INNER PRODUCT INEQUALITIES FOR TWO EQUIVALENT NORMS AND APPLICATIONS

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Abstract. Some inequalities for two inner products $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$ which generate the equivalent norms $\|\cdot\|_1$ and $\|\cdot\|_2$ with applications for invertible bounded linear operators, positive definite self-adjoint operators, integral and discrete inequalities are given.

1. Introduction

Let $(H,\langle\cdot,\cdot\rangle)$ be an inner product over the real or complex number field $\mathbb{K}$. The following inequality

$$|\langle x,y\rangle| \leq \|x\|\,\|y\|\,, \qquad x,y\in H \tag{1.1}$$

is well known in the literature as *Schwarz's inequality.* It plays an essential role in obtaining various results in the Geometry of Inner Product Spaces as well as in its applications in Operator Theory, Approximation Theory and other fields.

Due to the fact that

$$|\mathrm{Re}\,\langle x,y\rangle| \leq \|x\|\,\|y\|\,, \qquad x,y\in H, \tag{1.2}$$

we can introduce the angle between the vectors x,y, denoted by $\Phi_{x,y}$, through the formula

$$\cos\Phi_{xy} := \frac{\mathrm{Re}\,\langle x,y\rangle}{\|x\|\,\|y\|}, \qquad x,y\neq 0. \tag{1.3}$$

As observed by Krein in 1969, [6] (see also [5, p. 56]), the following interesting inequality holds:

$$\Phi_{xz}\leq\Phi_{xy}+\Phi_{yz} \text{ for any } x,y,z\in H\backslash\{0\}\,.$$

We now recall some inequalities in which the quantity $|\langle x,y\rangle|/(\|x\|\,\|y\|)$ for different vectors is involved:

$$\begin{aligned} 3\cdot\left|\frac{\langle x,y\rangle\langle y,z\rangle\langle z,x\rangle}{\|x\|^2\|y\|^2\|z\|^2}\right| &\leq \left|\frac{\langle x,y\rangle}{\|x\|\,\|y\|}\right|^2+\left|\frac{\langle y,z\rangle}{\|y\|\,\|z\|}\right|^2+\left|\frac{\langle z,x\rangle}{\|z\|\,\|x\|}\right|^2 \\ &\leq 1+2\cdot\left|\frac{\langle x,y\rangle\langle y,z\rangle\langle z,x\rangle}{\|x\|^2\|y\|^2\|z\|^2}\right| \qquad [2,\text{ p. } 37] \end{aligned} \tag{1.4}$$

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$$1 \geq \left| \frac{\langle x, y\rangle}{\|x\| \|y\|} - \frac{\langle x, z\rangle \langle z, y\rangle}{\|x\| \|y\| \|z\|^2} \right| + \left| \frac{\langle x, z\rangle \langle z, y\rangle}{\|x\| \|y\| \|z\|^2} \right| \geq \left| \frac{\langle x, y\rangle}{\|x\| \|y\|} \right| \qquad [2, \text{ p. } 38] \tag{1.5}$$

$$\left| \frac{\langle x, y\rangle \langle y, z\rangle}{\|x\| \|z\| \|y\|^2} \right| \leq \frac{1}{2} \cdot \left[1 + \left| \frac{\langle x, z\rangle}{\|x\| \|z\|} \right| \right] \qquad \text{(Buzano's inequality, [2, p. 49])} \tag{1.6}$$

$$\left| \frac{\langle x, y\rangle \langle y, z\rangle}{\|x\| \|z\| \|y\|^2} - \frac{1}{1+\eta} \cdot \frac{\langle x, z\rangle}{\|x\| \|z\|} \right| \leq \frac{1}{\sqrt{2}\sqrt{1 + \operatorname{Re} \eta}} \qquad [2, \text{ p. } 51] \tag{1.7}$$

where $\eta \in \mathbb{K}$ and $|\eta| = 1$, $\operatorname{Re} \eta \neq -1$, and

$$\left| \frac{\langle x, y\rangle \langle y, z\rangle}{\|x\| \|z\| \|y\|^2} \right| \leq \left| \frac{\langle x, y\rangle \langle y, z\rangle}{\|x\| \|z\| \|y\|^2} - \frac{1}{2} \frac{\langle x, z\rangle}{\|x\| \|z\|} \right| + \frac{1}{2} \cdot \left| \frac{\langle x, z\rangle}{\|x\| \|z\|} \right| \leq \frac{1}{2} \cdot \left[1 + \left| \frac{\langle x, z\rangle}{\|x\| \|z\|} \right| \right] \qquad [2, \text{ p. } 52], \tag{1.8}$$

where $x, y, z \in H \backslash \{0\}$.

We notice that (1.8) is a refinement of Buzano's inequality (1.6).

For other inequalities of this type, see [1], [4], [7], [8] and [9].

Motivated by the above results, the main aim of the present paper is to compare the quantities

$$\frac{|\langle x, y\rangle|_1}{\|x\|_1 \|y\|_1} \left(\frac{\operatorname{Re} \langle x, y\rangle_1}{\|x\|_1 \|y\|_1} \right) \qquad \text{and} \qquad \frac{|\langle x, y\rangle|_2}{\|x\|_2 \|y\|_2} \left(\frac{\operatorname{Re} \langle x, y\rangle_2}{\|x\|_2 \|y\|_2} \right)$$

in the case when the inner products $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ defined on H^2 generate two *equivalent norms*, i.e., we recall that $\|\cdot\|_1$ and $\|\cdot\|_2$ are *equivalent* if there exists the constants $m, M > 0$ such that

$$m \|x\|_2 \leq \|x\|_1 \leq M \|x\|_2, \qquad \text{for any } \ x \in H. \tag{1.9}$$

Applications for invertible bounded linear operators, positive definite self-adjoint operators, integral and discrete inequalities are also given.

2. The Results

The following result may be stated.

Theorem 1. *Assume that the inner products $\langle \cdot, \cdot \rangle_i$, $i \in \{1, 2\}$ on the real or complex linear space H generate the norms $\|\cdot\|_i$, $i \in \{1, 2\}$ which satisfy the following condition:*

$$m \|x\|_2 \leq \|x\|_1 \leq M \|x\|_2 \qquad \textit{for any } \ x \in H, \tag{2.1}$$

where $0 < m \leq M < \infty$ are given constants.

If $x, y \in H \backslash \{0\}$ satisfy the condition $\operatorname{Re} \langle x, y\rangle_2 \geq 0$, then

$$\frac{m^2}{M^2} - 1 + \frac{\operatorname{Re} \langle x, y\rangle_1}{\|x\|_1 \|y\|_1} \leq \frac{\operatorname{Re} \langle x, y\rangle_2}{\|x\|_2 \|y\|_2} \leq \frac{\operatorname{Re} \langle x, y\rangle_1}{\|x\|_1 \|y\|_1} + \frac{M^2}{m^2} - 1. \tag{2.2}$$

If $\operatorname{Re}\langle x,y\rangle_2<0$, *then*

$$1-\frac{m^2}{M^2}+\frac{\operatorname{Re}\langle x,y\rangle_1}{\|x\|_1\|y\|_1}\le\frac{\operatorname{Re}\langle x,y\rangle_2}{\|x\|_2\|y\|_2}\le\frac{\operatorname{Re}\langle x,y\rangle_1}{\|x\|_1\|y\|_1}+1-\frac{M^2}{m^2}. \tag{2.3}$$

Proof. For any inner product $\langle\cdot,\cdot\rangle$ on H, we have

$$1-\frac{\operatorname{Re}\langle x,y\rangle}{\|x\|\|y\|}=\frac{1}{2}\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|^2,\quad x,y\in H\setminus\{0\}. \tag{2.4}$$

Utilising the assumption (2.1), we have successively:

$$\begin{aligned}1-\frac{\operatorname{Re}\langle x,y\rangle_1}{\|x\|_1\|y\|_1}&=\frac{1}{2}\left\|\frac{x}{\|x\|_1}-\frac{y}{\|y\|_1}\right\|_1^2\le\frac{M^2}{2}\left\|\frac{x}{\|x\|_1}-\frac{y}{\|y\|_1}\right\|_2\\&=\frac{M^2}{2}\left[\frac{\|x\|_2^2}{\|x\|_1^2}+\frac{\|y\|_2^2}{\|y\|_1^2}-\frac{2\operatorname{Re}\langle x,y\rangle_2}{\|x\|_1\|y\|_1}\right]\\&\le\frac{M^2}{2}\left[\frac{1}{m^2}+\frac{1}{m^2}-2\frac{\operatorname{Re}\langle x,y\rangle_2}{\|x\|_1\|y\|_1}\right]\\&=M^2\left[\frac{1}{m^2}-\frac{\operatorname{Re}\langle x,y\rangle_2}{\|x\|_1\|y\|_1}\right]=:I.\end{aligned} \tag{2.5}$$

Now, if $\operatorname{Re}\langle x,y\rangle_2\ge 0$, then

$$-\frac{\operatorname{Re}\langle x,y\rangle_2}{\|x\|_1\|y\|_1}\le-\frac{\operatorname{Re}\langle x,y\rangle_2}{M^2\|x\|_2\|y\|_2}$$

which implies that

$$I\le M^2\left[\frac{1}{m^2}-\frac{\operatorname{Re}\langle x,y\rangle_2}{M^2\|x\|_2\|y\|_2}\right]=\frac{M^2}{m^2}-\frac{\operatorname{Re}\langle x,y\rangle_2}{\|x\|_2\|y\|_2}. \tag{2.6}$$

Utilising (2.5) and (2.6) we deduce

$$1-\frac{\operatorname{Re}\langle x,y\rangle_1}{\|x\|_1\|y\|_1}\le\frac{M^2}{m^2}-\frac{\operatorname{Re}\langle x,y\rangle_2}{\|x\|_2\|y\|_2},$$

which produces the second inequality in (2.2).

By (2.4) and (2.1) we also have

$$\begin{aligned}1-\frac{\operatorname{Re}\langle x,y\rangle_1}{\|x\|_1\|y\|_1}&\ge\frac{m^2}{2}\left\|\frac{x}{\|x\|_1}-\frac{y}{\|y\|_1}\right\|_2^2\\&=\frac{m^2}{2}\left[\frac{\|x\|_2^2}{\|x\|_1^2}+\frac{\|y\|_2^2}{\|y\|_1^2}-2\frac{\operatorname{Re}\langle x,y\rangle_2}{\|x\|_1\|y\|_1}\right]\\&\ge\frac{m^2}{2}\left[\frac{1}{M^2}+\frac{1}{M^2}-2\frac{\operatorname{Re}\langle x,y\rangle_2}{\|x\|_1\|y\|_1}\right]\\&=m^2\left[\frac{1}{M^2}-\frac{\operatorname{Re}\langle x,y\rangle_2}{\|x\|_1\|y\|_1}\right]=:J.\end{aligned} \tag{2.7}$$

Due to the fact that $\operatorname{Re}\langle x,y\rangle_2\ge 0$, we have

$$-\frac{\operatorname{Re}\langle x,y\rangle_2}{\|x\|_1\|y\|_1}\ge-\frac{\operatorname{Re}\langle x,y\rangle_2}{m^2\|x\|_2\|y\|_2},$$

which implies that

$$J \geq m^2 \left[\frac{1}{M^2} - \frac{\operatorname{Re}\langle x, y \rangle_2}{m^2 \|x\|_2 \|y\|_2} \right] = \frac{m^2}{M^2} - \frac{\operatorname{Re}\langle x, y \rangle_2}{\|x\|_2 \|y\|_2}. \tag{2.8}$$

By making use of (2.7) and (2.8) we get

$$1 - \frac{\operatorname{Re}\langle x, y \rangle_1}{\|x\|_1 \|y\|_1} \geq \frac{m^2}{M^2} - \frac{\operatorname{Re}\langle x, y \rangle_2}{\|x\|_2 \|y\|_2},$$

which is clearly equivalent to the first inequality in (2.2).

Finally, if $\operatorname{Re}\langle x, y \rangle_2 < 0$, then $\operatorname{Re}\langle x, -y \rangle_2 > 0$ and writing the inequality (2.2) for $-y$ instead of y, we easily deduce (2.3). □

Corollary 1. *Let $A \in B(H)$ be an invertible operator on the Hilbert space $(H; \langle \cdot, \cdot \rangle)$. If $x, y \in H \setminus \{0\}$ are such that $\operatorname{Re}\langle x, y \rangle \geq 0$, then:*

$$\begin{aligned} 1 - \|A\|^2 \left\|A^{-1}\right\|^2 + \frac{\operatorname{Re}\langle x, y \rangle}{\|x\| \|y\|} &\leq \frac{\operatorname{Re}\langle Ax, Ay \rangle}{\|Ax\| \|Ay\|} \\ &\leq \frac{\operatorname{Re}\langle x, y \rangle}{\|x\| \|y\|} + 1 - \frac{1}{\|A\|^2 \|A^{-1}\|^2}. \end{aligned} \tag{2.9}$$

If $x, y \in H \setminus \{0\}$ and $\operatorname{Re}\langle x, y \rangle < 0$, then

$$\begin{aligned} \|A\|^2 \left\|A^{-1}\right\|^2 - 1 + \frac{\operatorname{Re}\langle x, y \rangle}{\|x\| \|y\|} &\leq \frac{\operatorname{Re}\langle Ax, Ay \rangle}{\|Ax\| \|Ay\|} \\ &\leq \frac{\operatorname{Re}\langle x, y \rangle}{\|x\| \|y\|} + \frac{1}{\|A\|^2 \|A^{-1}\|^2} - 1. \end{aligned} \tag{2.10}$$

Proof. Since $A \in B(H)$ is invertible, then

$$\frac{1}{\|A^{-1}\|} \cdot \|x\| \leq \|Ax\| \leq \|A\| \|x\|, \qquad \text{for any } x \in H.$$

Applying Theorem 1 for $\langle x, y \rangle_1 := \langle Ax, Ay \rangle$, $\langle x, y \rangle_2 := \langle x, y \rangle$ and $m = \frac{1}{\|A^{-1}\|}$, $M = \|A\|$ and doing the necessary calculations, we deduce the desired result. □

Corollary 2. *Let $A \in B(H)$ be a self-adjoint operator on the Hilbert space $(H; \langle \cdot, \cdot \rangle)$ which satisfies the condition*

$$\gamma I \leq A \leq \Gamma I, \tag{2.11}$$

in the operator order of $B(H)$, where $0 < \gamma \leq \Gamma < \infty$ are given.

If $x, y \in H \setminus \{0\}$ are such that $\operatorname{Re}\langle x, y \rangle \geq 0$, then

$$1 - \frac{\Gamma}{\gamma} + \frac{\operatorname{Re}\langle x, y \rangle}{\|x\| \|y\|} \leq \frac{\operatorname{Re}\langle Ax, y \rangle}{[\langle Ax, x \rangle \langle Ay, y \rangle]^{\frac{1}{2}}} \leq \frac{\operatorname{Re}\langle x, y \rangle}{\|x\| \|y\|} + 1 - \frac{\gamma}{\Gamma}. \tag{2.12}$$

If $x, y \in H \setminus \{0\}$ are such that $\operatorname{Re}\langle x, y \rangle < 0$, then

$$\frac{\Gamma}{\gamma} - 1 + \frac{\operatorname{Re}\langle x, y \rangle}{\|x\| \|y\|} \leq \frac{\operatorname{Re}\langle Ax, y \rangle}{[\langle Ax, x \rangle \langle Ay, y \rangle]^{\frac{1}{2}}} \leq \frac{\operatorname{Re}\langle x, y \rangle}{\|x\| \|y\|} + \frac{\gamma}{\Gamma} - 1. \tag{2.13}$$

Proof. From (2.11) we have

$$\gamma \langle x, x \rangle \leq \langle Ax, x \rangle \leq \Gamma \langle x, x \rangle, \qquad x \in H,$$

which implies that $\sqrt{\gamma} \|x\| \leq [\langle Ax, x \rangle]^{1/2} \leq \sqrt{\Gamma} \|x\|$, $x \in H$.

Now, if we apply Theorem 1 for $\langle x,y\rangle_1 := \langle Ax,y\rangle$, $\langle x,y\rangle_2 := \langle x,y\rangle$, $x,y \in H$ and $m=\sqrt{\gamma}$, $M=\sqrt{\Gamma}$, then we obtain the desired result. □

The following lemma is of interest in itself.

Lemma 1. *Assume that the inner products $\langle\cdot,\cdot\rangle_i$, $i\in\{1,2\}$ defined on H satisfy the condition (2.1). Then for any $x,y\in H$ we have*

$$m^2\left[\|x\|_2\|y\|_2-|\langle x,y\rangle_2|\right]\leq\|x\|_1\|y\|_1-|\langle x,y\rangle|_1 \leq M^2\left[\|x\|_2\|y\|_2-|\langle x,y\rangle_2|\right] \tag{2.14}$$

and

$$m^2\left[\|x\|_2\|y\|_2-\operatorname{Re}\langle x,y\rangle_2\right]\leq\|x\|_1\|y\|_1-\operatorname{Re}\langle x,y\rangle_1 \leq M^2\left[\|x\|_2\|y\|_2-\operatorname{Re}\langle x,y\rangle_2\right], \tag{2.15}$$

respectively.

Proof. We use the following result obtained by Dragomir and Mond in [3] (see also [2, p. 9]):

If $[\cdot,\cdot]_1$, $[\cdot,\cdot]_2$ are two hermitian forms on H with $[x,x]_1^{1/2}\leq[x,x]_2^{1/2}$ for any $x\in H$, then

$$[x,x]_1^{1/2}[y,y]_1^{1/2}-|[x,y]_1|\leq[x,x]_2^{1/2}[y,y]_2^{1/2}-|[x,y]_2| \tag{2.16}$$

and

$$[x,x]_1^{1/2}[y,y]_1^{1/2}-\operatorname{Re}[x,y]_1\leq[x,x]_2^{1/2}[y,y]_2^{1/2}-\operatorname{Re}[x,y]_2 \tag{2.17}$$

for any $x,y\in H$.

Now, if we apply (2.16) and (2.17) firstly for $[\cdot,\cdot]_2:=\langle\cdot,\cdot\rangle_1$, $[\cdot,\cdot]_1=m^2\langle\cdot,\cdot\rangle_2$ and then for $[\cdot,\cdot]_2=M^2\langle\cdot,\cdot\rangle_2$, $[\cdot,\cdot]_1:=\langle\cdot,\cdot\rangle_1$, we deduce the desired results (2.14) and (2.15). □

The following result may be stated as well.

Theorem 2. *Assume that the inner products $\langle\cdot,\cdot\rangle_i$, $i\in\{1,2\}$ satisfy the condition (2.1). Then for any $x,y\in H\backslash\{0\}$, we have the inequalities:*

$$\frac{m^2}{M^2}-1+\frac{|\langle x,y\rangle_1|}{\|x\|_1\|y\|_1}\leq\frac{|\langle x,y\rangle_2|}{\|x\|_2\|y\|_2}\leq\frac{|\langle x,y\rangle_1|}{\|x\|_1\|y\|_1}+\frac{M^2}{m^2}-1. \tag{2.18}$$

Proof. Dividing the inequality (2.14) by $\|x\|_1\|y\|_1\neq0$, we obtain

$$m^2\left[\frac{\|x\|_2\|y\|_2}{\|x\|_1\|y\|_1}-\frac{|\langle x,y\rangle_2|}{\|x\|_1\|y\|_1}\right]\leq1-\frac{|\langle x,y\rangle_1|}{\|x\|_1\|y\|_1} \leq M^2\left[\frac{\|x\|_2\|y\|_2}{\|x\|_1\|y\|_1}-\frac{|\langle x,y\rangle_2|}{\|x\|_1\|y\|_1}\right], \tag{2.19}$$

for any $x,y\in H\backslash\{0\}$.

Observe, by (2.1) that:

$$\frac{\|x\|_2\|y\|_2}{\|x\|_1\|y\|_1}\leq\frac{1}{m^2}\quad\text{and}\quad-\frac{|\langle x,y\rangle_2|}{\|x\|_1\|y\|_1}\leq-\frac{|\langle x,y\rangle_2|}{M^2\|x\|_2\|y\|_2}.$$

Utilising the second inequality in (2.19), we deduce

$$1-\frac{|\langle x,y\rangle_1|}{\|x\|_1\|y\|_1}\leq\frac{M^2}{m^2}-\frac{|\langle x,y\rangle_2|}{\|x\|_2\|y\|_2},$$

which is equivalent with the second inequality in (2.18).

In addition, we have

$$\frac{1}{M^2} \leq \frac{\|x\|_2 \|y\|_2}{\|x\|_1 \|y\|_1} \quad \text{and} \quad -\frac{|\langle x,y\rangle_2|}{m^2 \|x\|_2 \|y\|_2} \leq -\frac{|\langle x,y\rangle_1|}{\|x\|_1 \|y\|_1}$$

which together with the first inequality in (2.19) produces the first part of (2.18). □

Remark 1. *On utilising the inequality (2.15) and an argument similar to the one in the proof of Theorem 2, we can reobtain the inequalities (2.2) and (2.3) from Theorem 1. The details are omitted.*

Corollary 3. *Let $A \in B(H)$ be an invertible operator on the Hilbert space $(H; \langle\cdot,\cdot\rangle)$. Then*

$$1 - \|A\|^2 \left\|A^{-1}\right\|^2 + \frac{|\langle x,y\rangle|}{\|x\| \|y\|} \leq \frac{|\langle Ax,Ay\rangle|}{\|Ax\| \|Ay\|} \leq \frac{|\langle x,y\rangle|}{\|x\| \|y\|} + 1 - \frac{1}{\|A\|^2 \|A^{-1}\|^2} \tag{2.20}$$

for any $x, y \in H\backslash \{0\}$.

The proof follows from Theorem 2 on choosing $\langle x,y\rangle_1 := \langle Ax,Ay\rangle$, $\langle x,y\rangle_2 := \langle x,y\rangle$, $x,y \in H$ and $m = \frac{1}{\|A^{-1}\|}$, $M = \|A\|$.

Corollary 4. *Let $A \in B(H)$ be a self-adjoint operator satisfying the condition (2.11). Then*

$$1 - \frac{\Gamma}{\gamma} + \frac{|\langle x,y\rangle|}{\|x\| \|y\|} \leq \frac{|\langle Ax,y\rangle|}{[\langle Ax,x\rangle \langle Ay,y\rangle]^{\frac{1}{2}}} \leq \frac{|\langle x,y\rangle|}{\|x\| \|y\|} + 1 - \frac{\gamma}{\Gamma}, \tag{2.21}$$

for any $x, y \in H\backslash \{0\}$.

The proof follows by Theorem 2 on choosing $\langle x,y\rangle_1 := \langle Ax,y\rangle$, $\langle x,y\rangle_2 := \langle x,y\rangle$, $x,y \in H$ and $m = \sqrt{\gamma}$, $M = \sqrt{\Gamma}$.

3. Applications for Integral Inequalities

Assume that $(K; \langle\cdot,\cdot\rangle)$ is a Hilbert space over the real or complex number field $\mathbb{K}$. If $\rho : [a,b] \subset \mathbb{R} \to [0,\infty)$ is a measurable function then we may consider the space $L^2_\rho([a,b]; K)$ of all functions $f : [a,b] \to K$ that are strongly measurable and $\int_a^b \rho(t) \|f(t)\|^2 dt < \infty$. It is well known that $L^2_\rho([a,b]; K)$ endowed with the inner product $\langle\cdot,\cdot\rangle_\rho$ defined by

$$\langle f,g\rangle_\rho := \int_a^b \rho(t) \langle f(t), g(t)\rangle \, dt$$

and generating the norm

$$\|f\|_\rho := \left(\int_a^b \rho(t) \|f(t)\|^2 dt\right)^{\frac{1}{2}}$$

is a Hilbert space over $\mathbb{K}$.

The following proposition can be stated.

Proposition 1. *Let $\rho_1, \rho_2 : [a,b] \to [0,\infty)$ be two measurable functions with the property that there exists $0 < \phi \leq \Phi < \infty$ so that*

$$\phi \leq \frac{\rho_1(t)}{\rho_2(t)} \leq \Phi \quad \textit{for a.e.} \quad t \in [a,b]. \tag{3.1}$$

Then for any $f, g \in L^2_{\rho_2}([a,b];K)$ we have the inequalities:

$$\begin{aligned} &\frac{\phi}{\Phi} - 1 + \frac{\left|\int_a^b \rho_1(t)\langle f(t), g(t)\rangle dt\right|}{\left(\int_a^b \rho_1(t)\|f(t)\|^2 dt\right)^{\frac{1}{2}} \left(\int_a^b \rho_2(t)\|g(t)\|^2 dt\right)^{\frac{1}{2}}} \\ &\leq \frac{\left|\int_a^b \rho_2(t)\langle f(t), g(t)\rangle dt\right|}{\left(\int_a^b \rho_2(t)\|f(t)\|^2 dt\right)^{\frac{1}{2}} \left(\int_a^b \rho_2(t)\|g(t)\|^2 dt\right)^{\frac{1}{2}}} \\ &\leq \frac{\left|\int_a^b \rho_1(t)\langle f(t), g(t)\rangle dt\right|}{\left(\int_a^b \rho_1(t)\|f(t)\|^2 dt\right)^{\frac{1}{2}} \left(\int_a^b \rho_2(t)\|g(t)\|^2 dt\right)^{\frac{1}{2}}} + \frac{\Phi}{\phi} - 1. \end{aligned} \tag{3.2}$$

Proof. From (3.1) we have

$$\begin{aligned} \sqrt{\phi}\left(\int_a^b \rho_2(t)\|f(t)\|^2 dt\right)^{\frac{1}{2}} &\leq \left(\int_a^b \rho_1(t)\|f(t)\|^2 dt\right)^{\frac{1}{2}} \\ &\leq \sqrt{\Phi}\left(\int_a^b \rho_2(t)\|f(t)\|^2 dt\right)^{\frac{1}{2}}. \end{aligned}$$

Applying Theorem 2 for $\langle\cdot,\cdot\rangle_i = \langle\cdot,\cdot\rangle_{\rho_i}$, $i \in \{1,2\}$ and $H = L^2_{\rho_1}([a,b];K) = L^2_{\rho_2}([a,b];K)$, we deduce the desired result. □

Remark 2. *A similar result can be stated if one uses Theorem 1. The details are omitted.*

4. Applications for Discrete Inequalities

Assume that $(K;\langle\cdot,\cdot\rangle)$ is a Hilbert space over the real or complex number field and $\mathbf{p} = (p_i)_{i\in\mathbb{N}}$ with $p_i \geq 0$, $i \in \mathbb{N}$ and $\sum_{i=1}^{\infty} p_i < \infty$. Define

$$\ell_p^2(K) := \left\{ \mathbf{x} := (x_i)_{i\in\mathbb{N}} \;\middle|\; x_i \in K,\ i \in \mathbb{N} \text{ and } \sum_{i=1}^{\infty} p_i \|x_i\|^2 < \infty \right\}.$$

It is well known that $\ell_p^2(K)$ endowed with the inner product

$$\langle \mathbf{x}, \mathbf{y}\rangle_p := \sum_{i=1}^{n} p_i \langle x_i, y_i\rangle$$

and generating the norm

$$\|\mathbf{x}\|_p := \left(\sum_{i=1}^{\infty} p_i \|x_i\|^2\right)^{\frac{1}{2}}$$

is a Hilbert space over $\mathbb{K}$.

Proposition 2. *Assume that* $\mathbf{p} = (p_i)_{i\in\mathbb{N}}$, $\mathbf{q} = (q_i)_{i\in\mathbb{N}}$ *are such that* $p_i, q_i \geq 0$, $i \in \mathbb{N}$, $\sum_{i=1}^{\infty} p_i, \sum_{i=1}^{\infty} q_i < \infty$ *and*

$$nq_i \leq p_i \leq Nq_i \quad \text{for any} \quad i \in \mathbb{N}, \tag{4.1}$$

where $0 < n \leq N < \infty$. *Then we have the inequality*

$$\begin{aligned} \frac{n}{N} - 1 + \frac{\left|\sum_{i=1}^{\infty} p_i \langle x_i, y_i \rangle\right|}{\left(\sum_{i=1}^{\infty} p_i \|x_i\|^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{\infty} p_i \|y_i\|^2\right)^{\frac{1}{2}}} \\ \leq \frac{\left|\sum_{i=1}^{\infty} q_i \langle x_i, y_i \rangle\right|}{\left(\sum_{i=1}^{\infty} q_i \|x_i\|^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{\infty} q_i \|y_i\|^2\right)^{\frac{1}{2}}} \\ \leq \frac{\left|\sum_{i=1}^{\infty} p_i \langle x_i, y_i \rangle\right|}{\left(\sum_{i=1}^{\infty} p_i \|x_i\|^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{\infty} p_i \|y_i\|^2\right)^{\frac{1}{2}}} + \frac{N}{n} - 1. \end{aligned} \tag{4.2}$$

A similar result can be stated if one uses Theorem 1. However the details are omitted.

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