := \min_{t \in [m,M]} f(t)$, $\Gamma := \max_{t \in [m,M]} f(t)$, $\delta := \min_{t \in [m,M]} g(t)$ *and* $\Delta := \max_{t \in [m,M]} g(t)$.

Proof. Utilising the inequality (5.2) we deduce the inequality

$$\left| \left\langle \sum_{k=1}^{n} p_k f(A_k) g(A_k) x, x \right\rangle \right| \leq \left| \left\langle \sum_{k=1}^{n} p_k f(A_k) x, x \right\rangle \right| \cdot \left| \left\langle \sum_{k=1}^{n} p_k g(A_k) x, x \right\rangle \right| + \frac{1}{4} (\Gamma - \gamma)(\Delta - \delta)$$

for each $x \in H$ with $\|x\| = 1$. Taking the supremum over $\|x\| = 1$ we deduce the desired inequality (5.9). ∎

Example 7. ***a.*** *Let A_j, $j \in \{1, ..., n\}$ be a selfadjoint operators with $Sp(A_j) \subseteq [m, M]$, $j \in \{1, ..., n\}$ for some scalars $m < M$ and $p_j \geq 0, j \in \{1, ..., n\}$ with* $\sum_{j=1}^{n} p_j = 1$.

If A_j, $j \in \{1, ..., n\}$ are positive $(m \geq 0)$ and $p, q > 0$, then

$$\left\| \sum_{k=1}^{n} p_k A_k^{p+q} \right\| \leq \left\| \sum_{k=1}^{n} p_k A_k^{p} \right\| \cdot \left\| \sum_{k=1}^{n} p_k A_k^{q} \right\| + \frac{1}{4} \cdot (M^p - m^p)(M^q - m^q). \tag{5.10}$$

If A_j, $j \in \{1, ..., n\}$ are positive definite $(m > 0)$ and $p, q < 0$, then

$$\left\| \sum_{k=1}^{n} p_k A_k^{p+q} \right\| \leq \left\| \sum_{k=1}^{n} p_k A_k^{p} \right\| \cdot \left\| \sum_{k=1}^{n} p_k A_k^{q} \right\| + \frac{1}{4} \cdot \frac{M^{-p} - m^{-p}}{M^{-p} m^{-p}} \frac{M^{-q} - m^{-q}}{M^{-q} m^{-q}}. \tag{5.11}$$

b. *Let A_j, $j \in \{1, ..., n\}$ be positive definite operators with $Sp(A_j) \subseteq [m, M]$, $j \in \{1, ..., n\}$ for some scalars $0 < m < M$ and $p_j \geq 0, j \in \{1, ..., n\}$ with* $\sum_{j=1}^{n} p_j = 1$.

If $p > 0$ then

$$\left\| \sum_{k=1}^{n} p_k A_k^{p} \ln A_k \right\| \leq \left\| \sum_{k=1}^{n} p_k A_k^{p} \right\| \cdot \left\| \sum_{k=1}^{n} p_k \ln A_k \right\| + \frac{1}{2} \cdot (M^p - m^p) \ln \sqrt{\frac{M}{m}}. \tag{5.12}$$

References

[1] G.A. Anastassiou, Grüss type inequalities for the Stieltjes integral. *Nonlinear Funct. Anal. Appl.* **12** (2007), no. 4, 583–593.

[2] G. A. Anastassiou, Chebyshev-Grüss type and comparison of integral means inequalities for the Stieltjes integral. *Panamer. Math. J.* **17** (2007), no. 3, 91–109.

[3] G. A. Anastassiou, Chebyshev-Grüss type inequalities via Euler type and Fink identities. *Math. Comput. Modelling* **45** (2007), no. 9-10, 1189–1200

[4] P. Cerone, On some results involving the Čebyšev functional and its generalisations. *J. Inequal. Pure Appl. Math.* **4** (2003), no. 3, Article 55, 17 pp.

[5] P. Cerone, On Chebyshev functional bounds. *Differential & difference equations and applications*, 267–277, Hindawi Publ. Corp., New York, 2006.

[6] P. Cerone, On a Čebyšev-type functional and Grüss-like bounds. *Math. Inequal. Appl.* **9** (2006), no. 1, 87–102.

[7] P. Cerone and S. S. Dragomir, A refinement of the Grüss inequality and applications. *Tamkang J. Math.* **38** (2007), no. 1, 37–49.
[8] P. Cerone and S. S. Dragomir, New bounds for the Čebyšev functional. *Appl. Math. Lett.* **18** (2005), no. 6, 603–611.
[9] P. Cerone and S. S. Dragomir, Chebychev functional bounds using Ostrowski seminorms. *Southeast Asian Bull. Math.* **28** (2004), no. 2, 219–228.
[10] S.S. Dragomir, Grüss inequality in inner product spaces, *The Australian Math Soc. Gazette*, **26** (1999), No. 2, 66-70.
[11] S.S. Dragomir, A generalization of Grüss' inequality in inner product spaces and applications, *J. Math. Anal. Appl.*, **237** (1999), 74-82.
[12] S.S. Dragomir, Some discrete inequalities of Grüss type and applications in guessing theory, *Honam Math. J.*, **21**(1) (1999), 145-156.
[13] S.S. Dragomir, Some integral inequalities of Grüss type, *Indian J. of Pure and Appl. Math.,* **31**(4) (2000), 397-415.
[14] S.S. Dragomir and G.L. Booth, On a Grüss-Lupaş type inequality and its applications for the estimation of p-moments of guessing mappings, *Mathematical Communications,* **5** (2000), 117-126.
[15] S.S. Dragomir, A Grüss type integral inequality for mappings of r-Hölder's type and applications for trapezoid formula, *Tamkang J. of Math.*, **31**(1) (2000), 43-47.
[16] S.S. Dragomir and I. Fedotov, An inequality of Grüss' type for Riemann-Stieltjes integral and applications for special means, *Tamkang J. of Math.*, **29**(4)(1998), 286-292.
[17] S.S. Dragomir, Čebyšev's type inequalities for functions of selfadjoint operators in Hilbert spaces, Preprint *RGMIA Res. Rep. Coll.*, **11**(e) (2008), Art. 9. [ONLINE: `http://www.staff.vu.edu.au/RGMIA/v11(E).asp`]
[18] A.M. Fink, A treatise on Grüss' inequality, *Analytic and Geometric Inequalities,* 93-113, Math. Appl. 478, Kluwer Academic Publ., 1999.
[19] G. Grüss, Über das Maximum des absoluten Betrages von $\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2}\int_a^b f(x)dx \int_a^b g(x)dx$, *Math. Z.* , **39**(1935), 215-226.
[20] T. Furuta, J. Mićić Hot, J. Pečarić and Y. Seo, *Mond-Pečarić Method in Operator Inequalities. Inequalities for Bounded Selfadjoint Operators on a Hilbert Space*, Element, Zagreb, 2005.
[21] A. Matković, J. Pečarić and I. Perić, A variant of Jensen's inequality of Mercer's type for operators with applications. *Linear Algebra Appl.* **418** (2006), no. 2-3, 551–564.
[22] D.S. Mitrinović, J.E. Pečarić and A.M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht, 1993.
[23] B. Mond and J. Pečarić, On some operator inequalities, *Indian J. Math.*, **35**(1993), 221-232.
[24] B. Mond and J. Pečarić, Classical inequalities for matrix functions, *Utilitas Math.*, **46**(1994), 155-166.
[25] B. G. Pachpatte, A note on Grüss type inequalities via Cauchy's mean value theorem. *Math. Inequal. Appl.* **11** (2008), no. 1, 75–80.
[26] J. Pečarić, J. Mićić and Y. Seo, Inequalities between operator means based on the Mond-Pečarić method. *Houston J. Math.* **30** (2004), no. 1, 191–207
[27] C.-J. Zhao and W.-S. Cheung, On multivariate Grüss inequalities. *J. Inequal. Appl.* **2008**, Art. ID 249438, 8 pp.

Research Group in Mathematical Inequalities & Applications, School of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: `sever.dragomir@vu.edu.au`

URL: `http://www.staff.vu.edu.au/rgmia/dragomir/`