

Sharp Function Estimates for Vector-Valued Multilinear Operator of Multiplier Operator

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Abstract: In this paper, we establish a sharp function estimate for the vector-valued multilinear operator of the multiplier. As the application, we obtain the weighted $L^p(1 < p < \infty)$ norm inequality for the multilinear operator.

Keywords: Vector-valued multilinear operator; Multiplier operator; Sharp estimate; BMO.

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1. Introduction and Results

Let $b \in BMO(R^n)$ and T be the Calderón-Zygmund singular integral operator. The commutator $[b, T]$ generated by b and T is defined by $[b, T](f)(x) = b(x)T(f)(x) - T(bf)(x)$. As the development of the Calderón-Zygmund singular integral operators, their commutators and multilinear operators have been well studied(see [1-7][15-18]). In [10], Hu and Yang proved a variant sharp function estimate for the multilinear singular integral operators. In [18], C.Pérez and R.Trujillo-Gonzalez obtained a sharp weighted estimates for the singular integral operators and their commutators. In [23], You proved that the commutator $[b, T]$ is bounded in $L^p(R^n)$ when T is a multiplier operator and $b \in \dot{\Lambda}_\beta(R^n)$. In [24][25], Zhang studied the $(L^p, \dot{F}_p^{\beta, \infty})$ - boundedness of the commutator of the multipliers. In this paper, we will introduce the vector-valued multilinear operator associated to the multiplier operator and study the sharp function inequality of the vector-valued multilinear operator. By using the sharp inequality, we obtain the weighted L^p - norm inequality for the vector-valued multilinear operator.

First, let us introduce some notations. In this paper, Q will denote a cube of R^n with sides

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parallel to the axes. For a cube Q and a locally integrable function b , let $b_Q = |Q|^{-1} \int_Q b(x) dx$. The sharp function of b is defined by

$$b^\#(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |b(y) - b_Q| dy.$$

It is well-known that (see [9])

$$b^\#(x) \approx \sup_{Q \ni x} \inf_{c \in \mathbf{C}} \frac{1}{|Q|} \int_Q |b(y) - c| dy$$

and

$$\|b - b_{2^k Q}\|_{BMO} \leq Ck \|b\|_{BMO} \text{ for } k \geq 1.$$

We say that b belongs to $BMO(R^n)$, if $b^\#$ belongs to $L^\infty(R^n)$ and define

$$\|b\|_{BMO} = \|b^\#\|_{L^\infty}.$$

Let M be the Hardy-Littlewood maximal operator defined by

$$M(f)(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)| dy,$$

we write that $M_p(f) = (M(f^p))^{1/p}$ for $0 < p < \infty$.

We denote the Muckenhoupt weights by A_1 (see [9]), that is

$$A_1 = \{w : M(w)(x) \leq Cw(x), a.e.\}.$$

A bounded measurable function k defined on $R^n \setminus \{0\}$ is called a multiplier. The multiplier operator T associated with k is defined by

$$T(f)(x) = k(x) \hat{f}(x), \text{ for } f \in S(R^n),$$

where \hat{f} denotes the Fourier transform of f and $S(R^n)$ is the Schwartz test function class. Now, we recall the definition of the class $M(s, l)$. Denote by $|x| \sim t$ the fact that the value of x lies in the annulus $\{x \in R^n : at < |x| < bt\}$, where $0 < a \leq 1 < b < \infty$ are values specified in each instance.

Definition 1. ([11]) Let $l \geq 0$ be a real number and $1 \leq s \leq 2$. we say that the multiplier k satisfies the condition $M(s, l)$, if

$$\left(\int_{|\xi| \sim R} |D^\alpha k(\xi)|^s d\xi \right)^{\frac{1}{s}} < CR^{n/s - |\alpha|}$$

for all $R > 0$ and multi-indices α with $|\alpha| \leq l$, when l is a positive integer, and, in addition, if

$$\left(\int_{|\xi| \sim R} |D^\alpha k(\xi) - D^\alpha k(\xi - z)|^s d\xi \right)^{\frac{1}{s}} \leq C \left(\frac{|z|}{R} \right)^\gamma R^{\frac{n}{s} - |\alpha|}$$

for all $|z| < R/2$ and all multi-indices α with $|\alpha| = [l]$, the integer part of l , i.e., $[l]$ is the greatest integer less than or equal to l , and $l = [l] + \gamma$ when l is not an integer.

Denote $D(R^n) = \{\phi \in S(R^n) : \text{supp}(\phi) \text{ is compact}\}$ and $\hat{D}_0(R^n) = \{\phi \in S(R^n) : \hat{\phi} \in D(R^n) \text{ and } \hat{\phi} \text{ vanishes in a neighbourhood of the origin}\}$. The following boundedness property of T on $L^p(R^n)$ is proved by Strömberg and Torkinsky (see [11-14]).

Lemma 1. ([11]) Let $k \in M(s, l)$, $1 \leq s \leq 2$, and $l > \frac{n}{s}$. Then the associated mapping T , defined a priori for $f \in \hat{D}_0(R^n)$, $T(f)(x) = (f * K)(x)$, extends to a bounded mapping from $L^p(R^n)$ into itself for $1 < p < \infty$ and $K(x) = \check{k}(x)$.

Definition 2. ([11]) For a real number $\tilde{l} \geq 0$ and $1 \leq \tilde{s} < \infty$, we say that K verifies the condition $\tilde{M}(\tilde{s}, \tilde{l})$, and write $K \in \tilde{M}(\tilde{s}, \tilde{l})$, if

$$\left(\int_{|x| \sim R} |D^{\tilde{\alpha}} K(x)|^{\tilde{s}} dx \right)^{\frac{1}{\tilde{s}}} \leq C R^{n/\tilde{s} - n - |\tilde{\alpha}|}, \quad R > 0$$

for all multi-indices $|\tilde{\alpha}| \leq \tilde{l}$ and, in addition, if

$$\left(\int_{|x| \sim R} |D^{\tilde{\alpha}} K(x) - D^{\tilde{\alpha}} K(x - z)|^{\tilde{s}} dx \right)^{\frac{1}{\tilde{s}}} \leq C \left(\frac{|z|}{R} \right)^v R^{\frac{n}{\tilde{s}} - n - u}, \quad \text{if } 0 < v < 1,$$

$$\left(\int_{|x| \sim R} |D^{\tilde{\alpha}} K(x) - D^{\tilde{\alpha}} K(x - z)|^{\tilde{s}} dx \right)^{\frac{1}{\tilde{s}}} \leq C \left(\frac{|z|}{R} \right) \left(\log \frac{R}{|z|} \right) R^{\frac{n}{\tilde{s}} - n - u}, \quad \text{if } v = 1,$$

for all $|z| < \frac{R}{2}$, $R > 0$, and all multi-indices $\tilde{\alpha}$ with $|\tilde{\alpha}| = u$, where u denotes the largest integer strictly less than \tilde{l} with $\tilde{l} = u + v$.

Lemma 2. ([11]) Suppose $k \in M(s, l)$, $1 \leq s \leq 2$. Given $1 \leq \tilde{s} < \infty$, let $r \geq 1$ be such that $\frac{1}{r} = \max\{\frac{1}{s}, 1 - \frac{1}{\tilde{s}}\}$. Then $K \in \tilde{M}(\tilde{s}, \tilde{l})$, where $\tilde{l} = l - \frac{n}{r}$.

Lemma 3. Let $1 \leq s \leq 2$, suppose that l is a positive real number with $l > n/r$, $1/r = \max\{1/s, 1 - 1/\tilde{s}\}$, and $k \in M(s, l)$. Then there is a positive constant a , such that

$$\left(\int_{B_k} |K(x - z) - K(x_Q - z)|^{\tilde{s}} dz \right)^{1/\tilde{s}} \leq C 2^{-ka} (2^k h)^{-n/\tilde{s}'}$$

Proof. We split our proof into two cases:

Case 1. $1 \leq s \leq 2$ and $0 < l - n/s \leq 1$. We choose a real number $1 < \tilde{s} < \infty$ such that $s \leq \tilde{s}$, and set $\tilde{l} = l - \frac{n}{s} > 0$. Since $k \in M(s, l)$, then by Lemma 3, there is $K \in \tilde{M}(\tilde{s}, \tilde{l})$.

When $\tilde{l} = l - \frac{n}{s} < 1$, noting that l is a positive real number and $l > \frac{n}{s}$. Applying the condition $K \in \tilde{M}(\tilde{s}, \tilde{l})$ for $v = l - \frac{n}{s}$ and $u = 0$, one has

$$\left(\int_{B_k} |K(x-z) - K(x_Q-z)|^{\tilde{s}} dz \right)^{\frac{1}{\tilde{s}}} \leq C 2^{-k(l-\frac{n}{s})} (2^k h)^{-\frac{n}{\tilde{s}'}}$$

let $a = l - \frac{n}{s}$,

$$\left(\int_{B_k} |K(x-z) - K(x_Q-z)|^{\tilde{s}} dz \right)^{\frac{1}{\tilde{s}}} \leq C 2^{-ka} (2^k h)^{-\frac{n}{\tilde{s}'}}$$

When $\tilde{l} = l - \frac{n}{s} = 1$, we choose $0 < \xi < 1$, such that $t^{1-\xi} \log(1/t) \leq C$ for $0 < t < 1/2$. Noting that $K \in \tilde{M}(\tilde{s}, \tilde{l})$, by Definition 2, for $u = 0, v = 1$,

$$\begin{aligned} & \left(\int_{B_k} |K(x-z) - K(x_Q-z)|^{\tilde{s}} dz \right)^{\frac{1}{\tilde{s}}} \\ & \leq C \left(\frac{|y-x_Q|}{2^k h} \right)^\xi \left(\frac{|y-x_Q|}{2^k h} \right)^{1-\xi} \left(\log \frac{2^k h}{|y-x_Q|} \right) (2^k h)^{n/\tilde{s}-n} \\ & \leq C 2^{-k\xi} (2^k h)^{-n/\tilde{s}'}, \end{aligned}$$

let $a = \xi$, then

$$\left(\int_{B_k} |K(x-z) - K(x_Q-z)|^{\tilde{s}} dz \right)^{\frac{1}{\tilde{s}}} \leq C 2^{-ka} (2^k h)^{-n/\tilde{s}'}$$

Case 2. $1 \leq s \leq 2$ and $l - n/s > 1$. Set $d = [l - n/s]$, if $l - n/s > 1$ is not an integer, and $d = l - n/s - 1$ if $l - n/s > 1$ is an integer. Choose $l_1 = l - d$; then $0 < l_1 - n/s \leq 1$ and $0 < l_1 < l$. So, from $k \in M(s, l)$ we know $k \in M(s, l_1)$. Set $\tilde{l} = l_1 - n/s$; by Lemma 3, $K \in \tilde{M}(\tilde{s}, \tilde{l})$. Repeating the proof of **Case 1**, except for replacing l by l_1 , we can obtain the same result under the assumption $l - n/s > 1$. We omit the details here.

Certainly when $0 < \tilde{s}' < s$, which is the same as the above.

Now we can define the vector-valued multilinear operator associated to the multiplier operator T . Let m_j be the positive integers ($j = 1, \dots, l$), $m_1 + \dots + m_l = m$ and b_j be the functions on R^n ($j = 1, \dots, l$). Set, for $1 \leq j \leq m$,

$$R_{m_j+1}(b_j; x, y) = b_j(x) - \sum_{|\alpha| \leq m_j} \frac{1}{\alpha!} D^\alpha b_j(y) (x-y)^\alpha.$$

By Lemma 1, $T(f)(x) = (K * f)(x)$ for $K(x) = \check{k}(x)$. Given the functions f_i defined on R^n , $i = 1, 2, \dots$, for $1 < r < \infty$, the the vector-valued multilinear operator associated to T is defined by

$$|T_b(f)(x)|_r = \left(\sum_{i=1}^{\infty} (T_b(f_i)(x))^r \right)^{1/r},$$

where

$$T_b(f_i)(x) = \int_{R^n} \frac{\prod_{j=1}^l R_{m_j+1}(b_j; x, y)}{|x-y|^m} K(x-y) f_i(y) dy.$$

Set

$$|T(f)(x)|_r = \left(\sum_{i=1}^{\infty} |T(f_i)(x)|^r \right)^{1/r} \quad \text{and} \quad |f(x)|_r = \left(\sum_{i=1}^{\infty} |f_i(x)|^r \right)^{1/r}.$$

Note that when $m = 0$, $|T_b|_r$ is just the vector-valued multilinear commutator of T and b_j (see [18]). While when $m > 0$, $|T_b|_r$ is non-trivial generalizations of the commutator. It is well known that multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors (see [2-5][7]). Hu and Yang (see [10]) proved a variant sharp estimate for the multilinear singular integral operators. In [17], Pérez and Trujillo-Gonzalez prove a sharp estimate for the multilinear commutator when $b_j \in Osc_{expL^{r_j}}(R^n)$. The main purpose of this paper is to prove a sharp function inequality for the vector-valued multilinear multiplier operator when $D^\alpha b_j \in BMO(R^n)$ for all α with $|\alpha| = m_j$. As the application, we obtain the $L^p(p > 1)$ norm inequality for the vector-valued multilinear operator.

We shall prove the following theorems.

Theorem 1. Let $1 < r < \infty$ and $D^\alpha b_j \in BMO(R^n)$ for all α with $|\alpha| = m_j$ and $j = 1, \dots, l$. Then there exists a constant $C > 0$ such that for every $f \in C_0^\infty(R^n)$, $1 < s < \infty$ and $\tilde{x} \in R^n$,

$$(|T_b(f)|_r)_s^\#(\tilde{x}) \leq C \prod_{j=1}^l \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) M_s(|f|_r)(\tilde{x}).$$

Theorem 2. Let $1 < r < \infty$ and $D^\alpha b_j \in BMO(R^n)$ for all α with $|\alpha| = m_j$ and $j = 1, \dots, l$. Then $|T_b|_r$ is bounded on $L^p(w)$ for any $w \in A_1$ and $1 < p < \infty$, that is

$$\| |T_b(f)|_r \|_{L^p(w)} \leq C \prod_{j=1}^l \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) \| |f|_r \|_{L^p(w)}.$$

2. Proofs of Theorems

To prove the theorems, we need the following lemmas.

Lemma 4.([4]) Let b be a function on R^n and $D^\alpha b \in L^q(R^n)$ for all α with $|\alpha| = m$ and some $q > n$. Then

$$|R_m(b; x, y)| \leq C |x-y|^m \sum_{|\alpha|=m} \left(\frac{1}{|\tilde{Q}(x, y)|} \int_{\tilde{Q}(x, y)} |D^\alpha b(z)|^q dz \right)^{1/q},$$

where \tilde{Q} is the cube centered at x and having side length $5\sqrt{n}|x-y|$.

Lemma 5.([8][11]) Let T be the multiplier operator. Then, for every $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$,

$$\| |T(f)|_r \|_{L^p} \leq C \| |f|_r \|_{L^p}.$$

Proof of Theorem 1. It suffices to prove for $f \in C_0^\infty(\mathbb{R}^n)$ and some constant C_0 , the following inequality holds:

$$\frac{1}{|Q|} \int_Q \| |T^A(f)(x)|_r - C_0 \| dx \leq C \prod_{j=1}^l \left(\sum_{|\alpha_j|=m_j} \| D^{\alpha_j} b_j \|_{BMO} \right) M_s(|f|_r)(x).$$

Without loss of generality, we may assume $l = 2$. Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Let $\tilde{Q} = 5\sqrt{n}Q$ and $\tilde{b}_j(x) = b_j(x) - \sum_{|\alpha|=m_j} \frac{1}{\alpha!} (D^\alpha b_j)_{\tilde{Q}} x^\alpha$, then $R_{m_j+1}(b_j; x, y) = R_{m_j+1}(\tilde{b}_j; x, y)$ and $D^\alpha \tilde{b}_j = D^\alpha b_j - (D^\alpha b_j)_{\tilde{Q}}$ for $|\alpha| = m_j$. We split $f = g + h = \{g_i\} + \{h_i\}$ for $g_i = f_i \chi_{\tilde{Q}}$ and $h_i = f_i \chi_{\mathbb{R}^n \setminus \tilde{Q}}$. Write

$$\begin{aligned} T_b(f_i)(x) &= \int_{\mathbb{R}^n} \frac{\prod_{j=1}^2 R_{m_j}(\tilde{b}_j; x, y)}{|x-y|^m} K(x-y) g_i(y) dy \\ &\quad - \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \int_{\mathbb{R}^n} \frac{R_{m_2}(\tilde{b}_2; x, y)(x-y)^{\alpha_1}}{|x-y|^m} D^{\alpha_1} \tilde{b}_1(y) K(x-y) g_i(y) dy \\ &\quad - \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \int_{\mathbb{R}^n} \frac{R_{m_1}(\tilde{b}_1; x, y)(x-y)^{\alpha_2}}{|x-y|^m} D^{\alpha_2} \tilde{b}_2(y) K(x-y) g_i(y) dy \\ &\quad + \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \frac{1}{\alpha_1! \alpha_2!} \int_{\mathbb{R}^n} \frac{(x-y)^{\alpha_1+\alpha_2} D^{\alpha_1} \tilde{b}_1(y) D^{\alpha_2} \tilde{b}_2(y)}{|x-y|^m} K(x-y) g_i(y) dy \\ &\quad + \int_{\mathbb{R}^n} \frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{b}_j; x, y)}{|x-y|^m} K(x-y) h_i(y) dy \\ &= T \left(\frac{\prod_{j=1}^2 R_{m_j}(\tilde{b}_j; x, \cdot)}{|x-\cdot|^m} g_i \right) (x) \\ &\quad - T \left(\sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \frac{R_{m_2}(\tilde{b}_2; x, \cdot)(x-\cdot)^{\alpha_1} D^{\alpha_1} \tilde{b}_1}{|x-\cdot|^m} g_i \right) (x) \\ &\quad - T \left(\sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \frac{R_{m_1}(\tilde{b}_1; x, \cdot)(x-\cdot)^{\alpha_2} D^{\alpha_2} \tilde{b}_2}{|x-\cdot|^m} g_i \right) (x) \\ &\quad + T \left(\sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \frac{1}{\alpha_1! \alpha_2!} \frac{(x-\cdot)^{\alpha_1+\alpha_2} D^{\alpha_1} \tilde{b}_1 D^{\alpha_2} \tilde{b}_2}{|x-\cdot|^m} g_i \right) (x) + T_b(h_i)(x), \end{aligned}$$

then, by Minkowski' inequality,

$$\frac{1}{|Q|} \int_Q \| |T_b(f)(x)|_r - |T_b(h)(x_0)|_r \| dx$$

$$\begin{aligned}
&\leq \frac{1}{|Q|} \int_Q \left(\sum_{i=1}^{\infty} |T_b(f_i)(x) - T_{\tilde{b}}(h_i)(x_0)|^r \right)^{1/r} dx \\
&\leq \frac{1}{|Q|} \int_Q \left(\sum_{i=1}^{\infty} \left| T \left(\frac{\prod_{j=1}^2 R_{m_j}(\tilde{b}_j; x, \cdot)}{|x - \cdot|^m} g_i \right) (x) \right|^r \right)^{1/r} dx \\
&\quad + \frac{1}{|Q|} \int_Q \left(\sum_{i=1}^{\infty} \left| T \left(\sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \frac{R_{m_2}(\tilde{b}_2; x, \cdot)(x - \cdot)^{\alpha_1} D^{\alpha_1} \tilde{b}_1}{|x - \cdot|^m} g_i \right) (x) \right|^r \right)^{1/r} dx \\
&\quad + \frac{1}{|Q|} \int_Q \left(\sum_{i=1}^{\infty} \left| T \left(\sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \frac{R_{m_1}(\tilde{b}_1; x, \cdot)(x - \cdot)^{\alpha_2} D^{\alpha_2} \tilde{b}_2}{|x - \cdot|^m} g_i \right) (x) \right|^r \right)^{1/r} dx \\
&\quad + \frac{1}{|Q|} \int_Q \left(\sum_{i=1}^{\infty} \left| T \left(\sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \frac{1}{\alpha_1! \alpha_2!} \frac{(x - \cdot)^{\alpha_1 + \alpha_2} D^{\alpha_1} \tilde{b}_1 D^{\alpha_2} \tilde{b}_2}{|x - \cdot|^m} g_i \right) (x) \right|^r \right)^{1/r} dx \\
&\quad + \frac{1}{|Q|} \int_Q \left(\sum_{i=1}^{\infty} |T_{\tilde{b}}(h_i)(x) - T_{\tilde{b}}(h_i)(x_0)|^r \right)^{1/r} dx \\
&:= I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}$$

Now, let us estimate I_1, I_2, I_3, I_4 and I_5 , respectively. First, for $x \in Q$ and $y \in \tilde{Q}$, by Lemma 4, we get

$$R_m(\tilde{b}_j; x, y) \leq C|x - y|^m \sum_{|\alpha_j|=m} \|D^{\alpha_j} b_j\|_{BMO},$$

thus, by the Hölder's inequality and L^s -boundedness of $|T|_r$ (Lemma 5), we obtain

$$\begin{aligned}
I_1 &\leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) \frac{1}{|Q|} \int_Q |T(g)(x)|_r dx \\
&\leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) \left(\frac{1}{|Q|} \int_{R^n} |T(g)(x)|_r^s dx \right)^{1/s} \\
&\leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) \left(\frac{1}{|Q|} \int_{R^n} |g(x)|_r^s dx \right)^{1/s} \\
&\leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} |f(x)|_r^s dx \right)^{1/s} \\
&\leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) M_s(|f|_r)(\tilde{x}).
\end{aligned}$$

For I_2 , denoting $s = pq$ for $1 < p < \infty, q > 1$ and $1/q + 1/q' = 1$, we have, by Lemma 5,

$$I_2 \leq C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2} b_2\|_{BMO} \sum_{|\alpha_1|=m_1} \frac{1}{|Q|} \int_Q |T(D^{\alpha_1} \tilde{b}_1 g)(x)|_r dx$$

$$\begin{aligned}
&\leq C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2} b_2\|_{BMO} \sum_{|\alpha_1|=m_1} \left(\frac{1}{|Q|} \int_{R^n} |T(D^{\alpha_1} \tilde{b}_1 g)(x)|_r^p dx \right)^{1/p} \\
&\leq C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2} b_2\|_{BMO} \sum_{|\alpha_1|=m_1} \left(\frac{1}{|Q|} \int_{R^n} |D^{\alpha_1} \tilde{b}_1(x)| |g(x)|_r^p dx \right)^{1/p} \\
&\leq C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2} b_2\|_{BMO} \sum_{|\alpha_1|=m_1} \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} |D^{\alpha_1} b_1(x) - (D^{\alpha} b_j)_{\tilde{Q}}|^{pq'} dx \right)^{1/pq'} \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} |f(x)|_r^{pq} dx \right)^{1/pq} \\
&\leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^{\alpha} b_j\|_{BMO} \right) M_s(|f|_r)(\tilde{x}).
\end{aligned}$$

For I_3 , similar to the proof of I_2 , we get

$$I_3 \leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^{\alpha} b_j\|_{BMO} \right) M_s(|f|_r)(\tilde{x}).$$

Similarly, for I_4 , denoting $s = pq_3$ for $1 < p < \infty$, $q_1, q_2, q_3 > 1$ and $1/q_1 + 1/q_2 + 1/q_3 = 1$, we obtain

$$\begin{aligned}
I_4 &\leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \frac{1}{|Q|} \int_Q |T(D^{\alpha_1} \tilde{b}_1 D^{\alpha_2} \tilde{b}_2 g)(x)|_r dx \\
&\leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \left(\frac{1}{|Q|} \int_{R^n} |T(D^{\alpha_1} \tilde{b}_1 D^{\alpha_2} \tilde{b}_2 g)(x)|_r^p dx \right)^{1/p} \\
&\leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \left(\frac{1}{|Q|} \int_{R^n} |D^{\alpha_1} \tilde{b}_1(x) D^{\alpha_2} \tilde{b}_2(x)| |g(x)|_r^p dx \right)^{1/p} \\
&\leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \prod_{j=1}^2 \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} |D^{\alpha_j} \tilde{b}_j(x)|^{pq_j} dx \right)^{1/pq_j} \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} |f(x)|_r^{pq_3} dx \right)^{1/pq_3} \\
&\leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^{\alpha} b_j\|_{BMO} \right) M_s(|f|_r)(\tilde{x}).
\end{aligned}$$

For I_5 , we write

$$\begin{aligned}
T_{\tilde{b}}(h_i)(x) - T_{\tilde{b}}(h_i)(x_0) &= \int_{R^n} (K(x-y) - K(x_0-y)) \frac{1}{|x-y|^m} \prod_{j=1}^2 R_{m_j}(\tilde{b}_j; x, y) h_i(y) dy \\
&+ \int_{R^n} \left(\frac{1}{|x-y|^m} - \frac{1}{|x_0-y|^m} \right) K(x_0-y) \prod_{j=1}^2 R_{m_j}(\tilde{b}_j; x, y) h_i(y) dy \\
&+ \int_{R^n} (R_{m_1}(\tilde{b}_1; x, y) - R_{m_1}(\tilde{b}_1; x_0, y)) \frac{R_{m_2}(\tilde{b}_2; x, y)}{|x_0-y|^m} K(x_0-y) h_i(y) dy \\
&+ \int_{R^n} (R_{m_2}(\tilde{b}_2; x, y) - R_{m_2}(\tilde{b}_2; x_0, y)) \frac{R_{m_1}(\tilde{b}_1; x_0, y)}{|x_0-y|^m} K(x_0-y) h_i(y) dy
\end{aligned}$$

$$\begin{aligned}
& - \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \int_{R^n} \left[\frac{R_{m_2}(\tilde{b}_2; x, y)(x-y)^{\alpha_1}}{|x-y|^m} K(x-y) - \frac{R_{m_2}(\tilde{b}_2; x_0, y)(x_0-y)^{\alpha_1}}{|x_0-y|^m} K(x_0-y) \right] \\
& \times D^{\alpha_1} \tilde{b}_1(y) h_i(y) dy \\
& - \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \int_{R^n} \left[\frac{R_{m_1}(\tilde{b}_1; x, y)(x-y)^{\alpha_2}}{|x-y|^m} K(x-y) - \frac{R_{m_1}(\tilde{b}_1; x_0, y)(x_0-y)^{\alpha_2}}{|x_0-y|^m} K(x_0-y) \right] \\
& \times D^{\alpha_2} \tilde{b}_2(y) h_i(y) dy \\
& + \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \frac{1}{\alpha_1! \alpha_2!} \int_{R^n} \left[\frac{(x-y)^{\alpha_1+\alpha_2}}{|x-y|^m} K(x-y) - \frac{(x_0-y)^{\alpha_1+\alpha_2}}{|x_0-y|^m} K(x_0-y) \right] \\
& \times D^{\alpha_1} \tilde{b}_1(y) D^{\alpha_2} \tilde{b}_2(y) h_i(y) dy \\
& = I_5^{(1)} + I_5^{(2)} + I_5^{(3)} + I_5^{(4)} + I_5^{(5)} + I_5^{(6)} + I_5^{(7)}.
\end{aligned}$$

By Lemma 4 and the following inequality(see [20])

$$|b_{Q_1} - b_{Q_2}| \leq C \log(|Q_2|/|Q_1|) \|b\|_{BMO} \text{ for } Q_1 \subset Q_2,$$

we know that, for $x \in Q$ and $y \in 2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}$,

$$\begin{aligned}
|R_m(\tilde{b}; x, y)| & \leq C|x-y|^m \sum_{|\alpha|=m} (\|D^\alpha b\|_{BMO} + |(D^\alpha b)_{\tilde{Q}(x,y)} - (D^\alpha b)_{\tilde{Q}}|) \\
& \leq Ck|x-y|^m \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO}.
\end{aligned}$$

Note that $|x-y| \sim |x_0-y|$ for $x \in Q$ and $y \in R^n \setminus \tilde{Q}$, we have

$$|I_5^{(1)}| \leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha b_j\|_{BMO} \right) \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} k^2 |K(x-y) - K(x_0-y)| |f_i(y)| dy,$$

thus, by the Minkowski' inequality and Lemma 3, we obtain

$$\begin{aligned}
\left(\sum_{i=1}^{\infty} \|I_5^{(1)}\|^r \right)^{1/r} & \leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha b_j\|_{BMO} \right) \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} k^2 |K(x-y) - K(x_0-y)| |f(y)|_r dy \\
& \leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha b_j\|_{BMO} \right) \sum_{k=0}^{\infty} k^2 \left(\int_{2^{k+1}\tilde{Q}} |f(y)|_r^s dy \right)^{1/s} \\
& \quad \times \left(\int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} |K(x-y) - K(x_0-y)|^{s'} dy \right)^{1/s'} \\
& \leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha b_j\|_{BMO} \right) \sum_{k=1}^{\infty} k^2 2^{-ka} \left(\frac{1}{|2^k\tilde{Q}|} \int_{2^k\tilde{Q}} |f(y)|_r^s dy \right)^{1/s}
\end{aligned}$$

$$\begin{aligned}
&\leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha b_j\|_{BMO} \right) \sum_{k=1}^{\infty} k^2 2^{-ka} M_s(|f|_r)(\tilde{x}) \\
&\leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha b_j\|_{BMO} \right) M_s(|f|_r)(\tilde{x}).
\end{aligned}$$

For $I_5^{(2)}$, by the Minkowski' inequality and Lemma 2, we obtain

$$\begin{aligned}
\left(\sum_{i=1}^{\infty} \|I_5^{(2)}\|^r \right)^{1/r} &\leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha b_j\|_{BMO} \right) \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} k^2 \frac{|x-x_0|}{|x_0-y|} |K(x_0-y)| |f(y)|_r dy \\
&\leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha b_j\|_{BMO} \right) \sum_{k=0}^{\infty} k^2 2^{-k} \left(\int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} |K(x_0-y)|^{s'} dy \right)^{1/s'} \\
&\quad \times \left(\int_{2^{k+1}\tilde{Q}} |f(y)|_r^s dy \right)^{1/s} \\
&\leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha b_j\|_{BMO} \right) \sum_{k=0}^{\infty} k^2 2^{-k} \left(\frac{1}{|2^k\tilde{Q}|} \int_{2^k\tilde{Q}} |f(y)|_r^s dy \right)^{1/s} \\
&\leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha b_j\|_{BMO} \right) M_s(|f|_r)(\tilde{x}).
\end{aligned}$$

For $I_5^{(3)}$, by the formula (see [5]):

$$R_{m_j}(\tilde{b}; x, y) - R_{m_j}(\tilde{b}; x_0, y) = \sum_{|\beta| < m_j} \frac{1}{\beta!} R_{m_j-|\beta|}(D^\beta \tilde{b}; x, x_0)(x-y)^\beta$$

and Lemma 4, we have

$$|R_{m_j}(\tilde{b}; x, y) - R_{m_j}(\tilde{b}; x_0, y)| \leq C \sum_{|\beta| < m_j} \sum_{|\alpha|=m_j} |x-x_0|^{m_j-|\beta|} |x-y|^{|\beta|} \|D^\alpha b\|_{BMO},$$

thus, by the Minkowski' inequality and Lemma 2, we obtain

$$\begin{aligned}
\left(\sum_{i=1}^{\infty} \|I_5^{(3)}\|^r \right)^{1/r} &\leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha b_j\|_{BMO} \right) \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} k^2 \frac{|x-x_0|}{|x_0-y|} |K(x_0-y)| |f(y)|_r dy \\
&\leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha b_j\|_{BMO} \right) \sum_{k=0}^{\infty} k^2 2^{-k} \left(\int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} |K(x_0-y)|^{s'} dy \right)^{1/s'} \\
&\quad \times \left(\int_{2^{k+1}\tilde{Q}} |f(y)|_r^s dy \right)^{1/s} \\
&\leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha b_j\|_{BMO} \right) \sum_{k=0}^{\infty} k^2 2^{-k} (2^k d)^{-n/s} (2^k d)^{n/s} M_s(|f|_r)(\tilde{x})
\end{aligned}$$

$$\leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha b_j\|_{BMO} \right) M_s(|f|_r)(\tilde{x}).$$

Similarly,

$$\left(\sum_{i=1}^{\infty} \|I_5^{(4)}\|^r \right)^{1/r} \leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) M_s(|f|_r)(\tilde{x}).$$

For $I_5^{(5)}$, similar to the proofs of $I_5^{(1)}$, $I_5^{(2)}$, $I_5^{(3)}$ and I_4 , we get, for $1 < s_1, s_2 < \infty$ with $1/s_1 + 1/s_2 + 1/s = 1$,

$$\begin{aligned} & \left(\sum_{i=1}^{\infty} \|I_5^{(5)}\|^r \right)^{1/r} \\ & \leq C \sum_{|\alpha_1|=m_1} \int_{(\tilde{Q})^c} \frac{|(x-y)^{\alpha_1}|}{|x-y|^m} |R_{m_2}(\tilde{b}_2; x, y) - R_{m_2}(\tilde{b}_2; x_0, y)| |K(x-y) D^{\alpha_1} \tilde{b}_1(y)| |f(y)|_r dy \\ & \quad + C \sum_{|\alpha_1|=m_1} \int_{(\tilde{Q})^c} \left| \frac{(x-y)^{\alpha_1}}{|x-y|^m} - \frac{(x_0-y)^{\alpha_1}}{|x_0-y|^m} \right| |R_{m_2}(\tilde{b}_2; x_0, y) K(x-y) D^{\alpha_1} \tilde{b}_1(y)| |f(y)|_r dy \\ & \quad + C \sum_{|\alpha_1|=m_1} \int_{(\tilde{Q})^c} \frac{|R_{m_2}(\tilde{b}_2; x_0, y)| |(x_0-y)^{\alpha_1}|}{|x_0-y|^m} |K(x-y) - K(x_0-y)| |D^{\alpha_1} \tilde{b}_1(y)| |f(y)|_r dy \\ & \leq C \sum_{|\alpha|=m_2} \|D^\alpha b_2\|_{BMO} \sum_{|\alpha_1|=m_1} \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} k \frac{|x-x_0|}{|x_0-y|} |K(x-y)| |D^{\alpha_1} \tilde{b}_1(y)| |f(y)|_r dy \\ & \quad + C \sum_{|\alpha|=m_2} \|D^\alpha b_2\|_{BMO} \sum_{|\alpha_1|=m_1} \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} k |K(x-y) - K(x_0-y)| |D^{\alpha_1} \tilde{b}_1(y)| |f(y)|_r dy \\ & \leq C \sum_{|\alpha|=m_2} \|D^\alpha b_2\|_{BMO} \sum_{|\alpha_1|=m_1} \sum_{k=0}^{\infty} k 2^{-k} \left(\int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} |K(x-y)|^{s_1} dy \right)^{1/s_1} \\ & \quad \times \left(\int_{2^{k+1}\tilde{Q}} |D^{\alpha_1} \tilde{b}_1(y)|^{s_2} dy \right)^{1/s_2} \left(\int_{2^{k+1}\tilde{Q}} |f(y)|_r^s dy \right)^{1/s} \\ & \quad + C \sum_{|\alpha|=m_2} \|D^\alpha b_2\|_{BMO} \sum_{|\alpha_1|=m_1} \sum_{k=0}^{\infty} k \left(\int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} |K(x-y) - K(x_0-y)|^{s_1} dy \right)^{1/s_1} \\ & \quad \times \left(\int_{2^{k+1}\tilde{Q}} |D^{\alpha_1} \tilde{b}_1(y)|^{s_2} dy \right)^{1/s_2} \left(\int_{2^{k+1}\tilde{Q}} |f(y)|_r^s dy \right)^{1/s} \\ & \leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha b_j\|_{BMO} \right) \sum_{k=1}^{\infty} k^2 (2^{-k} + 2^{-ka}) \left(\frac{1}{|2^k\tilde{Q}|} \int_{2^k\tilde{Q}} |f(y)|_r^s dy \right)^{1/s} \\ & \leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha b_j\|_{BMO} \right) M_s(|f|_r)(\tilde{x}). \end{aligned}$$

Similarly,

$$\left(\sum_{i=1}^{\infty} \|I_5^{(6)}\|^r \right)^{1/r} \leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) M_s(|f|_r)(\tilde{x}).$$

For $I_5^{(7)}$, similar to the proofs of $I_5^{(5)}$ and I_4 , we get, for $1 < s_1, s_2, s_3 < \infty$ with $1/s_1 + 1/s_2 + 1/s_3 + 1/s = 1$,

$$\begin{aligned} & \left(\sum_{i=1}^{\infty} \|I_5^{(7)}\|^r \right)^{1/r} \\ & \leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \int_{(\tilde{Q})^c} \left| \frac{(x-y)^{\alpha_1+\alpha_2}}{|x-y|^m} - \frac{(x_0-y)^{\alpha_1+\alpha_2}}{|x_0-y|^m} \right| |K(x-y) D^{\alpha_1} \tilde{b}_1(y) D^{\alpha_2} \tilde{b}_2(y)| |f(y)|_r dy \\ & \quad + C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \int_{(\tilde{Q})^c} \frac{|(x_0-y)^{\alpha_1+\alpha_2}|}{|x_0-y|^m} |K(x-y) - K(x_0-y)| |D^{\alpha_1} \tilde{b}_1(y) D^{\alpha_2} \tilde{b}_2(y)| |f(y)|_r dy \\ & \leq C \sum_{|\alpha_1|=m_1, |\alpha|=m_2} \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} \frac{|x-x_0|}{|x_0-y|} |K(x-y)| |D^{\alpha_1} \tilde{b}_1(y) D^{\alpha_2} \tilde{b}_2(y)| |f(y)|_r dy \\ & \quad + C \sum_{|\alpha_1|=m_1, |\alpha|=m_2} \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} |K(x-y) - K(x_0-y)| |D^{\alpha_1} \tilde{b}_1(y) D^{\alpha_2} \tilde{b}_2(y)| |f(y)|_r dy \\ & \leq C \sum_{|\alpha_1|=m_1, |\alpha|=m_2} \sum_{k=0}^{\infty} 2^{-k} \left(\int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} |K(x-y)|^{s_1} dy \right)^{1/s_1} \left(\int_{2^{k+1}\tilde{Q}} |D^{\alpha_1} \tilde{b}_1(y)|^{s_2} dy \right)^{1/s_2} \\ & \quad \times \left(\int_{2^{k+1}\tilde{Q}} |D^{\alpha_2} \tilde{b}_2(y)|^{s_3} dy \right)^{1/s_3} \left(\int_{2^{k+1}\tilde{Q}} |f(y)|_r^s dy \right)^{1/s} \\ & \quad + C \sum_{|\alpha_1|=m_1, |\alpha|=m_2} \sum_{k=0}^{\infty} \left(\int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} |K(x-y) - K(x_0-y)|^{s_1} dy \right)^{1/s_1} \\ & \quad \times \left(\int_{2^{k+1}\tilde{Q}} |D^{\alpha_1} \tilde{b}_1(y)|^{s_2} dy \right)^{1/s_2} \left(\int_{2^{k+1}\tilde{Q}} |D^{\alpha_2} \tilde{b}_2(y)|^{s_3} dy \right)^{1/s_3} \left(\int_{2^{k+1}\tilde{Q}} |f(y)|_r^s dy \right)^{1/s} \\ & \leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^{\alpha} b_j\|_{BMO} \right) \sum_{k=1}^{\infty} k^2 (2^{-k} + 2^{-ka}) \left(\frac{1}{|2^k\tilde{Q}|} \int_{2^k\tilde{Q}} |f(y)|_r^s dy \right)^{1/s} \\ & \leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^{\alpha} b_j\|_{BMO} \right) M_s(|f|_r)(\tilde{x}). \end{aligned}$$

Thus

$$I_5 \leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^{\alpha} b_j\|_{BMO} \right) M_s(|f|_r)(\tilde{x}).$$

This completes the proof of Theorem 1.

Proof of Theorem 2. We choose $1 < s < p$ in Theorem 1 and by [8], we get

$$\|T_b(f)\|_{L^p(w)} \leq \|M(T_b(f))\|_{L^p(w)} \leq C \|(T_b(f))^{\#}\|_{L^p(w)}$$

$$\begin{aligned}
&\leq C \prod_{j=1}^l \left(\sum_{|\alpha|=m_j} \|D^\alpha b_j\|_{BMO} \right) \|M_s(|f|_r)\|_{L^p(w)} \\
&\leq C \prod_{j=1}^l \left(\sum_{|\alpha|=m_j} \|D^\alpha b_j\|_{BMO} \right) \| |f|_r \|_{L^p(w)}.
\end{aligned}$$

This finishes the proof.

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