# ON SOME INEQUALITIES FOR MEANS AND THE SECOND GAUTSCHI-KERSHAW'S INEQUALITY

CHAO-PING CHEN

ABSTRACT. We establish some inequalities for means, and we present new bounds for the second Gautschi-Kershaw's inequality.

## 1. INTRODUCTION

Let $r, s \in \mathbb{R}$ and let $a, b > 0$. The Stolarsky mean $E_{r,s}(a, b)$ of order $(r, s)$ of $a$ and $b$ with $a \neq b$ are defined as

$$E_{r,s}(a,b) = \begin{cases} \left(\dfrac{r}{s} \cdot \dfrac{b^s - a^s}{b^r - a^r}\right)^{1/(s-r)}, & rs(r-s) \neq 0, \\ \exp\left(-\dfrac{1}{r} + \dfrac{a^r \ln a - b^r \ln b}{a^r - b^r}\right), & r = s \neq 0, \\ \left(\dfrac{1}{r} \cdot \dfrac{b^r - a^r}{\ln b - \ln a}\right)^{1/r}, & r \neq 0, s = 0, \\ \sqrt{ab}, & r = s = 0, \end{cases} \tag{1}$$

with $E_{r,s}(a, a) = a$ (see [51, 52]), while the Gini mean $G_{r,s}(a, b)$ of order $(r, s)$ of $a$ and $b$ are defined in [18] as

$$G_{r,s}(a,b) = \begin{cases} \left(\dfrac{a^s + b^s}{a^r + b^r}\right)^{1/(s-r)}, & r \neq s, \\ \exp\left(\dfrac{a^r \ln a + b^r \ln b}{a^r + b^r}\right), & r = s \neq 0, \\ \sqrt{ab}, & r = s = 0. \end{cases} \tag{2}$$

K. B. Stolarsky [51], Leach and Sholander [27] showed that $E_{r,s}(a, b)$ are for $a \neq b$ strictly increasing with both $r$ and $s$. For $a \neq b$, $G_{r,s}(a, b)$ are also strictly increasing with both $r$ and $s$, see [36, 42]. Leach and Sholander [29] and Páles [38] solved the problem of comparison of Stolarsky mean. The problem of comparison of Gini mean was completely solved by Páles [39] (see also the paper by P. Czinder and Zs. Ples [12]). A problem of comparability of Gini and Stolarsky means was addressed by Neuman and Páles [35]. Minkowski-type inequality for Stolarsky and Gini means can be found in [11, 31, 32].

---

2000 *Mathematics Subject Classification.* 26D99, 26D15.

*Key words and phrases.* Stolarsky mean, Gini mean, Muirhead mean, inequality, series expansion.

This work was supported by Natural scientific research plan project of education department of Henan Province (2008A110007), Project of the plan of science and technology of education department of Henan Province (2007110011).

Since $E_{r,s}(a,b)$ are for $a \neq b$ strictly increasing with both $r$ and $s$, for the particular choices of the parameters $r$ and $s$, we obtain the following chain of inequalities:

$$E_{-2,-1}(a,b) < E_{0,0}(a,b) < E_{1,0}(a,b) < E_{1,1}(a,b) < E_{2,1}(a,b) \quad \text{for} \quad a \neq b,$$

that is,

$$H(a,b) < G(a,b) < L(a,b) < I(a,b) < A(a,b) \quad \text{for} \quad a \neq b,$$

where $H$, $G$, $L$, $I$ and $A$ are the harmonic, geometric, logarithmic, identric and arithmetic means, respectively.

It is worth mentioning that

$$G_{0,r}(a,b) = E_{r,2r}(a,b) = M_r(a,b) = \begin{cases} \left(\dfrac{a^r + b^r}{2}\right)^{1/r}, & r \neq 0, \\ \sqrt{ab}, & r = 0. \end{cases}$$

Thus the classes of Gini and Stolarsky means contain both the power means. Alzer and Ruscheweyh [2] have proven that the joint elements in the classes of Gini and Stolarsky means are exactly the power means.

Let $r, s \in \mathbb{R}$ and $a, b > 0$, the generalized Muirhead mean $\sum_{r,s}(a,b)$ of $a$ and $b$ is defined by (see, for instance, [5, p. 333] or [4])

$$\sum\nolimits_{r,s}(a,b) = \left(\frac{a^r b^s + a^s b^r}{2}\right)^{1/(r+s)}, r + s \neq 0.$$

T. Trif [53] investigated the monotonicity of $\sum_{r,s}(a,b)$ with respect to $r$ or $s$. Likewise, a comparison theorem and a Minkowski-type inequality involving the generalized Muirhead mean $\sum_{r,s}(a,b)$ are established.

In the special case when $r + s = 1$, i.e., $r = \alpha, s = 1 - \alpha$, the Muirhead (or symmetric) mean is obtained:

$$\sum\nolimits_{\alpha,1-\alpha}(a,b) = \frac{a^\alpha b^{1-\alpha} + a^{1-\alpha} b^\alpha}{2}.$$

Following A. O. Pittenger [41], we write symmetric mean into the from

$$S_\delta(a,b) = \frac{a^{\frac{1+\sqrt{\delta}}{2}} b^{\frac{1-\sqrt{\delta}}{2}} + a^{\frac{1-\sqrt{\delta}}{2}} b^{\frac{1+\sqrt{\delta}}{2}}}{2}.$$

It is shown in [22] that $S_\delta$ is increasing in $\delta$ and that for $a \neq b$

$$M_0(a,b) < S_\delta(a,b) < M_1(a,b)$$

provided $0 < \delta < 1$.

The following result is known

$$S_{1/3}(a,b) < L(a,b) < M_{1/3}(a,b). \tag{3}$$

The first inequality in (3) has been established by A. O. Pittenger [41], while the second one was proven in [30, 41]. The first inequality in (3) improves a result of B. C. Carlson [9], who proved $S_{1/4}(a,b) < L(a,b)$.

E. Neuman [34] established the integral representation

$$L(a,b) = \int_0^1 a^t b^{1-t} dt. \tag{4}$$

By applying the Gausss quadrature formula with two knots (see [13, pp. 343-344], [14, p. 36])

$$\int_0^1 f(t)dt = \frac{1}{2}f\left(\frac{1}{2}+\frac{1}{2\sqrt{3}}\right)+\frac{1}{2}f\left(\frac{1}{2}-\frac{1}{2\sqrt{3}}\right)+\frac{1}{4320}f^{(4)}(\xi), \quad 0<\xi<1 \quad (5)$$

to the function $f(t)=a^t b^{1-t}$, T. Trif [53] presented a very short proof of the first inequality in (3).

Motivated by the technique of T. Trif, we here present a very short proof of the second inequality in (3). To this aim, by applying Simpson's $\frac{3}{8}$ rule (see [6, 26])

$$\begin{aligned}\int_a^b f(t)dt = \frac{b-a}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]\\ -\frac{(b-a)^5}{6480}f^{(4)}(\xi) \quad \text{for some } \xi \text{ between } a \text{ and } b,\end{aligned} \quad (6)$$

with $a=0, b=1$ and $f(t)=a^t b^{1-t}(0\le t\le 1)$, we get

$$L(a,b)=M_{1/3}(a,b)-\frac{1}{6480}a^{\xi}b^{1-\xi}(\ln a-\ln b)^4.$$

This yields the second inequality in (3).

We remark that F. Burk [7] obtained the second inequality in (3) by applying (6) to the function $f(t)=e^t$, replacing $a$ and $b$ with $\ln a$ and $\ln b$, respectively.

This paper is organized as follows. In Section 2 we establish some inequalities for means. In Section 3 we presents new bounds, and the series representation for the ratio $\frac{\ln\Gamma(y)-\ln\Gamma(x)}{y-x}$, where $\Gamma$ denotes the gamma function.

## 2. Inequalities for means

**Theorem 1.** *Let $a,b>0$ with $a\ne b$, then*

$$L<\frac{1}{3}H+\frac{2}{3}A, \quad (7)$$

$$\frac{1}{L}<\frac{1}{3}\frac{1}{H}+\frac{2}{3}\frac{1}{A}, \quad (8)$$

$$I^2<\frac{1}{3}G^2+\frac{2}{3}A^2. \quad (9)$$

*Proof.* B. C. Carlson [9] has established the integral representation

$$L(a,b)=\left[\int_0^1\frac{1}{ta+(1-t)b}dt\right]^{-1}. \quad (10)$$

Use the change of variable $u=ta+(1-t)b$, (10) can be written as

$$L(a,b)=\left[\frac{1}{b-a}\int_a^b\frac{1}{u}du\right]^{-1}. \quad (11)$$

By applying (5) to the function $f(t) = \frac{1}{ta+(1-t)b}$, we obtain

$$\begin{aligned}\frac{1}{L(a,b)} &= \frac{1}{2}\left[\frac{1}{(\frac{1}{2}+\frac{1}{2\sqrt{3}})a+(\frac{1}{2}-\frac{1}{2\sqrt{3}})b} + \frac{1}{(\frac{1}{2}-\frac{1}{2\sqrt{3}})a+(\frac{1}{2}+\frac{1}{2\sqrt{3}})b}\right] \\ &\qquad + \frac{(b-a)^4}{180[\xi a+(1-\xi)b]^5} \\ &= \frac{1}{\frac{1}{3}H+\frac{2}{3}A} + \frac{(b-a)^4}{180[\xi a+(1-\xi)b]^5} \qquad (0<\xi<1) \\ &> \frac{1}{\frac{1}{3}H+\frac{2}{3}A}.\end{aligned}$$

This proves (7).

By applying the composite Simpson rule (see [21])

$$\begin{aligned}\int_a^b f(t)dt &= \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right] \\ &\qquad - \frac{(b-a)^5}{2880}f^{(4)}(\xi) \quad \text{for some } \xi \text{ between } a \text{ and } b,\end{aligned} \tag{12}$$

to the function $f(t) = \frac{1}{t}$, we obtain

$$\frac{1}{L(a,b)} = \frac{1}{6}\left[\frac{1}{b}+\frac{4}{(a+b)/2}+\frac{1}{a}\right] - \frac{(b-a)^4}{120\xi^5} < \frac{1}{3H}+\frac{2}{3A}.$$

This proves (8).

It is easy to see that

$$\ln I(a,b) = \frac{1}{b-a}\int_a^b \ln t dt = \int_0^1 \ln[ta+(1-t)b]dt. \tag{13}$$

By applying (5) to the function $f(t) = \ln[ta+(1-t)b]$, we obtain

$$\begin{aligned}\ln I(a,b) &= \frac{1}{2}\ln\left[\left(\frac{1}{2}+\frac{1}{2\sqrt{3}}\right)a+\left(\frac{1}{2}-\frac{1}{2\sqrt{3}}\right)b\right] \\ &\qquad + \frac{1}{2}\ln\left[\left(\frac{1}{2}-\frac{1}{2\sqrt{3}}\right)a+\left(\frac{1}{2}+\frac{1}{2\sqrt{3}}\right)b\right] - \frac{(b-a)^4}{720[\xi a+(1-\xi)b]^4} \\ &= \frac{1}{2}\ln\left(\frac{1}{6}a^2+\frac{2}{3}ab+\frac{1}{6}b^2\right) - \frac{(b-a)^4}{720[\xi a+(1-\xi)b]^4} \\ &= \frac{1}{2}\ln\left(\frac{1}{3}G^2+\frac{2}{3}A^2\right) - \frac{(b-a)^4}{720[\xi a+(1-\xi)b]^4} \qquad (0<\xi<1) \\ &< \frac{1}{2}\ln\left(\frac{1}{3}G^2+\frac{2}{3}A^2\right),\end{aligned}$$

thus, (9) holds. The proof of Theorem 1 is complete. □

*Remark* 1. The following result is known

$$\sqrt[3]{AG^2} < L < \frac{1}{3}A + \frac{2}{3}G. \tag{14}$$

The first inequality in (14) was established by E. B. leach and M. C. Sholander [28], while the second one was proven by B. C. Carlson [9]. The second inequality in (14) can be obtained by applying (12) with $a = 0, b = 1$ and $f(t) = x^t y^{1-t}$.

The inequality

$$I > \sqrt[3]{GA^2} \tag{15}$$

can be concluded by applying (12) to the function $f(t) = \ln t$, see [49, 50]. A stronger inequality than (15) is (cf.[49])

$$I > \frac{1}{3}G + \frac{2}{3}A. \tag{16}$$

## 3. The second Gautschi-Kershaw's inequality

In 1959 W. Gautschi [17] presented the remarkable inequality:

$$n^{1-s} < \frac{\Gamma(n+1)}{\Gamma(n+s)} < \exp[(1-s)\psi(n+1)], \quad 0 < s < 1,\, n = 1, 2, \dots, \tag{17}$$

where $\psi = \Gamma'/\Gamma$ denotes the logarithmic derivative of the gamma function. In 1983 D. Kershaw [24] gave the following closer bounds:

$$\left(x + \frac{s}{2}\right)^{1-s} < \frac{\Gamma(x+1)}{\Gamma(x+s)} < \left(x - \frac{1}{2} + \sqrt{x + \frac{1}{4}}\right)^{1-s}, \tag{18}$$

$$\exp[(1-s)\psi(x+\sqrt{s})] < \frac{\Gamma(x+1)}{\Gamma(x+s)} < \exp\left[(1-s)\psi\left(x + \frac{s+1}{2}\right)\right] \tag{19}$$

for real $x > 0$ and $0 < s < 1$. Inequalities (18) and (19) are respectively called the first and the second Gautschi-Kershaw's inequality in the literature.

C. Giordano et al. [19] and B. Palumbo [40] gave a unified treatment and some extensions of Gautschi-Kershaw type inequalities. For each $s > 0$, $x > 0$, the inequality (18) is valid for $0 < s < 1$ or $s > 2$, while the reverse inequality is valid for $1 < s < 2$; and the inequality (19) is valid for $0 < s < 1$, while the reverse inequality is valid for $s > 1$.

The inequality (19) can be written as

$$\psi(x+\sqrt{s}) < \frac{\ln\Gamma(x+1) - \ln\Gamma(x+s)}{1-s} < \psi\left(x + \frac{s+1}{2}\right). \tag{20}$$

In 2005, D. Kershaw [25] proved that for $s, t > 0$,

$$\psi(x+\sqrt{st}) < \frac{\ln\Gamma(x+t) - \ln\Gamma(x+s)}{t-s} < \psi\left(x + \frac{s+t}{2}\right). \tag{21}$$

N. Elezović and J. Pečarić [16, Lemma 1] proved that for $s, t > 0$,

$$\psi(L(s,t)) \le \frac{\ln\Gamma(t) - \ln\Gamma(s)}{t-s}. \tag{22}$$

N. Batir [3, Theorem 2.7] established an extended form of (22): Let $x$ and $y$ be positive real numbers and $n$ be a positive integer. Then

$$(-1)^n\psi^{(n+1)}\left(\frac{x+y}{2}\right) < \frac{(-1)^n[\psi^{(n)}(x) - \psi^{(n)}(y)]}{x-y} < (-1)^n\psi^{(n+1)}\left(S_{-(n+1)}(x,y)\right). \tag{23}$$

Recently, F. Qi et al. [44] presented closer lower bound:

$$(-1)^n\psi^{(n+1)}(I(x,y)) < \frac{(-1)^n[\psi^{(n)}(x) - \psi^{(n)}(y)]}{x-y}. \tag{24}$$

There have been a lot of literature about the second Gautschi-Kershaw's inequality, please refer to [1, 8, 10, 15, 20, 23, 33], [43]-[48].

Our Theorem 2 establishes new bounds, and the series representation for the ratio $\frac{\ln\Gamma(y)-\ln\Gamma(x)}{y-x}$.

**Theorem 2.** *Let $x, y > 0$ with $x \neq y$, then*

$$\frac{1}{3}A\left(\psi(x),\psi(y)\right)+\frac{2}{3}\psi\left(A(x,y)\right)-\frac{(y-x)^4}{2880}\psi^{(4)}(\max(x,y)) < \frac{\ln\Gamma(y)-\ln\Gamma(x)}{y-x}$$
$$< \frac{1}{3}A\left(\psi(x),\psi(y)\right)+\frac{2}{3}\psi\left(A(x,y)\right)-\frac{(y-x)^4}{2880}\psi^{(4)}(\min(x,y)), \tag{25}$$

$$\frac{\ln\Gamma(y)-\ln\Gamma(x)}{y-x}=\sum_{k=0}^{\infty}\frac{1}{(2k+1)!}\left(\frac{y-x}{2}\right)^{2k}\psi^{(2k)}\left(\frac{x+y}{2}\right). \tag{26}$$

*Proof.* It is known that

$$\psi^{(n)}(x)=(-1)^{n+1}\int_0^\infty\frac{t^n}{1-e^{-t}}e^{-xt}\,\mathrm{d}t,\quad n=1,2,\ldots. \tag{27}$$

By (12) and (27), we get

$$\begin{aligned}\frac{\ln\Gamma(y)-\ln\Gamma(x)}{y-x}&=\frac{1}{y-x}\int_x^y\psi(t)dt\\&=\frac{1}{6}\left[\psi(x)+4\psi\left(\frac{x+y}{2}\right)+\psi(y)\right]-\frac{(y-x)^4}{2880}\psi^{(4)}(\xi)\\&=\frac{1}{3}A\left(\psi(x),\psi(y)\right)+\frac{2}{3}\psi\left(A(x,y)\right)-\frac{(y-x)^4}{2880}\psi^{(4)}(\xi)\end{aligned}$$

for some $\xi$ between $x$ and $y$. This yields (25).

By applying the following result (see [37]):

$$\frac{1}{y-x}\int_x^y f(t)dt=\sum_{k=0}^{\infty}\frac{1}{(2k+1)!}\left(\frac{y-x}{2}\right)^{2k}f^{(2k)}\left(\frac{x+y}{2}\right), \tag{28}$$

we obtain

$$\begin{aligned}\frac{\ln\Gamma(y)-\ln\Gamma(x)}{y-x}&=\frac{1}{y-x}\int_x^y\psi(t)dt\\&=\sum_{k=0}^{\infty}\frac{1}{(2k+1)!}\left(\frac{y-x}{2}\right)^{2k}\psi^{(2k)}\left(\frac{x+y}{2}\right).\end{aligned}$$

The proof of Theorem 2 is complete. □

*Remark* 2. Since $\psi^{(2k)}(x)<0, k=1,2,\ldots$, by (26) we can obtain the second inequality in (21), replacing $y$ and $x$ with $x+t$ and $x+s$, respectively .

The following Theorem 3 presents an extended form of (25).

**Theorem 3.** *Let $x, y > 0$ with $x \neq y$, then for all integers $n \geq 0$,*

$$\begin{aligned}
&(-1)^n\left[\frac{1}{3}A\left(\psi^{(n+1)}(x), \psi^{(n+1)}(y)\right) + \frac{2}{3}\psi^{(n+1)}\left(A(x,y)\right) - \frac{(y-x)^4}{2880}\psi^{(n+5)}(\min(x,y))\right] \\
&< \frac{(-1)^n[\psi^{(n)}(y) - \psi^{(n)}(x)]}{y-x} \\
&< (-1)^n\left[\frac{1}{3}A\left(\psi^{(n+1)}(x), \psi^{(n+1)}(y)\right) + \frac{2}{3}\psi^{(n+1)}\left(A(x,y)\right) - \frac{(y-x)^4}{2880}\psi^{(n+5)}(\max(x,y))\right],
\end{aligned} \tag{29}$$

$$\frac{\psi^{(n)}(y) - \psi^{(n)}(x)}{y-x} = \sum_{k=0}^{\infty}\frac{1}{(2k+1)!}\left(\frac{y-x}{2}\right)^{2k}\psi^{(2k+n+1)}\left(\frac{x+y}{2}\right). \tag{30}$$

*Proof.* By (12) and (27), we get

$$\begin{aligned}
&\frac{\psi^{(n)}(y) - \psi^{(n)}(x)}{y-x} = \frac{1}{y-x}\int_x^y \psi^{(n+1)}(t)dt \\
&= \frac{1}{6}\left[\psi^{(n+1)}(x) + 4\psi^{(n+1)}\left(\frac{x+y}{2}\right) + \psi^{(n+1)}(y)\right] - \frac{(y-x)^4}{2880}\psi^{(n+5)}(\xi) \\
&= \frac{1}{3}A\left(\psi^{(n+1)}(x), \psi^{(n+1)}(y)\right) + \frac{2}{3}\psi^{(n+1)}\left(A(x,y)\right) - \frac{(y-x)^4}{2880}\psi^{(n+5)}(\xi)
\end{aligned}$$

for some $\xi$ between $x$ and $y$.

It is easy to see that if $n$ is an even number, then

$$\begin{aligned}
&\frac{1}{3}A\left(\psi^{(n+1)}(x), \psi^{(n+1)}(y)\right) + \frac{2}{3}\psi^{(n+1)}\left(A(x,y)\right) - \frac{(y-x)^4}{2880}\psi^{(n+5)}(\min(x,y)) \\
&< \frac{\psi^{(n)}(y) - \psi^{(n)}(x)}{y-x} \\
&< \frac{1}{3}A\left(\psi^{(n+1)}(x), \psi^{(n+1)}(y)\right) + \frac{2}{3}\psi^{(n+1)}\left(A(x,y)\right) - \frac{(y-x)^4}{2880}\psi^{(n+5)}(\max(x,y)),
\end{aligned} \tag{31}$$

while the reverse inequality holds if $n$ is an odd number. Hence, (29) is valid for all integers $n \geq 0$.

By applying (28), we obtain

$$\begin{aligned}
&\frac{\psi^{(n)}(y) - \psi^{(n)}(x)}{y-x} = \frac{1}{y-x}\int_x^y \psi^{(n+1)}(t)dt \\
&= \sum_{k=0}^{\infty}\frac{1}{(2k+1)!}\left(\frac{y-x}{2}\right)^{2k}\psi^{(2k+n+1)}\left(\frac{x+y}{2}\right).
\end{aligned}$$

The proof of Theorem 3 is complete. □

*Remark* 3. Since $\psi^{(2k-1)}(x) > 0$ and $\psi^{(2k)}(x) < 0, k = 1, 2, \ldots$, by (30) we can obtain the first inequality in (23).

## References


[1] H. Alzer, *Some gamma function inequalities*, Math. comp. **60** (1993), 337-346.
[2] H. Alzer and S. Ruscheweyh, *On the intersection of two-parameter mean value families*, Proc. Amer. Math. Soc. **129** (2001), no. 9, 2655–2662.
[3] N. Batir, *On some properties of digamma and polygamma functions*, J. Math. Anal. Appl. **328** (2007), 452-465.
[4] J. L. Brenner and B. C. Carlson, *Homogeneous mean values: weights and asymptotics*, J. Math. Anal. Appl. **123** (1987), no. 1, 265-280.
[5] P. S. Bullen, D. S. Mitrinović and P. M. Vasić, *Means and their inequalities*, D. Reidel Publishing Co., Dordrecht, 1988.
[6] R. L. Burden, J. D. Faires and A. C. Reynolds, *Numerical analysis*, second ed., Prindle, Weber & Schmidt, 1981.
[7] F. Burk, *The Geometric, Logarithmic, and Arithmetic Mean Inequality*, Amer. Math. Monthly **94** (1987), no. 6, 527-528.
[8] J. Bustoz, M.E.M. Ismail, *On gamma function inequalities*, Math. Comp. **47** (1986), 659-667.
[9] B. C. Carlson, *The logarithmic mean*, Amer. Math. Monthly **79** (1972), 615-618.
[10] Ch.-P. Chen, *Monotonicity and convexity for the gamma function*, J. Inequal. Pure Appl. Math. **6** (2005), no. 4, Article 100, 6 pp.
[11] P. Czinder and Zs. Ples, *A general Minkowski-type inequality for two variable Gini means*, Publ. Math. Debrecen **57** (2000), no. 1-2, 203-216.
[12] P. Czinder and Zs. Ples, *Local monotonicity properties of two-variable Gini means and the comparison theorem revisited*, J. Math. Anal. Appl. **301** (2005), no. 2, 427-438.
[13] P. J. Davis, *Interpolation and approximation*, Blaisdell Publishing Co. Ginn and Co. New York-Toronto-London 1963.
[14] P. J. Davis and P. Rabinowitz, *Numerical integration*, Blaisdell Publishing Co. Ginn and Co., Waltham, Mass.-Toronto, Ont.-London 1967.
[15] N. Elezović, C. Giordano and J. Pečarić, *The best bounds in Gautschi's inequalitity*, Math. Inequal. Appl. **3** (2000), 239-252.
[16] N. Elezović and J. Pečarić, *Differential and integral f-means and applications to digamma function*, Math. Inequal. Appl. **3** (2000), no.2, 189-196.
[17] W. Gautschi, *Some elementary inequalities relating to the gamme and incomplete gamma function*, J. Math. Phys. **38** (1959), 77-81.
[18] C. Gini, *Di una formula comprensiva delle medie*, Metron **13** (1938), 3-22.
[19] C. Giordano, A. Laforgia, J. Pečarić, *Unified treatment of Gautschi-Kershaw type inequalities for the gamma function*, J. comput. Appl. Math. **99** (1998), 166-175.
[20] C. Giordano, A. Laforgia, J. Pečarić, *Monotonicity properties for some functions involving the ratio of two gamma functions*, in: A. Bellacicco, A. Laforgia (Eds.), Funzioni Speciali e Applicazioin, Franco Angeli, Milano, 1998, pp. 35-42.
[21] G. Hämmerlin and K.-H Hoffmann, *Numerical mathematics*, Translated from the German by Larry Schumaker. Undergraduate Texts in Mathematics. Readings in Mathematics. Springer-Verlag, New York, 1991.
[22] G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, Cambridge-London, 1952.
[23] M.E.H. Ismail, M.E. Muldoon, *Inequalities and monotonicity properties for gamma and q-gamma functions*, in:R.V.M. Zahar (Ed.), Approximation and computation:A Festschrift in Honour of Walter Gautschi, ISNM, Vol. 119, Birkhaüser, Basel. 1994, pp. 309-323.
[24] D. Kershaw, *Some extensions of W. Gautschi's inequalities for gamma function*, Math. Comp. **41** (1983), No. 164, 607–611.
[25] D. Kershaw, *Upper and lower bounds for a ratio involving the gamma function*, Anal. Appl. (Singap.) **3** (2005), no. 3, 293-295.
[26] K. S. Kunz, *Numerical analysis*, New York, 1957.
[27] E. B. Leach and M. C. Sholander, *Extended mean values*, Amer. Math. Monthly **85** (1978), 84–90.
[28] E. B. Leach and M. C. Sholander, *Extended mean values. II* J. Math. Anal. Appl. **92** (1983), no. 1, 207-223.
[29] E. B. Leach and M. C. Sholander, *Multivariable extended mean values*, J. Math. Anal. Appl. **104** (1984), no. 2, 390-407.

[30] T. P. Lin, *The power mean and the logarithmic mean*, Amer. Math. Monthly **81** (1974), 879-883.
[31] L. Losonczi and Zs. Ples, *Minkowski's inequality for two variable Gini means*, Acta Sci. Math. (Szeged) **62** (1996), no. 3-4, 413-425.
[32] L. Losonczi and Zs. Ples, *Minkowski's inequality for two variable difference means*, Proc. Amer. Math. Soc. **126** (1998), no. 3, 779-789.
[33] M.E. Muldoon, *Some monotonicity properties and characterizations of the gamma function*, Aequationes Math. **18** (1978), 54-63.
[34] E. Neuman, *The weighted logarithmic mean*, J. Math. Anal. Appl. **188** (1994), no. 3, 885-900.
[35] E. Neuman, Zs. Pĺes, *On comparison of Stolarsky and Gini means*, J. Math. Anal. Appl. **278** (2003), 274-284.
[36] E. Neuman and J. Sándor, *Inequalities involving Stolarsky and Gini means*, Math. Pannon. **14** (2003), no. 1, 29-44.
[37] E. Neuman and J. Sándor, *On the Ky Fan inequality and related inequalities, II*, Bull. Austral. Math. Soc. **72** (2005), no. 1, 87-107.
[38] Zs. Páles, *Inequalities for differences of powers*, J. Math. Anal. Appl. **131** (1988)., 271–281.
[39] Zs. Páles, *Inequalities for sums of powers*, J. Math. Anal. Appl., **131** (1988), 265-270.
[40] B. Palumbo, *A generalization of some inequalities for the gamma function*, J. comput. Appl. Math. **88** (1998), 255-268.
[41] A. O. Pittenger, *The symmetric, logarithmic and power means*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 681 (1980), 19-23.
[42] F. Qi, *Generalized abstracted mean values*, JIPAM. J. Inequal. Pure Appl. Math. **1** (2000), no. 1, Article 4, 9 pp. (electronic).
[43] F. Qi, B.-N. Guo, Ch.-P. Chen, *The best bounds in Gautschi-Kershaw inequalitities*, Math. Inequal. Appl. **9** (2006), no. 3, 427-436.
[44] F. Qi, S. Guo and Sh.-X. Chen, *A new upper bound in the second Kershaw's double inequality and its generalizations*, J. Comput. Appl. Math. **220** (2008), 111-118.
[45] F. Qi, B.-N. Guo, *Wendels and Gautschis inequalities: Refinements, extensions, and a class of logarithmically completely monotonic functions*, Appl. Math. Comput. (2008), doi: 10.1016/j.amc.2008.07.005
[46] F. Qi, *A class of logarithmically completely monotonic functions and application to the best bounds in the second GautschiCKershaws inequality*, J. Comput. Appl. Math. (2008), doi: 10.1016/j.cam.2008.05.030
[47] F. Qi, *A new lower bound in the second Kershaw's double inequality*, J. Comput. Appl. Math. **214** (2008), no. 2, 610-616.
[48] F. Qi and B.-N. Guo, *A class of logarithmically completely monotonic functions and the best bounds in the second Kershaw's double inequality*, J. Comput. Appl. Math. **212** (2008), no. 2, 444-456.
[49] J. Sándor, *A note on some inequalities for means*, Arch. Math. (Basel) **56** (1991), no. 5, 471-473.
[50] J. Sándor, *Two inequalities for means*, Internat. J. Math. Math. Sci. **18** (1995), no. 3, 621-623.
[51] K. B. Stolarsky, *Generalizations of the logarithmic mean*, Math. Mag. **48** (1975), 87–92.
[52] K. B. Stolarsky, *The power and generalized logarithmic means*, Amer. Math. Monthly, **87** (1980), 545-548.
[53] T. Trif, *Monotonicity, comparison and Minkowski's inequality for generalized Muirhead means in two variables*, Mathematica 48(71) (2006), no. 1, 99-110.

(Ch.-P. Chen) College of Mathematics and Informatics, Henan Polytechnic University, Jiaozuo City, Henan 454010, China
*E-mail address*: chenchaoping@sohu.com