Refinements of the Cauchy-Bunyakovsky-Schwarz Inequality for Functions of Selfadjoint Operators in Hilbert Spaces

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Abstract. Some inequalities for continuous functions of selfadjoint operators in Hilbert spaces that improve the Cauchy-Bunyakovsky-Schwarz inequality, are given.

1. Introduction

In [**1**], Daykin, Elizer and Carlitz obtained the following refinement of the Cauchy-Bunyakovsky-Schwarz inequality, which, in the version from [**5**, p. 87], can be stated as:

$$\text{(DEC)}\qquad \left(\sum_{i=1}^{n} a_i b_i\right)^2 \le \sum_{i=1}^{n} \varphi(a_i, b_i) \sum_{i=1}^{n} \psi(a_i, b_i) \le \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2,$$

where and $a_i, b_i \in [0,\infty)$ for each $i \in \{1, ..., n\}$ and (φ, ψ) is a pair of functions defined on $[0,\infty) \times [0,\infty)$ and satisfying the conditions

(i) $\varphi(a,b)\psi(a,b) = a^2b^2$ for any $a, b \in [0,\infty)$;

(ii) $\varphi(ka, kb) = k^2\varphi(a,b)$ for any $a, b, k \in [0,\infty)$;

(iii) $\frac{b\varphi(a,1)}{a\varphi(b,1)} + \frac{a\varphi(b,1)}{b\varphi(a,1)} \le \frac{a}{b} + \frac{b}{a}$ for any $a, b \in (0,\infty)$.

As examples of such pairs of functions, which will be called for simplicity *(DEC)-pairs,* we can indicate the following functions: $\varphi(a,b) = a^2 + b^2$, $\psi(a,b) = \frac{a^2b^2}{a^2+b^2}$ and $\varphi(a,b) = a^{1+\alpha}b^{1-\alpha}$, $\psi(a,b) = a^{1-\alpha}b^{1+\alpha}$ with $\alpha \in [0,1]$. The first pair generates the famous *Milne's inequality:*

$$(1.1)\qquad \left(\sum_{i=1}^{n} a_i b_i\right)^2 \le \sum_{i=1}^{n} \left(a_i^2 + b_i^2\right) \sum_{i=1}^{n} \frac{a_i^2 b_i^2}{a_i^2 + b_i^2} \le \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2,$$

while the second generates the *Callebaut's inequality:*

$$(1.2)\qquad \left(\sum_{i=1}^{n} a_i b_i\right)^2 \le \sum_{i=1}^{n} a_i^{1+\alpha} b_i^{1-\alpha} \sum_{i=1}^{n} a_i^{1-\alpha} b_i^{1+\alpha} \le \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2.$$

It is an open problem for the author to find other nice and simple examples of such pair of functions.

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In order to state the operator version of this result we recall the Gelfand functional calculus.

Let A be a selfadjoint linear operator on a complex Hilbert space $(H;\langle .,.\rangle)$. The *Gelfand map* establishes a $*$-isometrically isomorphism Φ between the set $C(Sp(A))$ of all *continuous functions* defined on the *spectrum* of A, denoted $Sp(A)$, an the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows (see for instance [**4**, p. 3]):

For any $f,g \in C(Sp(A))$ and any $\alpha,\beta \in \mathbb{C}$ we have

(i) $\Phi(\alpha f+\beta g)=\alpha\Phi(f)+\beta\Phi(g)$;

(ii) $\Phi(fg)=\Phi(f)\Phi(g)$ and $\Phi(\bar{f})=\Phi(f)^*$;

(iii) $\|\Phi(f)\|=\|f\|:=\sup_{t\in Sp(A)}|f(t)|$;

(iv) $\Phi(f_0)=1_H$ and $\Phi(f_1)=A$, where $f_0(t)=1$ and $f_1(t)=t$, for $t\in Sp(A)$.

With this notation we define

$$f(A):=\Phi(f) \text{ for all } f\in C(Sp(A))$$

and we call it the *continuous functional calculus* for a selfadjoint operator A.

If A is a selfadjoint operator and f is a real valued continuous function on $Sp(A)$, then $f(t)\geq 0$ for any $t\in Sp(A)$ implies that $f(A)\geq 0$, *i.e.* $f(A)$ is a positive operator on H. Moreover, if both f and g are real valued functions on $Sp(A)$ then the following important property holds:

$$\text{(P)} \qquad f(t)\geq g(t) \text{ for any } t\in Sp(A) \text{ implies that } f(A)\geq g(A)$$

in the operator order of $B(H)$.

For a recent monograph devoted to various inequalities for functions of selfadjoint operators, see [**4**] and the references therein.

For other results conserning functions of selfadjoint operators, see [**2**], [**3**], [**7**], [**6**] and [**8**].

2. A Two Operators Version

The following result may be stated:

Theorem 1. *Let (φ,ψ) be a (DEC)-pair of continuous functions on $[0,\infty)\times[0,\infty)$. If A,B are selfadjoint operators on the Hilbert space $(H;\langle .,.\rangle)$ with $Sp(A)$, $Sp(B)\subseteq[m,M]$ for some scalars $m<M$ and if f and g are continuous on $[m,M]$ and with values in $[0,\infty)$, then we have the inequality*

$$\begin{aligned}(2.1)\quad 2\langle f(A)g(A)x,x\rangle\langle f(B)g(B)y,y\rangle &\leq \langle\varphi(f(A),g(A))x,x\rangle\langle\psi(f(B),g(B))y,y\rangle\\ &\quad+\langle\psi(f(A),g(A))x,x\rangle\langle\varphi(f(B),g(B))y,y\rangle\\ &\leq\left\langle f^2(A)x,x\right\rangle\left\langle g^2(B)y,y\right\rangle+\left\langle g^2(A)x,x\right\rangle\left\langle f^2(B)y,y\right\rangle\end{aligned}$$

for any $x,y\in H$ with $\|x\|=\|y\|=1$.

Proof. We observe that from the property (iii) we have the inequality

$$2\leq\frac{b\varphi(a,1)}{a\varphi(b,1)}+\frac{a\varphi(b,1)}{b\varphi(a,1)}\leq\frac{a}{b}+\frac{b}{a},$$

for any $a,b>0$.

If in this inequality we choose $a=\frac{u}{v}$ and $b=\frac{z}{w}$, then we get

$$2\leq\frac{zv\varphi\left(\frac{u}{v},1\right)}{uw\varphi\left(\frac{z}{w},1\right)}+\frac{uw\varphi\left(\frac{z}{w},1\right)}{zv\varphi\left(\frac{u}{v},1\right)}\leq\frac{uw}{vz}+\frac{vz}{uw}. \tag{2.2}$$

From the property (ii) we have

$$zv\varphi\left(\frac{u}{v},1\right)=\frac{z}{v}\varphi\left(u,v\right)\text{ and }uw\varphi\left(\frac{z}{w},1\right)=\frac{u}{w}\varphi\left(z,w\right)$$

which give from (2.2) that

$$2\leq\frac{zw\varphi\left(u,v\right)}{uv\varphi\left(z,w\right)}+\frac{uv\varphi\left(z,w\right)}{zw\varphi\left(u,v\right)}\leq\frac{uw}{vz}+\frac{vz}{uw}, \tag{2.3}$$

for any $u,v,z,w>0$.

Utilising the property (i) we have

$$\varphi\left(z,w\right)=\frac{z^{2}w^{2}}{\psi\left(z,w\right)}\text{ and }\varphi\left(u,v\right)=\frac{u^{2}v^{2}}{\psi\left(u,v\right)},$$

which, from (2.3), produces the inequality

$$2\leq\frac{\varphi\left(u,v\right)\psi\left(z,w\right)}{zwuv}+\frac{\varphi\left(z,w\right)\psi\left(u,v\right)}{uvzw}\leq\frac{uw}{vz}+\frac{vz}{uw},$$

i.e., the inequality

$$2uvzw\leq\varphi\left(u,v\right)\psi\left(z,w\right)+\varphi\left(z,w\right)\psi\left(u,v\right)\leq u^{2}w^{2}+v^{2}z^{2}, \tag{2.4}$$

for any $u,v,z,w\geq0$.

Now, if we choose $u=f\left(s\right),v=g\left(s\right),z=f\left(t\right)$ and $w=g\left(t\right)$ in (2.4) then we get

$$\begin{aligned} &2f\left(s\right)g\left(s\right)f\left(t\right)g\left(t\right)\\ &\leq\varphi\left(f\left(s\right),g\left(s\right)\right)\psi\left(f\left(t\right),g\left(t\right)\right)+\varphi\left(f\left(t\right),g\left(t\right)\right)\psi\left(f\left(s\right),g\left(s\right)\right)\\ &\leq f^{2}\left(s\right)g^{2}\left(t\right)+g^{2}\left(s\right)f^{2}\left(t\right)\end{aligned} \tag{2.5}$$

for any $s,t\in\left[m,M\right]$.

Further, if we fix $t\in\left[m,M\right]$ and apply the property (P) for the operator A, then we get the inequality

$$\begin{aligned} &2f\left(t\right)g\left(t\right)\left\langle f\left(A\right)g\left(A\right)x,x\right\rangle\\ &\leq\psi\left(f\left(t\right),g\left(t\right)\right)\left\langle\varphi\left(f\left(A\right),g\left(A\right)\right)x,x\right\rangle+\varphi\left(f\left(t\right),g\left(t\right)\right)\left\langle\psi\left(f\left(A\right),g\left(A\right)\right)x,x\right\rangle\\ &\leq g^{2}\left(t\right)\left\langle f^{2}\left(A\right)x,x\right\rangle+f^{2}\left(t\right)\left\langle g^{2}\left(A\right)x,x\right\rangle\end{aligned} \tag{2.6}$$

for any $x\in H$ with $\|x\|=1$.

Now, if we fix $x\in H$ with $\|x\|=1$ and apply the same property (P) for the inequality (2.6) and the operator B, then we get the desired inequality (2.1). ∎

The following particular case is of interest:

Corollary 1. *Let (φ,ψ) be a (DEC)-pair of continuous functions on $[0,\infty)\times[0,\infty)$. If A is a selfadjoint operator on the Hilbert space $(H;\langle.,.\rangle)$ with $Sp\left(A\right)\subseteq$*

$[m,M]$ for some scalars $m<M$ and if f and g are continuous on $[m,M]$ and with values in $[0,\infty)$, then we have the inequality

$$\begin{aligned}(2.7)\quad &\langle f(A)\,g(A)\,x,x\rangle^2\\ &\le \langle \varphi(f(A),g(A))\,x,x\rangle\langle \psi(f(A),g(A))\,x,x\rangle \le \langle f^2(A)\,x,x\rangle\langle g^2(A)\,x,x\rangle\end{aligned}$$

for any $x\in H$, $\|x\|=1$.

Remark 1. ***a.*** *If A is a selfadjoint operator on the Hilbert space $(H;\langle .,.\rangle)$ with $Sp(A)\subseteq[m,M]$ for some scalars $m<M$ and if f and g are continuous on $[m,M]$ and with values in $[0,\infty)$, then we have the inequality*

$$\begin{aligned}(2.8)\quad &\langle f(A)\,g(A)\,x,x\rangle^2\\ &\le \left\langle \left[f^{1+\alpha}(A)\,g^{1-\alpha}(A)\right]x,x\right\rangle\left\langle\left[f^{1-\alpha}(A)\,g^{1+\alpha}(A)\right]x,x\right\rangle\\ &\le \left\langle f^2(A)\,x,x\right\rangle\left\langle g^2(A)\,x,x\right\rangle\end{aligned}$$

for any $x\in H$ with $\|x\|=1$, where $\alpha\in[0,1]$.

b. *If A is a selfadjoint operator with $Sp(A)\subseteq[m,M]$ for some scalars $m<M$ and if f and g are continuous on $[m,M]$ with values in $[0,\infty)$ and such that $f^2(A)+g^2(A)$ is invertible, then we have the inequality*

$$\begin{aligned}(2.9)\quad &\langle f(A)\,g(A)\,x,x\rangle^2\\ &\le \left\langle\left[f^2(A)+g^2(A)\right]x,x\right\rangle\left\langle\left[\left[f^2(A)\,g^2(A)\right]\left[f^2(A)+g^2(A)\right]^{-1}\right]x,x\right\rangle\\ &\le \left\langle f^2(A)\,x,x\right\rangle\left\langle g^2(A)\,x,x\right\rangle\end{aligned}$$

for any $x\in H$, $\|x\|=1$.

The above two inequalities provide various particular cases that are of interest. We give here some examples as follows:

Example 1. ***a.*** *Assume that A is a positive operator on the Hilbert space H and $p,q>0$. Then for each $x\in H$ with $\|x\|=1$ we have the inequality*

$$\begin{aligned}(2.10)\quad \left\langle A^{p+q}x,x\right\rangle^2 &\le \left\langle A^{p+q+\alpha(p-q)}x,x\right\rangle\left\langle A^{p+q-\alpha(p-q)}x,x\right\rangle\\ &\le \left\langle A^{2p}x,x\right\rangle\left\langle A^{2q}x,x\right\rangle\end{aligned}$$

where $\alpha\in[0,1]$.

If A is positive definite then the inequality (2.10) also holds for $p,q<0$, $p>0$, $q<0$ or $p<0$, $q>0$.

b. *Assume that A is a selfadjoint operator and $n,r\in\mathbb{R}$. Then for each $x\in H$ with $\|x\|=1$ we have the inequality*

$$\begin{aligned}(2.11)\quad &\langle \exp[(n+r)A]\,x,x\rangle^2\\ &\le \langle \exp[n+r+\alpha(n-r)]\,Ax,x\rangle\langle\exp[n+r-\alpha(n-r)]\,Ax,x\rangle\\ &\le\langle\exp(2nA)\,x,x\rangle\langle\exp(2rA)\,x,x\rangle\end{aligned}$$

where $\alpha\in[0,1]$.

Another example conserning the thrigonometric operators $\sin(A)$ and $\cos(A)$ is as follows:

Example 2. *Let A be a selfadjoint operator with $Sp(A) \subseteq \left[0, \frac{\pi}{2}\right]$. Then we have the inequality*

$$\langle \sin(A)\cos(A)x, x\rangle^2 \leq \left\langle \left[\sin^2(A)\cos^2(A)\right]x, x\right\rangle \leq \left\langle \sin^2(A)x, x\right\rangle \left\langle \cos^2(A)x, x\right\rangle \tag{2.12}$$

for any $x \in H, \|x\| = 1$.

3. Some Versions for $2n$ Operators

The following multiple operator version of Theorem 1 holds:

Theorem 2. *Let (φ, ψ) be a (DEC)-pair of continuous functions on $[0,\infty) \times [0,\infty)$. If A_j, B_j are selfadjoint operators with $Sp(A_j), Sp(B_j) \subseteq [m, M]$ for $j \in \{1, ..., n\}$ and for some scalars $m < M$ and if f and g are continuous on $[m, M]$ and with values in $[0, \infty)$, then we have the inequality*

$$\begin{aligned}
&2\sum_{j=1}^{n}\langle f(A_j)g(A_j)x_j, x_j\rangle \sum_{j=1}^{n}\langle f(B_j)g(B_j)y_j, y_j\rangle \\
&\quad\leq \sum_{j=1}^{n}\langle \varphi(f(A_j), g(A_j))x_j, x_j\rangle \sum_{j=1}^{n}\langle \psi(f(B_j), g(B_j))y_j, y_j\rangle \\
&\quad+ \sum_{j=1}^{n}\langle \psi(f(A_j), g(A_j))x_j, x_j\rangle \sum_{j=1}^{n}\langle \varphi(f(B_j), g(B_j))y_j, y_j\rangle \\
&\leq \sum_{j=1}^{n}\left\langle f^2(A_j)x_j, x_j\right\rangle \sum_{j=1}^{n}\left\langle g^2(B_j)y_j, y_j\right\rangle + \sum_{j=1}^{n}\left\langle g^2(A_j)x_j, x_j\right\rangle \sum_{j=1}^{n}\left\langle f^2(B_j)y_j, y_j\right\rangle
\end{aligned} \tag{3.1}$$

for each $x_j, y_j \in H, j \in \{1, ..., n\}$ with $\sum_{j=1}^{n}\|x_j\|^2 = \sum_{j=1}^{n}\|y_j\|^2 = 1$.

Proof. As in [**4**, p. 6], if we put

$$\widetilde{A} := \begin{pmatrix} A_1 & . & . & . & 0 \\ & . & & & \\ & & . & & \\ & & & . & \\ 0 & . & . & . & A_n \end{pmatrix}, \; \widetilde{B} := \begin{pmatrix} B_1 & . & . & . & 0 \\ & . & & & \\ & & . & & \\ & & & . & \\ 0 & . & . & . & B_n \end{pmatrix}$$

$$\widetilde{x} = \begin{pmatrix} x_1 \\ . \\ . \\ . \\ x_n \end{pmatrix} \text{ and } \widetilde{y} = \begin{pmatrix} y_1 \\ . \\ . \\ . \\ y_n \end{pmatrix}$$

then we have $Sp\left(\widetilde{A}\right), Sp\left(\widetilde{B}\right) \subseteq [m, M]$, $\|\widetilde{x}\| = \|\widetilde{y}\| = 1$,

$$\left\langle f\left(\widetilde{A}\right)g\left(\widetilde{A}\right)\widetilde{x}, \widetilde{x}\right\rangle = \sum_{j=1}^{n}\langle f(A_j)g(A_j)x_j, x_j\rangle,$$

$$\left\langle f\left(\widetilde{A}\right)g\left(\widetilde{A}\right)\widetilde{y}, \widetilde{y}\right\rangle = \sum_{j=1}^{n}\langle f(A_j)g(A_j)y_j, y_j\rangle$$

and so on.

Applying Theorem 1 for $\widetilde{A}$, $\widetilde{B}$, $\widetilde{x}$ and $\widetilde{y}$ we deduce the desired result (3.1). ∎

As a particular case of interest we can state the following corollary:

Corollary 2. *Let (φ,ψ) be a (DEC)-pair of continuous functions on $[0,\infty)\times[0,\infty)$. If A_j are selfadjoint operators with $Sp(A_j)\subseteq[m,M]$ for $j\in\{1,...,n\}$ and for some scalars $m<M$ and if f and g are continuous on $[m,M]$ and with values in $[0,\infty)$, then we have the inequality*

$$\left(\sum_{j=1}^{n}\langle f(A_j)\,g(A_j)\,x_j,x_j\rangle\right)^2 \leq \sum_{j=1}^{n}\langle\varphi(f(A_j),g(A_j))\,x_j,x_j\rangle\sum_{j=1}^{n}\langle\psi(f(A_j),g(A_j))\,x_j,x_j\rangle \leq \sum_{j=1}^{n}\left\langle f^2(A_j)\,x_j,x_j\right\rangle\sum_{j=1}^{n}\left\langle g^2(A_j)\,x_j,x_j\right\rangle \tag{3.2}$$

for each $x_j\in H, j\in\{1,...,n\}$ with $\sum_{j=1}^{n}\|x_j\|^2=1$.

Remark 2. ***a.*** *Let (φ,ψ) be a (DEC)-pair of continuous functions on $[0,\infty)\times[0,\infty)$. If A_j are selfadjoint operators with $Sp(A_j)\subseteq[m,M]$ for $j\in\{1,...,n\}$ and for some scalars $m<M$ and if f and g are continuous on $[m,M]$ and with values in $[0,\infty)$, then we have the inequality*

$$\left(\sum_{j=1}^{n}\langle f(A_j)\,g(A_j)\,x_j,x_j\rangle\right)^2 \leq \sum_{j=1}^{n}\left\langle\left[f^{1+\alpha}(A_j)\,g^{1-\alpha}(A_j)\right]x_j,x_j\right\rangle\sum_{j=1}^{n}\left\langle\left[f^{1-\alpha}(A_j)\,g^{1+\alpha}(A_j)\right]x_j,x_j\right\rangle \leq \sum_{j=1}^{n}\left\langle f^2(A_j)\,x_j,x_j\right\rangle\sum_{j=1}^{n}\left\langle g^2(A_j)\,x_j,x_j\right\rangle \tag{3.3}$$

for each $x_j\in H, j\in\{1,...,n\}$ with $\sum_{j=1}^{n}\|x_j\|^2=1$, where $\alpha\in[0,1]$.

b. *If A_j are selfadjoint operators with $Sp(A_j)\subseteq[m,M]$ for $j\in\{1,...,n\}$ and for some scalars $m<M$ and if f and g are continuous on $[m,M]$ with values in $[0,\infty)$ and such that $f^2(A_j)+g^2(A_j)$ are invertible for each, $j\in\{1,...,n\}$ then we*

have the inequality

$$\left(\sum_{j=1}^{n}\langle f(A_j)\,g(A_j)\,x_j,x_j\rangle\right)^2 \leq \sum_{j=1}^{n}\left\langle\left[f^2(A_j)+g^2(A_j)\right]x_j,x_j\right\rangle \times\sum_{j=1}^{n}\left\langle\left[\left[f^2(A_j)\,g^2(A_j)\right]\left[f^2(A_j)+g^2(A_j)\right]^{-1}\right]x_j,x_j\right\rangle \leq \sum_{j=1}^{n}\left\langle f^2(A)\,x_j,x_j\right\rangle\sum_{j=1}^{n}\left\langle g^2(A)\,x_j,x_j\right\rangle \tag{3.4}$$

for each $x_j\in H, j\in\{1,...,n\}$ *with* $\sum_{j=1}^{n}\|x_j\|^2=1$.

Some particular inequalitties similar to those from Example 1 and Example 2 may be stated, however we do not mention them in here.

Another version for n operators is the following one:

Theorem 3. *Let* (φ,ψ) *be a (DEC)-pair of continuous functions on* $[0,\infty)\times[0,\infty)$. *If* A_j, B_j *are selfadjoint operators with* $Sp(A_j), Sp(B_j)\subseteq[m,M]$ *for* $j\in\{1,...,n\}$ *and for some scalars* $m<M$, $p_j\geq 0, q_j\geq 0, j\in\{1,...,n\}$ *with* $\sum_{j=1}^{n}p_j=\sum_{j=1}^{n}q_j=1$ *and if* f *and* g *are continuous on* $[m,M]$ *with values in* $[0,\infty)$, *then we have the inequality*

$$2\left\langle\sum_{j=1}^{n}p_jf(A_j)\,g(A_j)\,x,x\right\rangle\left\langle\sum_{j=1}^{n}q_jf(B_j)\,g(B_j)\,y,y\right\rangle \leq \left\langle\sum_{j=1}^{n}p_j\varphi(f(A_j),g(A_j))\,x,x\right\rangle\left\langle\sum_{j=1}^{n}q_j\psi(f(B_j),g(B_j))\,y,y\right\rangle + \left\langle\sum_{j=1}^{n}p_j\psi(f(A_j),g(A_j))\,x,x\right\rangle\left\langle\sum_{j=1}^{n}q_j\varphi(f(B_j),g(B_j))\,y,y\right\rangle \leq \left\langle\sum_{j=1}^{n}p_jf^2(A_j)\,x,x\right\rangle\left\langle\sum_{j=1}^{n}q_jg^2(B_j)\,y,y\right\rangle + \left\langle\sum_{j=1}^{n}p_jg^2(A_j)\,x,x\right\rangle\left\langle\sum_{j=1}^{n}q_jf^2(B_j)\,y,y\right\rangle \tag{3.5}$$

for each $x,y\in H$ *with* $\|x\|=\|y\|=1$.

Proof. Follows from Theorem 2 on choosing $x_j=\sqrt{p_j}\cdot x,\ y_j=\sqrt{q_j}\cdot y, j\in\{1,...,n\}$, where $p_j\geq 0, q_j\geq 0, j\in\{1,...,n\}$, $\sum_{j=1}^{n}p_j=\sum_{j=1}^{n}q_j=1$ and $x,y\in H$ with $\|x\|=\|y\|=1$. ∎

Corollary 3. *Let* (φ,ψ) *be a (DEC)-pair of continuous functions on* $[0,\infty)\times[0,\infty)$. *If* A_j *are selfadjoint operators with* $Sp(A_j)\subseteq[m,M]$ *for* $j\in\{1,...,n\}$ *and for some scalars* $m<M$, $p_j\geq 0, j\in\{1,...,n\}$ *with* $\sum_{j=1}^{n}p_j=1$ *and if* f *and* g *are*

continuous on $[m,M]$ *with values in* $[0,\infty)$ *, then we have the inequality*

$$\left\langle \sum_{j=1}^{n} p_j f(A_j) g(A_j) x, x \right\rangle^2 \leq \left\langle \sum_{j=1}^{n} p_j \varphi(f(A_j), g(A_j)) x, x \right\rangle \left\langle \sum_{j=1}^{n} p_j \psi(f(A_j), g(A_j)) x, x \right\rangle \leq \left\langle \sum_{j=1}^{n} p_j f^2(A_j) x, x \right\rangle \left\langle \sum_{j=1}^{n} p_j g^2(A_j) x, x \right\rangle, \tag{3.6}$$

for each $x \in H$*,with* $\|x\| = 1$*.*

Finally for the section, we can state the following particular inequalities of interest:

Remark 3. ***a.*** *Let* (φ, ψ) *be a (DEC)-pair of continuous functions on* $[0,\infty) \times [0,\infty)$ *. If* A_j *are selfadjoint operators with* $Sp(A_j) \subseteq [m,M]$ *for* $j \in \{1,...,n\}$ *and for some scalars* $m < M$ *and if* f *and* g *are continuous on* $[m,M]$ *and with values in* $[0,\infty)$ *, then we have the inequality*

$$\left\langle \sum_{j=1}^{n} p_j f(A_j) g(A_j) x, x \right\rangle^2 \leq \left\langle \sum_{j=1}^{n} p_j \left[f^{1+\alpha}(A_j) g^{1-\alpha}(A_j)\right] x, x \right\rangle \left\langle \sum_{j=1}^{n} p_j \left[f^{1-\alpha}(A_j) g^{1+\alpha}(A_j)\right] x, x \right\rangle \leq \left\langle \sum_{j=1}^{n} p_j f^2(A_j) x, x \right\rangle \left\langle \sum_{j=1}^{n} p_j g^2(A_j) x, x \right\rangle \tag{3.7}$$

for each $p_j \geq 0, j \in \{1,...,n\}$ *with* $\sum_{j=1}^{n} p_j = 1$ *and* $x \in H$ *with* $\|x\| = 1$ *where* $\alpha \in [0,1]$*.*

b. *If* A_j *are selfadjoint operators with* $Sp(A_j) \subseteq [m,M]$ *for* $j \in \{1,...,n\}$ *and for some scalars* $m < M$ *and if* f *and* g *are continuous on* $[m,M]$ *with values in* $[0,\infty)$ *and such that* $f^2(A_j) + g^2(A_j)$ *are invertible for each* $j \in \{1,...,n\}$ *then we have the inequality*

$$\left\langle \sum_{j=1}^{n} p_j f(A_j) g(A_j) x, x \right\rangle^2 \leq \left\langle \sum_{j=1}^{n} p_j \left[f^2(A_j) + g^2(A_j)\right] x, x \right\rangle \times \left\langle \sum_{j=1}^{n} p_j \left[\left[f^2(A_j) g^2(A_j)\right] \left[f^2(A_j) + g^2(A_j)\right]^{-1}\right] x, x \right\rangle \leq \left\langle \sum_{j=1}^{n} p_j f^2(A_j) x, x \right\rangle \left\langle \sum_{j=1}^{n} p_j g^2(A_j) x, x \right\rangle, \tag{3.8}$$

for each $p_j \geq 0, j \in \{1,...,n\}$ *with* $\sum_{j=1}^{n} p_j = 1$ *and* $x \in H$ *with* $\|x\| = 1$*.*

4. Related Results for Two Operators

The following result that provides another refinement for the Cauchy-Bunyakovsky-Schwarz inequality may be stated as well:

Theorem 4. *Let (φ,ψ) be a (DEC)-pair of continuous functions on $[0,\infty)\times[0,\infty)$. If A,B are selfadjoint operators on the Hilbert space $(H;\langle .,.\rangle)$ with $Sp(A)$, $Sp(B)\subseteq[m,M]$ for some scalars $m<M$ and if f and g are continuous on $[m,M]$ and with values in $[0,\infty)$, then we have the inequality*

$$\begin{aligned}(4.1)\quad & 2\langle f(A)g(A)x,x\rangle\langle f(B)g(B)y,y\rangle \\ &\le \langle\Gamma_1(B)(A,x)y,y\rangle+\langle\Gamma_2(B)(A,x)y,y\rangle \\ &\le \left\langle f^2(A)g^2(A)x,x\right\rangle+\left\langle f^2(B)g^2(B)y,y\right\rangle\end{aligned}$$

for any $x,y\in H$ with $\|x\|=\|y\|=1$ where

$$\Gamma_1(t)(A,x):=\langle\varphi(f(A),g(t))\psi(f(t),g(A))x,x\rangle$$

and

$$\Gamma_2(t)(A,x):=\langle\varphi(f(t),g(A))\psi(f(A),g(t))x,x\rangle$$

for $t\in[m,M]$.

Proof. We know that the following inequality holds

$$2uvzw\le\varphi(u,v)\psi(z,w)+\varphi(z,w)\psi(u,v)\le u^2w^2+v^2z^2 \tag{4.2}$$

for any $u,v,z,w\ge 0$.

Now, if we choose $u=f(s),v=g(t),z=f(t)$ and $w=g(s)$ in (4.2) then we get

$$\begin{aligned}(4.3)\quad & 2f(s)g(s)f(t)g(t) \\ &\le\varphi(f(s),g(t))\psi(f(t),g(s))+\varphi(f(t),g(s))\psi(f(s),g(t)) \\ &\le f^2(s)g^2(s)+g^2(t)f^2(t)\end{aligned}$$

for any $s,t\in[m,M]$.

Further, if we fix $t\in[m,M]$ and apply the property (P) for the operator A, then we get the inequality

$$\begin{aligned}(4.4)\quad & 2f(t)g(t)\langle f(A)g(A)x,x\rangle \\ &\le\langle\varphi(f(A),g(t))\psi(f(t),g(A))x,x\rangle+\langle\varphi(f(t),g(A))\psi(f(A),g(t))x,x\rangle \\ &\le\left\langle f^2(A)g^2(A)x,x\right\rangle+g^2(t)f^2(t)\end{aligned}$$

for any $x\in H$ with $\|x\|=1$. This inequality can be written in terms of the functions $\Gamma_1(.)(A,x)$ and $\Gamma_1(.)(A,x)$ as

$$\begin{aligned}(4.5)\quad & 2f(t)g(t)\langle f(A)g(A)x,x\rangle \\ &\le\Gamma_1(t)(A,x)+\Gamma_2(t)(A,x) \\ &\le\left\langle f^2(A)g^2(A)x,x\right\rangle+g^2(t)f^2(t)\end{aligned}$$

for any $t\in[m,M]$ and any $x\in H$ with $\|x\|=1$.

Now, if we fix $x\in H$ with $\|x\|=1$ and apply the same property (P) for the inequality (4.5) for the operator B then we get the desired inequality (4.1). ∎

The following particular case is of interest

Corollary 4. *Let (φ,ψ) be a (DEC)-pair of continuous functions on $[0,\infty)\times[0,\infty)$. If A is a selfadjoint operator on the Hilbert space $(H;\langle .,.\rangle)$ with $Sp(A)\subseteq[m,M]$ for some scalars $m<M$ and if f and g are continuous on $[m,M]$ and with values in $[0,\infty)$, then we have the inequality*

$$\langle f(A)g(A)x,x\rangle^2\le\langle\Gamma(B)(A,x)x,x\rangle\le\left\langle f^2(A)g^2(A)x,x\right\rangle \tag{4.6}$$

for any $x\in H$ with $\|x\|=1$ where

$$\Gamma(t)(A,x):=\langle\varphi(f(A),g(t))\psi(f(t),g(A))x,x\rangle$$

for $t\in[m,M]$.

Remark 4. *If $\varphi(a,b)=a^{1+\alpha}b^{1-\alpha}$, $\psi(a,b)=a^{1-\alpha}b^{1+\alpha}$ with $\alpha\in[0,1]$ then*

$$\Gamma_1(t)(A,x)=f^{1-\alpha}(t)\,g^{1-\alpha}(t)\left\langle f^{1+\alpha}(A)\,g^{1+\alpha}(A)x,x\right\rangle$$

and

$$\Gamma_2(t)(A,x):=f^{1+\alpha}(t)\,g^{1+\alpha}(t)\left\langle f^{1-\alpha}(A)\,g^{1-\alpha}(A)x,x\right\rangle$$

and from (4.1) we get the inequality

$$\begin{aligned}2\langle f(A)g(A)x,x\rangle\langle f(B)g(B)y,y\rangle&\\ \le\left\langle f^{1+\alpha}(A)g^{1+\alpha}(A)x,x\right\rangle\left\langle f^{1-\alpha}(B)g^{1-\alpha}(B)y,y\right\rangle&\\ +\left\langle f^{1-\alpha}(A)g^{1-\alpha}(A)x,x\right\rangle\left\langle f^{1+\alpha}(B)g^{1+\alpha}(B)y,y\right\rangle&\\ \le\left\langle f^2(A)g^2(A)x,x\right\rangle+\left\langle f^2(B)g^2(B)y,y\right\rangle&\end{aligned} \tag{4.7}$$

for any $x,y\in H$ with $\|x\|=\|y\|=1$ provided that A is a selfadjoint operator on the Hilbert space $(H;\langle .,.\rangle)$ with $Sp(A)\subseteq[m,M]$ for some scalars $m<M$ and if f and g are continuous on $[m,M]$ and with values in $[0,\infty)$.

In particular we have the inequality

$$\begin{aligned}\langle f(A)g(A)x,x\rangle^2&\le\left\langle f^{1+\alpha}(A)g^{1+\alpha}(A)x,x\right\rangle\left\langle f^{1-\alpha}(A)g^{1-\alpha}(A)x,x\right\rangle\\&\le\left\langle f^2(A)g^2(A)x,x\right\rangle\end{aligned} \tag{4.8}$$

for any $x\in H$ with $\|x\|=1$.

The above two inequalities provide various particular cases that are of interest. We give here some examples as follows:

Example 3. ***a.*** *Assume that A is a positive operator on the Hilbert space H and $p>0$. Then for each $x\in H$ with $\|x\|=1$ we have the inequality*

$$\langle A^px,x\rangle^2\le\left\langle A^{(1+\alpha)p}x,x\right\rangle\left\langle A^{(1-\alpha)p}x,x\right\rangle\le\left\langle A^{2p}x,x\right\rangle \tag{4.9}$$

where $\alpha\in[0,1]$.

If A is positive definite then the inequality (4.9) also holds for $p<0$.

b. *Assume that A is a selfadjoint operator and $r\in\mathbb{R}$. Then for each $x\in H$ with $\|x\|=1$ we have the inequality*

$$\begin{aligned}\langle\exp(rA)x,x\rangle^2&\le\langle\exp[(1+\alpha)rA]x,x\rangle\langle\exp[(1-\alpha)rA]x,x\rangle\\&\le\langle\exp(2rA)x,x\rangle\end{aligned} \tag{4.10}$$

where $\alpha\in[0,1]$.

Similar results can be stated for $2n$ operators, however the details are omitted.

The following different inequality may be stated as well:

Theorem 5. *Let* (φ,ψ) *be a (DEC)-pair of continuous functions on* $[0,\infty)\times[0,\infty)$. *If* A,B *are selfadjoint operators on the Hilbert space* $(H;\langle .,.\rangle)$ *with* $Sp(A)$, $Sp(B)\subseteq[m,M]$ *for some scalars* $m<M$ *and if* f *and* g *are continuous on* $[m,M]$ *and with values in* $[0,\infty)$, *then we have the inequality*

$$(2\langle f(A)g(A)x,x\rangle\langle f(B)g(B)y,y\rangle \\ \leq)\ \langle\varphi(f(A),g(A))x,x\rangle\langle\psi(f(B),g(B))y,y\rangle \\ +\langle\psi(f(A),g(A))x,x\rangle\langle\varphi(f(B),g(B))y,y\rangle \\ \leq\left\langle f^2(A)x,x\right\rangle\left\langle f^2(B)y,y\right\rangle+\left\langle g^2(A)x,x\right\rangle\left\langle g^2(B)y,y\right\rangle \tag{4.11}$$

for any $x,y\in H$ *with* $\|x\|=\|y\|=1$.

Proof. We know that the following inequality holds

$$2uvzw\leq\varphi(u,v)\psi(z,w)+\varphi(z,w)\psi(u,v)\leq u^2w^2+v^2z^2 \tag{4.12}$$

for any $u,v,z,w\geq 0$.

Further, if we choose $u=f(s)$, $v=g(s)$, $z=g(t)$ and $w=f(t)$ in (4.12) then we get

$$2f(s)g(s)f(t)g(t) \\ \leq\varphi(f(s),g(s))\psi(f(t),g(t))+\varphi(f(t),g(t))\psi(f(s),g(s)) \\ \leq f^2(s)f^2(t)+g^2(s)g^2(t) \tag{4.13}$$

for any $s,t\in[m,M]$.

Now, if we fix $t\in[m,M]$ and apply the property (P) for the operator A then we get the inequality

$$2f(t)g(t)\langle f(A)g(A)x,x\rangle \\ \leq\psi(f(t),g(t))\langle\varphi(f(A),g(A))x,x\rangle+\varphi(f(t),g(t))\langle\psi(f(A),g(A))x,x\rangle \\ \leq f^2(t)\left\langle f^2(A)x,x\right\rangle+g^2(t)\left\langle g^2(A)x,x\right\rangle \tag{4.14}$$

for any $x\in H$ with $\|x\|=1$.

Now, if we fix $x\in H$ with $\|x\|=1$ and apply the same property (P) for the inequality (4.14) for the operator B then we get the desired inequality (4.11). ∎

In particular, we have

Corollary 5. *Let* (φ,ψ) *be a (DEC)-pair of continuous functions on* $[0,\infty)\times[0,\infty)$. *If* A *is a selfadjoint operators on the Hilbert space* $(H;\langle .,.\rangle)$ *with* $Sp(A)\subseteq[m,M]$ *for some scalars* $m<M$ *and if* f *and* g *are continuous on* $[m,M]$ *and with values in* $[0,\infty)$, *then we have the inequality*

$$\left(2\langle f(A)g(A)x,x\rangle^2\leq\right)2\langle\varphi(f(A),g(A))x,x\rangle\langle\psi(f(A),g(A))x,x\rangle \\ \leq\left\langle f^2(A)x,x\right\rangle^2+\left\langle g^2(A)x,x\right\rangle^2 \tag{4.15}$$

for any $x\in H$, $\|x\|=1$.

Remark 5. *We observe that the inequality (4.15) is not as good as the second inequality in (2.7).*

Remark 6. *Consider now the following two bounds*

$$B_2 := \left\langle f^2(A)x,x\right\rangle \left\langle f^2(B)y,y\right\rangle + \left\langle g^2(A)x,x\right\rangle \left\langle g^2(B)y,y\right\rangle$$

and

$$B_1 := \left\langle f^2(A)x,x\right\rangle \left\langle g^2(B)y,y\right\rangle + \left\langle g^2(A)x,x\right\rangle \left\langle f^2(B)y,y\right\rangle$$

for the quntity

$$\langle \varphi(f(A),g(A))x,x\rangle \langle \psi(f(B),g(B))y,y\rangle + \langle \psi(f(A),g(A))x,x\rangle \langle \varphi(f(B),g(B))y,y\rangle$$

that have been obtained in Theorem 5 and Theorem 1, respectively. We observe that

$$B_2 - B_1 = \left\langle \left[f^2(A) - g^2(A)\right]x,x\right\rangle \left(\left\langle \left[f^2(B) - g^2(B)\right]y,y\right\rangle\right), \tag{4.16}$$

for any $x,y \in H$ with $\|x\| = \|y\| = 1$.

Utilising the equality (4.16) we can observe, for instance, that, if $f^2(A) \geq g^2(A)$ and $f^2(B) \geq g^2(B)$ in the operator order of $B(H)$, then B_1 is a better bound than B_2. The conclusion is the other way around if, for instance, $f^2(A) \geq g^2(A)$ and $g^2(B) \geq f^2(B)$ in the operator order of $B(H)$.

Similar results can be stated for $2n$ operators, however the details are omitted.

Remark 7. *One can choose the variables $u,v,z,w \geq 0$ in other different ways in the inequality*

$$2uvzw \leq \varphi(u,v)\psi(z,w) + \varphi(z,w)\psi(u,v) \leq u^2w^2 + v^2z^2 \tag{4.17}$$

to get similar results as those pointed out above. The details are left to the interested reader.

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