Refinements and sharpness of some inequalities for Mathieu type series

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Abstract. In this paper the classical Qi's type inequalities [14] for Mathieu type series as well as for its alternating version [12] are studied. Some new refinements and sharpness of the inequalities given in [12,14,18,19] will be presented.

1. Introduction and preliminaries

The following familiar infinite series

$$S(r)=\sum_{n=1}^{\infty}\frac{2n}{\left(n^2+r^2\right)^2}\qquad \left(r\in R^+\right)\tag{1.1}$$

is named after Emile Leonard Mathieu (1835-1890), who investigated it in his 1890 work [9] on elasticity of solid bodies. Bounds for this series are needed for the solution of boundary value problems for the biharmonic equations in a two-dimensional rectangular domain (see [15], p.258, eq. (54)). Inequality

$$S(r)<\frac{1}{r^2}\tag{1.2}$$

was conjectured in 1890 by Mathieu [9] and proved only in 1952 by L. Berg [2]. A very elegant and at the same time elementary proof of the inequalities

$$\frac{1}{r^2+\frac{1}{2}}<S(r)<\frac{1}{r^2}\tag{1.3}$$

is given by E. Makai [10]. These inequalities are called Mathieu's.
H.W. Gould and T. A. Chapman [6] have proved in 1962 the inequalities

$$\frac{1}{r^2+1}-\frac{1}{r^2+(m+1)^2}\le\sum_{n=1}^{m}\frac{2n}{\left(n^2+r^2\right)^2}\le\frac{1}{r^2}-\frac{1}{r^2+m^2},\qquad (\forall m\in \mathbf{N})\tag{1.4}$$

from which we can derive weaker inequalities than (1.3).
A remarkably useful integral representation for $S(r)$ in the elegant form

$$S(r)=\frac{1}{r}\int_0^{\infty}\frac{t\sin(rt)}{e^t-1}dt\tag{1.5}$$

was given by Emersleben [5].
Alternative version of (1.1)

$$\tilde{S}(r)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2n}{\left(n^2+r^2\right)^2}\qquad \left(r\in R^+\right)\tag{1.6}$$

was recently introduced by Pogany et.al in [12].
Let F be a Laplace transform of f, i.e.

$$F(p)=L(f(t)).\tag{1.7}$$

Using the relations (see [13], pp.651)

$$\sum_{n=1}^{\infty}F(n)=\int_0^{\infty}\frac{f(x)}{e^x-1}dx\tag{1.8}$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}F(n)=-\int_0^{\infty}\frac{f(x)}{e^x+1}dx\,, \tag{1.9}$$

we gave [12] an integral representation for $\tilde{S}(r)$:

$$\tilde{S}(r)=\frac{1}{r}\int_0^{\infty}\frac{t\sin(rt)}{e^t+1}dt\,. \tag{1.10}$$

Several interesting problems and solutions dealing with integral representations and bounds for the following slight generalization of the Mathieu series with a fractional power

$$S_\mu(r)=\sum_{n=1}^{\infty}\frac{2n}{(n^2+r^2)^\mu}\qquad (r\in R^+;\mu>1) \tag{1.11}$$

can be found in the works by Diananda [4], Tomovski and Trencevski [17] and Cerone and Lenard [3]. Feng Qi introduced [14] the series of following type:

$$S_{\mu+1}(r,\alpha)=\sum_{n=1}^{\infty}\frac{2n^{\alpha/2}}{(n^\alpha+r^2)^{\mu+1}}\qquad (r>0,\mu>0,\alpha>0) \tag{1.12}$$

It is obvious that $S_\mu(r,2)=S_\mu(r)$. Motivated essentially by the works of Cerone and Lenard [3] (and Qi [14]) we defined in [16] a family of generalized Mathieu series

$$S_\mu^{(\alpha,\beta)}(r;a)=S_\mu^{(\alpha,\beta)}\left(r;\{a_k\}_{k=1}^{\infty}\right)=\sum_{n=1}^{\infty}\frac{2a_n^\beta}{(a_n^\alpha+r^2)^\mu}\qquad (r,\alpha,\beta,\mu\in R^+) \tag{1.13}$$

where it is tacitly assumed that the positive sequence

$$a=\{a_k\}_{k=1}^{\infty}=\{a_1,a_2,a_3,....\}\qquad \left(\lim_{k\to\infty}a_k=\infty\right).$$

is chosen such that the infinite series in definition (1.13) converges, that is, that the following auxiliary series

$$\sum_{n=1}^{\infty}\frac{1}{a_n^{\mu\alpha-\beta}}$$

is convergent.

Comparing the definitions (1.1), (1.11) and (1.13), we see that $S_2(r)=S(r)$ and $S_\mu(r)=S_\mu^{(2,1)}(r,\{k\})$. Furthermore, the special cases $S_2^{(2,1)}(r;\{a_k\})$, $S_\mu(r)=S_\mu^{(2,1)}(r;\{k\})$, $S_\mu^{(2,1)}\left(r;\{k^\gamma\}\right)$, $S_\mu^{(\alpha,\alpha/2)}(r;\{k\})=S_\mu(r,\alpha)$ were investigated by Qi [14]; Diananda [3]; Tomovski [18] and Cerone-Lenard [3].

Let

$$\tilde{S}_\mu^{(\alpha,\beta)}(r;a)=\tilde{S}_\mu^{(\alpha,\beta)}\left(r;\{a_k\}_{k=1}^{\infty}\right)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2a_n^\beta}{(a_n^\alpha+r^2)^\mu}\quad (r,\alpha,\beta,\mu\in R^+) \tag{1.14}$$

be an alternating variant of (1.13), where the positive sequence $\{a_n\}_{n=1}^{\infty}$ satisfied the same conditions of the definition (1.13). In [12] we obtained several integral representations for (1.13) and its alternating variant in terms of the generalized hypergeometric functions and the Bessel function of first kind. Using these integral representations it is shown [19] that the series $\tilde{S}_\mu^{(\alpha,\beta)}\left(r;\{n^\gamma\}_{n=1}^{\infty}\right)$ under some conditions for parameters α,β,γ and μ are bounded with constants which do not depend on α,β and γ but only depend on r and μ, i.e.

$$\tilde{S}_\mu^{(\alpha,\beta)}\left(r;\{n^\gamma\}\right)\le\frac{2}{(1+r^2)^\mu}=M(r,\mu) \tag{1.15}$$

2. Refinements of the Qi type inequalities

In this part firstly we shall refine some inequalities of Qi type, given in [14] and [12]. We need the following proposition.

Proposition 2.1 For $x > 0$,

$$\frac{x}{2e^x} < \frac{x}{e^x+1} < \frac{x}{e^x} \qquad (2.1)$$

The following Theorem appears erroneously (without the term $-3e^{-\pi/(2r)}$ on the right hand sided inequality) in the works by Feng Qi [14] and we will present here the correct formulation.

Theorem 2.1 For any positive number $r > 0$,

$$L \le S(r) \le R, \qquad (2.2)$$

where

$$L = \frac{4(1+r^2)\left(e^{-\pi/r} + e^{-\pi/(2r)}\right) - 4r^2 - 1}{\left(e^{-\pi/r} - 1\right)(1+r^2)(1+4r^2)},$$

$$R = \frac{(1+4r^2)\left(e^{-\pi/r} - e^{-\pi/(2r)}\right) - 4(1+r^2) - 3e^{-\pi/(2r)}}{\left(e^{-\pi/r} - 1\right)(1+r^2)(1+4r^2)}.$$

Remark 2.1 This theorem was published six years after appearance of the very important paper by Alzer et al. [1]. Namely they proved the following theorem.

Theorem 2.2. For any positive number $r > 0$,

$$\frac{1}{k_1 + r^2} < S(r) < \frac{1}{k_2 + r^2} \qquad (2.3)$$

where $k_1 = \frac{1}{2\zeta(3)}$ and $k_2 = \frac{1}{6}$ are the best possible constants. Here ζ denotes Riemann-Zeta function and $\zeta(3) \approx 1.20205$ is the Apéry's constant.

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Remark 2.2 When $0 < r < 0{,}72149\ldots$, the upper bound in (2.2) is better than that in (2.3) (see figure 1).

Qi in [14] remarked that when $0 < r < 2{,}9002\ldots$, the lower bound in (2.2) is positive, and then is useful. But, it is not better than that in (2.3).

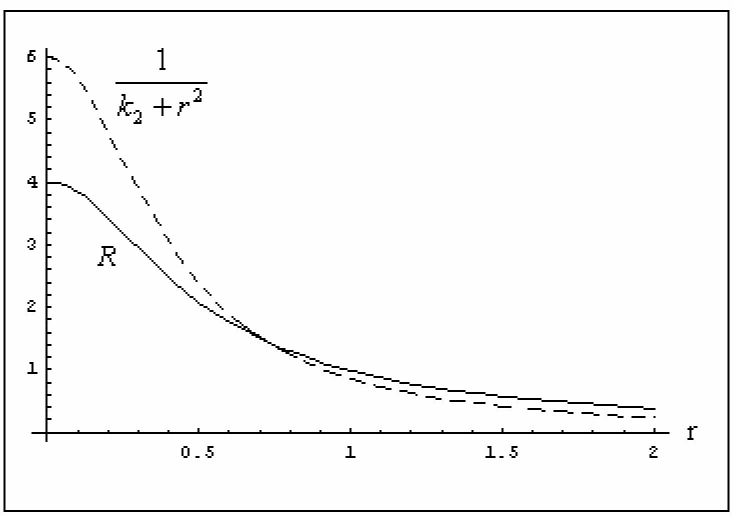

figure 1.

The proof of the next Theorem is similar to those of Theorem 2.1 in [14] and it is sufficient to be presented here one of them.

Theorem 2.3 For all $r > 0$,

$$\tilde{L} < \tilde{S}(r) < \tilde{R}, \tag{2.4}$$

where

$$\tilde{L} = \frac{2}{(1+r^2)^2} - \frac{1}{(1+r^2)^2(1-e^{-\pi/r})} - \frac{\pi e^{-\pi/r}}{2r(1+r^2)(1-e^{-\pi/r})^2}$$

and

$$\tilde{R} = \frac{1}{(1+r^2)^2} + \frac{1}{(1+r^2)^2(1-e^{-\pi/r})} + \frac{\pi e^{-\pi/r}}{2r(1+r^2)(1-e^{-\pi/r})^2}$$

Proof. Since

$$\tilde{S}(r) = \frac{1}{r}\sum_{k=0}^{\infty}\left(\int_{2k\pi/r}^{(2k+1)\pi/r} \frac{x}{e^x+1}\sin(rx)dx + \int_{(2k+1)\pi/r}^{(2k+2)\pi/r} \frac{x}{e^x+1}\sin(rx)dx\right), \tag{2.5}$$

by applying the elementary two- sided inequality (2.1), we obtain

$$\sum_{k=0}^{\infty}\int_{2k\pi/r}^{(2k+1)\pi/r} \frac{x}{e^x+1}\sin(rx)dx \le \sum_{k=0}^{\infty}\int_{2k\pi/r}^{(2k+1)\pi/r} xe^{-x}\sin(rx)dx$$

$$\sum_{k=0}^{\infty}\int_{(2k+1)\pi/r}^{(2k+2)\pi/r} \frac{x}{e^x+1}\sin(rx)dx \le \frac{1}{2}\sum_{k=0}^{\infty}\int_{(2k+1)\pi/r}^{(2k+2)\pi/r} xe^{-x}\sin(rx)dx\ .$$

Applying the well-known integral formula (see [13], p.446)

$$\int_0^{\infty} xe^{-x}\sin(rx)dx = \frac{2r}{(1+r^2)^2}$$

we get

$$\tilde{S}(r) \le \frac{1}{r}\left(\sum_{k=0}^{\infty}\int_{2k\pi/r}^{(2k+1)\pi/r} xe^{-x}\sin(rx)dx + \frac{1}{2}\sum_{k=0}^{\infty}\int_{(2k+1)\pi/r}^{(2k+2)\pi/r} xe^{-x}\sin(rx)dx\right) =$$

$$= \frac{1}{r}\left(\frac{1}{2}\int_0^{\infty} xe^{-x}\sin(rx)dx + \frac{1}{2}\sum_{k=0}^{\infty}\int_{2k\pi/r}^{(2k+1)\pi/r} xe^{-x}\sin(rx)dx\right)$$

$$= \frac{1}{(1+r^2)^2} + \frac{1}{2r}\sum_{k=0}^{\infty}\int_{2k\pi/r}^{(2k+1)\pi/r} xe^{-x}\sin(rx)dx = \tilde{R}.$$

On the other hand,

$$\tilde{S}(r) \ge \frac{1}{r}\left(\frac{1}{2}\sum_{k=0}^{\infty}\int_{2k\pi/r}^{(2k+1)\pi/r} xe^{-x}\sin(rx)dx + \sum_{k=0}^{\infty}\int_{(2k+1)\pi/r}^{(2k+2)\pi/r} xe^{-x}\sin(rx)dx\right) =$$

$$= \frac{1}{r}\left(\int_0^{\infty} xe^{-x}\sin(rx)dx - \frac{1}{2}\sum_{k=0}^{\infty}\int_{2k\pi/r}^{(2k+1)\pi/r} xe^{-x}\sin(rx)dx\right)$$

$$= \frac{2}{(1+r^2)^2} - \frac{1}{2r}\sum_{k=0}^{\infty}\int_{2k\pi/r}^{(2k+1)\pi/r} xe^{-x}\sin(rx)dx = \tilde{L}.$$

The proof is complete.

Remark 2.3 Since

$$\frac{1}{1-e^{-\frac{\pi}{r}}} > 1 \qquad (\forall r > 0),$$

$$\frac{\pi e^{-\frac{\pi}{r}}}{2r(1+r^2)\left(1-e^{-\frac{\pi}{r}}\right)^2} > 0 \qquad (\forall r > 0)$$

we obtain that

$$\tilde{R} > \frac{1}{(1+r^2)^2} + \frac{1}{(1+r^2)^2} = M(r,2),$$

i.e. the inequality (1.15) for $\mu = 2,\ \alpha = 2,\ \beta = 1$ is stronger than the right hand sided inequality (2.4). Comparing the inequalities (1.15) and (2.4) we get the following double inequalities for alternating Mathieu series $\tilde{S}(r)$.

Corollary 2.1 If $r > 0$ then the following two-sided inequality holds

$$M(r,2) - \frac{1}{(1+r^2)^2(1-e^{-\pi/r})} - \frac{\pi e^{-\pi/r}}{2r(1+r^2)(1-e^{-\pi/r})^2} \le \tilde{S}(r) \le M(r,2). \tag{2.6}$$

Theorem 2.4 [12] For all $r > 0$,

$$\frac{4\zeta(3)-3}{2(3r^2+2)(2\zeta(3)r^2+1)} < \tilde{S}(r) < \frac{12-\zeta(3)}{2(6r^2+1)(\zeta(3)r^2+2)}. \tag{2.7}$$

Remark 2.4 It is easy to show that when $0 < r < 0{,}28668$, the upper bound in (2.6) is better than the upper bound in (2.7). Namely, let

$$D(r) = M(r,2) - \frac{12-\zeta(3)}{2(6r^2+1)(\zeta(3)r^2+2)} = \frac{2}{(1+r^2)^2} - \frac{12-\zeta(3)}{2(6r^2+1)(\zeta(3)r^2+2)}$$

After simplifying, we have

$$D(r) = \frac{(25\zeta(3)-12)r^4 + 6(\zeta(3)+4)r^2 + (\zeta(3)-4)}{2(1+r^2)^2(6r^2+1)(\zeta(3)r^2+2)}$$

Let $G(r) = (25\zeta(3)-12)r^4 + 6(\zeta(3)+4)r^2 + (\zeta(3)-4)$

For $r > 0$, the function $G(r)$ is increasing and the positive root of the equation $G(r) = 0$ is

$$r_0 = \sqrt{\frac{-3(\zeta(3)+4) + \sqrt{-16\zeta^2(3) + 184\zeta(3) + 94}}{25\zeta(3)-12}} \approx 0{,}28668$$

Therefore, $D(r) < 0 \Leftrightarrow G(r) < 0 \Leftrightarrow r \in (0, r_0)$.

When $0 < r < 0{,}912173\ldots$, the lower bound in (2.6) is positive, and then is useful. It is better than the lower bound in (2.7) when $0 < r < 0{,}808583\ldots$ (see figure 2)

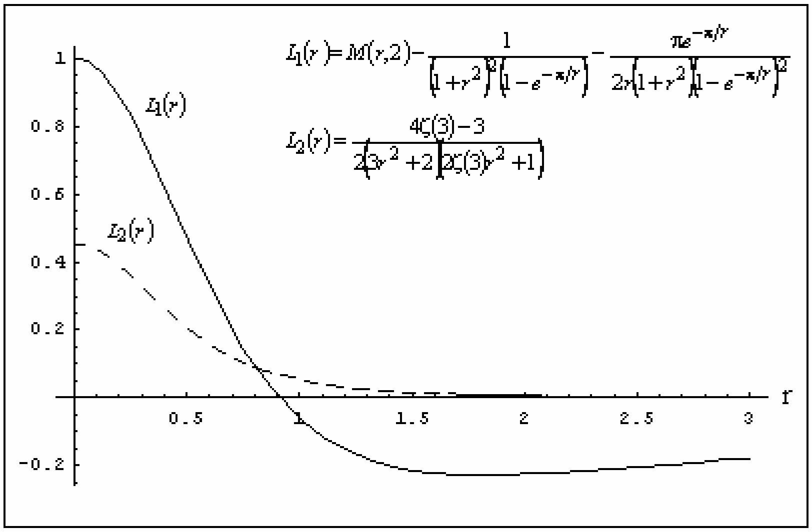

figure 2.

Applying the same technique as Qi did in [14], that is, by using the fact that $x^{-1}(e^x+1)$ is an increasing function for $x > c_0 \approx 1{,}27846$ (c_0 is a root of the equation $e^x - xe^x + 1 = 0$), the well-known inequality $\frac{x}{e^x+1} < \frac{2}{e^{x/2}}$ would yield the following result, which appears erroneously in the works by Pogany et al. [12, Theorem 4] and we will present here the correct formulation.

Theorem 2.5

(a) For all $r \in \left(0, \frac{\pi}{c_0}\right)$,

$$\tilde{S}(r) \le \frac{1}{r}\int_0^{\pi/r} \frac{x\sin(rx)}{e^x+1}\,dx \qquad (2.8)$$

(b) For all $r > 0$,

$$\frac{1}{r}\int_0^{\pi/r} \frac{x\sin(rx)}{e^x+1}\,dx \le \frac{2\left(1+e^{-\pi/(2r)}\right)}{r^2+\frac{1}{4}} \qquad (2.9)$$

3. Inequalities for generalized Mathieu series

In this section we shall refine double inequalities given in [18] and prove that the inequality (1.15) for $\alpha = 2,\ \beta = 1$ is sharpened than inequlity of [12] given by

$$\tilde{S}_{\mu}^{(2,1)}\left(r;\{n^{\gamma}\}\right) \le \frac{b_L \tilde{C}_{\mu}(r)\Gamma\left(\mu+\frac{1}{2}\right)}{\left(\mu-\frac{3}{2}\right)^{1/3}} = N_b(r,\mu) \qquad \left(\mu > \frac{3}{2}\right) \qquad (3.1)$$

where $\mu \ge \frac{1}{2}\left(1+r^2\right)$, $b_L = 0{,}674885...$, and $\tilde{C}_{\mu}(r) = \frac{\sqrt{\pi}}{(2r)^{\mu-\frac{3}{2}}\Gamma(\mu)}$.

Theorem 3.1 [18] For $r>0, \mu>0, \alpha>0$ the following integral representation for $S_{\mu+1}(r, \alpha)$ holds

$$S_{\mu+1}(r,\alpha)=\frac{2}{\Gamma(\mu+1)}\int_0^\infty x^{\mu}e^{-r^2x}g(x)dx, \tag{3.2}$$

where $g(x)=\sum_{n=1}^{\infty} n^{\alpha/2}e^{-n^{\alpha}x}$.

Theorem 3. 2 If $r>0, \mu>0, \alpha>0$, then

$$\frac{2}{\alpha r^{2\mu-2/\alpha+1}}B\left(\frac{1}{\alpha}+\frac{1}{2},\mu-\frac{1}{\alpha}+\frac{1}{2}\right)-\frac{2\Gamma\left(\mu+\frac{1}{2}\right)}{\sqrt{2e}\Gamma(\mu+1)r^{2\mu+1}}<S_{\mu+1}(r,\alpha)<$$

$$\frac{2}{\left(\mu\alpha+\frac{\alpha}{2}-1\right)\left(r^2+1\right)^{\mu+1}}{}_2F_1\left(1,\mu+1;\mu-\frac{1}{\alpha}+\frac{3}{2};\frac{r^2}{1+r^2}\right)+\frac{2\Gamma\left(\mu+\frac{1}{2}\right)}{\sqrt{2e}\Gamma(\mu+1)r^{2\mu+1}}, \tag{3.3}$$

where ${}_2F_1$ denotes the Gauss hypergeometric function and B is the Euler Beta function.

Proof. Let $f(x)=x^{\alpha/2}e^{-x^{\alpha}t}, x>0, t>0$. Since $\int_0^\infty f(x)dx=\frac{\Gamma\left(\frac{1}{\alpha}+\frac{1}{2}\right)}{\alpha t^{1/\alpha+1/2}}$ and $\max_{x\in R^+} f(x)=\frac{1}{\sqrt{2et}}$,

applying the trapezoidal quadrature rule

$$\int_0^\infty f(x)dx-\max_{x\in R^+} f(x)<\sum_{n=1}^{\infty} f(n)<\int_0^\infty f(x)dx+\max_{x\in R^+} f(x)-\int_0^1 f(x)dx, \tag{3.4}$$

we obtain

$$\frac{\Gamma\left(\frac{1}{\alpha}+\frac{1}{2}\right)}{\alpha t^{1/\alpha+1/2}}-\frac{1}{\sqrt{2et}}<g(t)<\frac{\Gamma\left(\frac{1}{\alpha}+\frac{1}{2},t\right)}{\alpha t^{1/\alpha+1/2}}+\frac{1}{\sqrt{2et}} \tag{3.5}$$

where $g(x)$ is the function defined in Theorem 3.1 and $\Gamma(a,x)=\int_x^\infty e^{-t}\,t^{a-1}\,dt$ is an incomplete gamma function. Hence by Theorem 3.1 we obtain

$$\frac{2\Gamma\left(\frac{1}{\alpha}+\frac{1}{2}\right)}{\alpha\Gamma(\mu+1)}\int_0^\infty t^{\mu-1/\alpha-1/2}e^{-r^2t}\,dt-\frac{2}{\sqrt{2e}\;\Gamma(\mu+1)}\int_0^\infty t^{\mu-1/2}e^{-r^2t}\,dt<S_{\mu+1}(r,\alpha)$$

$$<\frac{2}{\alpha\Gamma(\mu+1)}\int_0^\infty \Gamma\left(\frac{1}{\alpha}+\frac{1}{2},t\right)t^{\mu-1/\alpha-1/2}e^{-r^2t}\,dt+\frac{2}{\sqrt{2e}\;\Gamma(\mu+1)}\int_0^\infty t^{\mu-1/2}e^{-r^2t}\,dt,$$

$$\frac{2\Gamma\left(\frac{1}{\alpha}+\frac{1}{2}\right)\Gamma\left(\mu-\frac{1}{\alpha}+\frac{1}{2}\right)}{\alpha\Gamma(\mu+1)r^{2\mu-2/\alpha+1}}-\frac{2\Gamma\left(\mu+\frac{1}{2}\right)}{\sqrt{2e}\;\Gamma(\mu+1)r^{2\mu+1}}<S_{\mu+1}(r,\alpha).$$

$$<\frac{2}{\alpha\Gamma(\mu+1)}\int_0^\infty \Gamma\left(\frac{1}{\alpha}+\frac{1}{2},t\right)t^{\mu-1/\alpha-1/2}e^{-r^2t}\,dt+\frac{2\Gamma\left(\mu+\frac{1}{2}\right)}{\sqrt{2e}\;\Gamma(\mu+1)r^{2\mu+1}}.$$

Applying the integral formula (see [7], page 663)

$$\int_0^\infty \Gamma\left(\frac{1}{\alpha}+\frac{1}{2},t\right)t^{\mu-\frac{1}{\alpha}-\frac{1}{2}}e^{-r^2t}dt = \frac{\Gamma(\mu+1)}{\left(\mu-\frac{1}{\alpha}+\frac{1}{2}\right)\left(r^2+1\right)^{\mu+1}}\,{}_2F_1\left(1,\mu+1;\mu-\frac{1}{\alpha}+\frac{3}{2};\frac{r^2}{1+r^2}\right)$$

$$(r>0,\mu>0,\alpha>0) \qquad (3.6)$$

and definition of the Beta function

$$B\left(\frac{1}{\alpha}+\frac{1}{2},\mu-\frac{1}{\alpha}+\frac{1}{2}\right)=\frac{\Gamma\left(\frac{1}{\alpha}+\frac{1}{2}\right)\Gamma\left(\mu-\frac{1}{\alpha}+\frac{1}{2}\right)}{\Gamma(\mu+1)} \qquad (3.7)$$

we finally get (3.3).

Specially for $\alpha=2$ and $\mu+1\to\mu$ we get two sided inequality for $S_\mu(r)$.

Corollary 3.1 For $r>0,\ \mu>1$,

$$\frac{1}{(\mu-1)r^{2\mu-2}}-\frac{2\Gamma\left(\mu-\frac{1}{2}\right)}{\sqrt{2e}\,\Gamma(\mu)r^{2\mu-1}}<S_\mu(r)<\frac{1}{(\mu-1)\left(r^2+1\right)^{\mu-1}}+\frac{2\Gamma\left(\mu-\frac{1}{2}\right)}{\sqrt{2e}\,\Gamma(\mu)r^{2\mu-1}}. \qquad (3.8)$$

The next result was proved recently by Tomovski and Hilfer in [19], but we shall give here a modified proof illustrated graphically.

Theorem 3.3 For $\alpha=2,\ \beta=1$ the inequality (1.15) is stronger than inequality (3.1).

Proof. Let

$$P(r,\mu)=\frac{M(r,\mu)}{N_b(r,\mu)}=\frac{(\mu-3/2)^{1/3}}{b_L\sqrt{2\pi}}\cdot\frac{\Gamma(\mu)}{\Gamma(\mu+1/2)}\frac{2^\mu r^{\mu-3/2}}{\left(1+r^2\right)^\mu}. \qquad (3.9)$$

We shall prove that $P(r,\mu)<1$ for $\mu>\frac{3}{2}$ and $\forall r>0$.

For the proof of this theorem we need Gautschi's inequality (see [8],[11, page 286])

$$\frac{\Gamma(\mu+\beta)}{\Gamma(\mu)}\geq\left(\mu-\frac{1-\beta}{2}\right)^\beta \qquad \left(\mu>\frac{1-\beta}{2},\ \beta\in[0,1]\right) \qquad (3.10)$$

which for $\beta=\frac{1}{2}$ gives

$$\frac{\Gamma(\mu)}{\Gamma(\mu+1/2)}\leq\frac{1}{\sqrt{\mu-1/4}} \qquad (\mu>\frac{1}{4}) \qquad (3.11)$$

The function $f(r)=\frac{r^{\mu-3/2}}{\left(1+r^2\right)^\mu}$ has a maximum at the point $r_0=\sqrt{\frac{\mu-3/2}{\mu+3/2}}$.

Then,

$$P(r,\mu)\leq P(r_0,\mu)=\frac{(\mu-3/2)^{1/3}}{b_L\sqrt{2\pi}}\cdot\frac{\Gamma(\mu)}{\Gamma(\mu+1/2)}\cdot\frac{2^\mu\left(\sqrt{\frac{\mu-3/2}{\mu+3/2}}\right)^{\mu-3/2}}{\left(1+\left(\sqrt{\frac{\mu-3/2}{\mu+3/2}}\right)^2\right)^\mu}=$$

$$
= \frac{(\mu-3/2)^{1/3}}{b_L\sqrt{2\pi}} \cdot \frac{\Gamma(\mu)}{\Gamma(\mu+1/2)} \cdot \frac{2^{\mu}\,\dfrac{(\mu-3/2)^{\mu/2-3/4}}{(\mu+3/2)^{\mu/2-3/4}}}{\dfrac{2^{\mu}\mu^{\mu}}{(\mu+3/2)^{\mu}}}
$$

$$
\le \frac{1}{b_L\sqrt{2\pi}} \cdot \frac{(\mu-3/2)^{\mu/2-3/4}\cdot(\mu+3/2)^{\mu/2+3/4}}{\mu^{\mu}} \cdot \frac{(\mu-3/2)^{1/3}}{(\mu-1/4)^{1/2}} \qquad (3.12)
$$

$P(r_0,\mu)\le 1$ for $\mu>3/2$ (see figure 3)

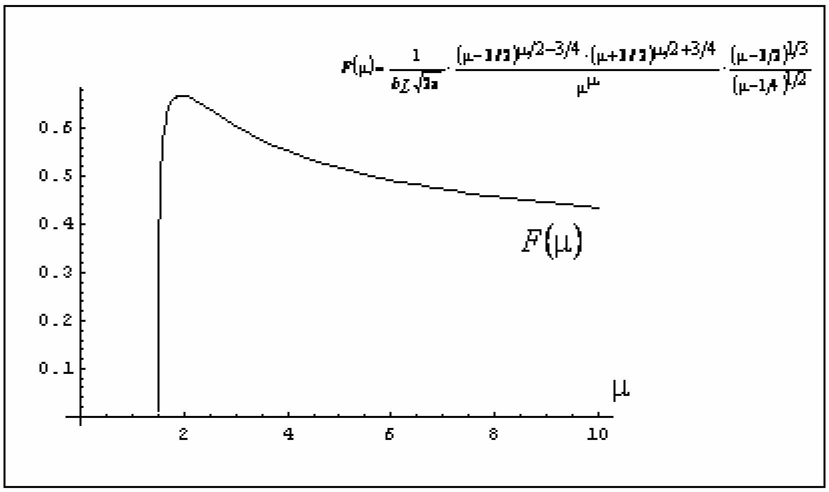

figure 3.

i) $\mu\ge 2$.

The function $f(x)=\dfrac{(x-3/2)^{1/3}}{(x-1/4)^{1/2}}$ has maximum at $x=4$.

Therefore, since (3.12), we have

$$
P(r,\mu)\le \frac{1}{b_L\sqrt{2\pi}} \cdot \frac{\left(\frac{5}{2}\right)^{1/3}}{\left(\frac{15}{4}\right)^{1/2}} \cdot L(\mu) \qquad (3.13)
$$

where

$$
L(\mu)=\frac{(\mu-3/2)^{\mu/2-3/4}\cdot(\mu+3/2)^{\mu/2+3/4}}{\mu^{\mu}}.
$$

Let $a=\mu-3/2$. It is clear that $a\ge 1/2$. Then

$$
L(\mu)=L(a+3/2)=\frac{a^{a/2}\cdot(a+3)^{(a+3)/2}}{\left(\frac{2a+3}{2}\right)^{\frac{2a+3}{2}}}
$$

$$= \sqrt{\frac{\left(1+\frac{3}{a}\right)^{a+3}}{\left(1+\frac{3}{2a}\right)^{2a+3}}} = \sqrt{\left(\frac{1+\frac{3}{a}}{1+\frac{3}{a}+\frac{9}{4a^2}}\right)^{a}} \cdot \sqrt{\left(\frac{1+\frac{3}{a}}{1+\frac{3}{2a}}\right)^{3}}$$

$$\leq \sqrt{\left(\frac{1+\frac{3}{a}}{1+\frac{3}{2a}}\right)^{3}} = \sqrt{\left(1+\frac{3}{2a+3}\right)^{3}} \leq \left(\frac{7}{4}\right)^{3/2} \tag{3.14}$$

Since (3.13) and (3.14) we have

$$P(r,\mu) \leq \frac{1}{b_L\sqrt{2\pi}} \cdot \frac{\left(\frac{5}{2}\right)^{1/3}}{\left(\frac{15}{4}\right)^{1/2}} \cdot \left(\frac{7}{4}\right)^{3/2} \leq 0{,}59113 \cdot \frac{1{,}35721}{1{,}93649} \cdot 2{,}31504 \leq 0{,}96 < 1$$

So, for $\mu \geq 2$, it holds $P(r,\mu) < 1$.

ii) $\frac{3}{2} < \mu \leq 2$ $\left(\frac{9}{6} < \mu \leq \frac{12}{6}\right)$

We will rewrite (3.12) in the following form

$$P(r,\mu) \leq \frac{1}{b_L\sqrt{2\pi}} \cdot \frac{(\mu - 3/2)^{\mu/2-5/12} \cdot (\mu + 3/2)^{\mu/2+3/4}}{\mu^{\mu} \cdot (\mu - 1/4)^{1/2}} \tag{3.15}$$

Since $0 < \mu - \frac{3}{2} < 1$ by (3.15), we have

a) ($\frac{3}{2} < \mu \leq \frac{5}{3}$)

$$P(r,\mu) < \frac{1}{b_L\sqrt{2\pi}} \cdot \frac{\left(\frac{1}{6}\right)^{1/3} \cdot \left(\frac{19}{6}\right)^{19/12}}{\left(\frac{3}{2}\right)^{3/2} \cdot \left(\frac{5}{4}\right)^{1/2}} < 0{,}9825 < 1$$

b) ($\frac{5}{3} < \mu \leq \frac{11}{6}$)

$$P(r,\mu) < \frac{1}{b_L\sqrt{2\pi}} \cdot \frac{\left(\frac{1}{3}\right)^{5/12} \cdot \left(\frac{20}{6}\right)^{20/12}}{\left(\frac{5}{3}\right)^{5/3} \cdot \left(\frac{17}{12}\right)^{1/2}} < 0{,}99764 < 1$$

c) ($\frac{11}{6} < \mu \leq 2$)

$$P(r,\mu) < \frac{1}{b_L\sqrt{2\pi}} \cdot \frac{\left(\frac{1}{2}\right)^{1/2} \cdot \left(\frac{7}{2}\right)^{7/4}}{\left(\frac{11}{6}\right)^{11/6} \cdot \left(\frac{19}{12}\right)^{1/2}} < 0{,}97931 < 1$$

This completes the proof.

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