Some new Hilbert-Pachpatte's inequalities with some sequences

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Abstract: Some new inequalities Hilbert-Pachpatte's inequalities with some sequences are established in this paper. The integral analogues of the main results are also given.

Key words: Hilbert's inequality; Hilbert-Pachpatte's inequality; Hölder's inequality

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1 Introduction

Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, a_m, $b_n \geq 0$. If $0 < \sum_{m=1}^{\infty} a_m^p < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$. Then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}\left\{\sum_{m=1}^{\infty} a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty} b_n^q\right\}^{\frac{1}{q}}, \tag{1.1}$$

where the constant $\frac{\pi}{\sin(\pi/p)}$ is the best possible.

Inequality (1.1) is the well known Hilbert's inequality[1]. It is well known that Hilbert's inequalities play a dominant role in analysis, so the literature on such inequalities and their applications is vast[2-6]. By defining the operator ∇, Pachpatte[7] gives new estimates on inequalities of this type.

In order to narrate conveniently, we give firstly notations and definitions[7]. Let R denote the set of real numbers, $N = \{1, 2, \cdots\}$, $N_0 = \{0, 1, 2, \cdots\}$, $N_\alpha = \{1, 2, \cdots, \alpha\}$, $\alpha \in N$. Define the operator ∇ by $\nabla u(t) = u(t) - u(t-1)$ for any function u defined on N_0. And define the operators $\nabla_1 v(s,t) = v(s,t) - v(s-1,t)$, $\nabla_2 v(s,t) = v(s,t) - v(s,t-1)$ and $\nabla_2\nabla_1 v(s,t) = \nabla_2(\nabla_1 v(s,t)) = \nabla_1(\nabla_2 v(s,t))$ for any function $v(s,t) : N_0 \times N_0 \to R$. we let $I = [0,\infty)$, $I_0 = (0,\infty)$, $I_\beta = [0,\beta)$, denote the subintervals of R. For any function $u : I \to R$; we denote by u' the derivatives of u; for the function $u(s,t) : I \times I \to R$, we denote the partial derivatives $(\partial/\partial s)u(s,t)$, $(\partial/\partial t)u(s,t)$, $(\partial^2/\partial s\partial t)u(s,t)$ and $(\partial^2/\partial t\partial s)u(s,t)$ by $D_1u(s,t)$, $D_2u(s,t)$, $D_1D_2u(s,t)$ and $D_2D_1u(s,t)$, respectively.

Now we repeat the result of [7] as follows.

Theorem 1.1 Let $p > 1$, $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. Let $a(s) : N_m \to R$, $b(t) : N_n \to R$, and $a(0) = b(0) = 0$. Then

$$\sum_{s=1}^{m}\sum_{t=1}^{n}\frac{|a(s)||b(t)|}{qs^{p-1}+pt^{q-1}} \leq M_1(p,q,m,n)\left\{\sum_{s=1}^{m}(m-s+1)|\nabla a(s)|^p\right\}^{\frac{1}{p}}\left\{\sum_{t=1}^{n}(n-t+1)|\nabla b(t)|^q\right\}^{\frac{1}{q}} \tag{1.2}$$

for $m, n \in N$, where $M_1(p,q,m,n) = \frac{1}{pq}m^{(p-1)/p}n^{(q-1)/q}$, for $m, n \in N$. Inequality (1.2) is the well known Hilbert-Pachpatte's inequality. Subsequently, a series of results are given[10-14].

The following theorem is obtained by Young-Ho Kim and Byung-Il Kim[8].

Theorem 1.2 Let $p > 1$, $q > 1$ be constants. Let $a(s) : N_m \to R$, $b(t) : N_n \to R$, and $a(0) = b(0) = 0$.

Then

$$\sum_{s=1}^{m}\sum_{t=1}^{n}\frac{|a(s)||b(t)|}{qs^{\frac{(p-1)(p+q)}{pq}}+pt^{\frac{(q-1)(p+q)}{pq}}}\le M_2(p,q,m,n)\left\{\sum_{s=1}^{m}(m-s+1)|\nabla a(s)|^p\right\}^{\frac{1}{p}}\left\{\sum_{t=1}^{n}(n-t+1)|\nabla b(t)|^q\right\}^{\frac{1}{q}} \tag{1.3}$$

for $m,n\in N$, where $M_2(p,q,m,n)=\frac{1}{p+q}m^{(p-1)/p}n^{(q-1)/q}$, for $m,n\in N$.

The purpose of this paper is to build some new inequalities with some sequences involving inequalities (1.2) and (1.3). The integral analogues of the main results are also given.

2 Main results

Theorem 2.1 Let $n\in N$, $n\ge 2$, $p_i>1,(i=1,2,\cdots,n)$, $\alpha=\sum_{i=1}^{n}\frac{1}{p_i}$ and $\beta_i=\prod_{j=1,j\ne i}^{n}p_j,(i=1,2,\cdots,n)$. Let $a_i(s_i):N_{m_i}\to R$ and $a_1(0)=a_2(0)=\cdots=a_n(0)=0$. Then

$$\sum_{s_1=1}^{m_1}\cdots\sum_{s_n=1}^{m_n}\frac{\prod_{i=1}^{n}|a_i(s_i)|}{\sum_{i=1}^{n}\beta_i s_i^{(p_i-1)\alpha}}\le M(p_1,\cdots,p_n,m_1,\cdots,m_n)\prod_{i=1}^{n}\left\{\sum_{s_i=1}^{m_i}(m_i-s_i+1)|\nabla a_i(s_i)|^{p_i}\right\}^{\frac{1}{p_i}} \tag{2.1}$$

for $m_1,m_2,\cdots,m_n\in N$, where $M(p_1,\cdots,p_n,m_1,\cdots,m_n)=\frac{1}{\sum_{i=1}^{n}\beta_i}\prod_{i=1}^{n}m_i^{\frac{p_i-1}{p_i}}$, for $m_1,m_2,\cdots,m_n\in N$.

Proof The idea of proof of Theorem 2.1 comes from Theorem 2.1 in Young-Ho Kim and Byung-Il Kim[8]. From the hypotheses of Theorem 2.1, it is easy to observe that the following identities hold

$$a_i(s_i)=\sum_{\tau_i=1}^{s_i}\nabla a_i(\tau_i),\ \ i=1,2,\cdots,n \tag{2.2}$$

for $s_i\in N_{m_i}$, $i=1,2,\cdots,n$. From (2.2) and using Hölder's inequality with indices p_i, $p_i/(p_i-1)$, we have

$$|a_i(s_i)|\le s_i^{\frac{p_i-1}{p_i}}\left\{\sum_{\tau_i=1}^{s_i}|\nabla a_i(\tau_i)|^{p_i}\right\}^{\frac{1}{p_i}},\ \ i=1,2,\cdots,n \tag{2.3}$$

for $s_i\in N_{m_i}$, $i=1,2,\cdots,n$. Using the inequality of means[9]

$$\left(\prod_{i=1}^{n}x_i^{\omega_i}\right)^{\frac{1}{\Omega_n}}\le\left(\frac{1}{\Omega_n}\sum_{i=1}^{n}\omega_i x_i^r\right)^{\frac{1}{r}} \tag{2.4}$$

for $r>0$, $\omega_i>0$, $\sum_{i=1}^{n}\omega_i=\Omega_n$. Let $x_i=s_i^{p_i-1}$, $\omega_i=\frac{1}{p_i}$, $i=1,2,\cdots,n$. and $r=\sum_{i=1}^{n}\omega_i$, from (2.3), one obtains

$$\begin{aligned}\prod_{i=1}^{n}|a_i(s_i)| &\le \prod_{i=1}^{n}s_i^{\frac{p_i-1}{p_i}}\left\{\sum_{\tau_i=1}^{s_i}|\nabla a_i(\tau_i)|^{p_i}\right\}^{\frac{1}{p_i}}\\ &\le \frac{1}{\alpha}\left\{\sum_{i=1}^{n}\frac{s_i^{(p_i-1)\alpha}}{p_i}\right\}\prod_{i=1}^{n}\left\{\sum_{\tau_i=1}^{s_i}|\nabla a_i(\tau_i)|^{p_i}\right\}^{\frac{1}{p_i}}\\ &= \frac{1}{\sum_{i=1}^{n}\beta_i}\left\{\sum_{i=1}^{n}\alpha_i s_i^{(p_i-1)\alpha}\right\}\prod_{i=1}^{n}\left\{\sum_{\tau_i=1}^{s_i}|\nabla a_i(\tau_i)|^{p_i}\right\}^{\frac{1}{p_i}}\end{aligned} \tag{2.5}$$

for $s_i\in N_{m_i}$, $i=1,2,\cdots,n$. We observe that

$$\frac{\prod_{i=1}^{n}|a_i(s_i)|}{\sum_{i=1}^{n}\alpha_i s_i^{(p_i-1)\alpha}}\le\frac{1}{\sum_{i=1}^{n}\beta_i}\prod_{i=1}^{n}\left\{\sum_{\tau_i=1}^{s_i}|\nabla a_i(\tau_i)|^{p_i}\right\}^{\frac{1}{p_i}} \tag{2.6}$$

for $s_i \in N_{m_i}$, $i = 1, 2, \cdots, n$. Taking the sum on both sides of (2.6) first over $s_i(i = 1, 2, \cdots, n)$ from 1 to m_i and then using Hölder's inequality and interchanging the order of summation, we observe that

$$\begin{aligned}
&\sum_{s_1=1}^{m_1} \cdots \sum_{s_n=1}^{m_n} \frac{\prod_{i=1}^n |a_i(s_i)|}{\sum_{i=1}^n \alpha_i s_i^{(p_i-1)\alpha}} \\
\le\ & \sum_{s_1=1}^{m_1} \cdots \sum_{s_n=1}^{m_n} \frac{1}{\sum_{i=1}^n \beta_i} \prod_{i=1}^n \left\{ \sum_{\tau_i=1}^{s_i} |\nabla a_i(\tau_i)|^{p_i} \right\}^{\frac{1}{p_i}} \\
=\ & \frac{1}{\sum_{i=1}^n \beta_i} \prod_{i=1}^n \left(\sum_{s_i=1}^{m_i} \left\{ \sum_{\tau_i=1}^{s_i} |\nabla a_i(\tau_i)|^{p_i} \right\}^{\frac{1}{p_i}} \right) \\
\le\ & \frac{1}{\sum_{i=1}^n \beta_i} \prod_{i=1}^n m_i^{\frac{p_i-1}{p_i}} \prod_{i=1}^n \left\{ \sum_{s_i=1}^{m_i} \sum_{\tau_i=1}^{s_i} |\nabla a_i(\tau_i)|^{p_i} \right\}^{\frac{1}{p_i}} \\
=\ & \frac{1}{\sum_{i=1}^n \beta_i} \prod_{i=1}^n m_i^{\frac{p_i-1}{p_i}} \prod_{i=1}^n \left\{ \sum_{s_i=1}^{m_i} (m_i - s_i + 1) |\nabla a_i(s_i)|^{p_i} \right\}^{\frac{1}{p_i}}.
\end{aligned}$$

This completes the proof of Theorem 2.1.

Remark 1 In Theorem 2.1, $n = 2$, our result is Theorem 1.2. In Theorem 2.1, $n = 2$ and $\alpha = 1$, our result is Theorem 1.1.

In the following theorems we establish the two independent variable versions of the inequalities involving some sequences.

Theorem 2.2 Let $n \in N$, $n \ge 2$, $p_i > 1, (i = 1, 2, \cdots, n)$, $\alpha = \sum_{i=1}^n \frac{1}{p_i}$ and $\beta_i = \prod_{j=1, j\ne i}^n p_j, (i = 1, 2, \cdots, n)$. Let $a_i(s_i, t_i) : N_{m_i} \times N_{n_i} \to R$ and $a_1(0, t) = a_2(0, t) = \cdots = a_n(0, t) = a_1(s, 0) = a_2(s, 0) = \cdots = a_n(s, 0) = 0$. Then

$$\begin{aligned}
\sum_{s_1=1}^{m_1} \sum_{t_1=1}^{n_1} \cdots \sum_{s_n=1}^{m_n} \sum_{t_n=1}^{n_n} \frac{\prod_{i=1}^n |a_i(s_i, t_i)|}{\sum_{i=1}^n \beta_i (s_i t_i)^{(p_i-1)\alpha}} &\le L(p_1, \cdots, p_n, m_1, \cdots, m_n, n_1, \cdots, n_n) \\
\times \prod_{i=1}^n \left\{ \sum_{s_i=1}^{m_i} \sum_{t_i=1}^{n_i} (m_i - s_i + 1)(n_i - t_i + 1) |\nabla_2 \nabla_1 a_i(s_i, t_i)|^{p_i} \right\}^{\frac{1}{p_i}} \qquad (2.7)
\end{aligned}$$

for $m_1, m_2, \cdots, m_n, n_1, \cdots, n_n \in N$, where $L(p_1, \cdots, p_n, m_1, \cdots, m_n, n_1 \cdots, n_n) = \frac{1}{\sum_{i=1}^n \beta_i} \prod\limits_{i=1}^n (m_i n_i)^{\frac{p_i-1}{p_i}}$, for $m_1, m_2, \cdots, m_n, n_1, \cdots, n_n \in N$.

Proof From the hypotheses of Theorem 2.2, it is easy to observe that the following identities hold

$$a_i(s_i, t_i) = \sum_{\tau_i=1}^{s_i} \sum_{\sigma_i=1}^{t_i} \nabla_2 \nabla_1 a_i(\tau_i, \sigma_i), \quad i = 1, 2, \cdots, n \qquad (2.8)$$

for $s_i \in N_{m_i}, t_i \in N_{n_i}$, $i = 1, 2, \cdots, n$. From (2.8) and using Hölder's inequality with indices p_i, $p_i/(p_i - 1)$, we have

$$|a_i(s_i, t_i)| \le (s_i t_i)^{\frac{p_i-1}{p_i}} \left\{ \sum_{\tau_i=1}^{s_i} \sum_{\sigma_i=1}^{t_i} |\nabla_2 \nabla_1 a_i(\tau_i, \sigma_i)|^{p_i} \right\}^{\frac{1}{p_i}}, \quad i = 1, 2, \cdots, n \qquad (2.9)$$

for $s_i \in N_{m_i}, t_i \in N_{n_i}$, $i = 1, 2, \cdots, n$. Let $x_i = (s_i t_i)^{p_i - 1}$, $\omega_i = \frac{1}{p_i}$, $i = 1, 2, \cdots, n$. and $r = \sum\limits_{i=1}^n \omega_i$, from (2.4) and (2.9), one obtains

$$\begin{aligned}\prod_{i=1}^{n}|a_i(s_i,t_i)| &\leq \prod_{i=1}^{n}(s_it_i)^{\frac{p_i-1}{p_i}}\left\{\sum_{\tau_i=1}^{s_i}\sum_{\sigma_i=1}^{t_i}|\nabla_2\nabla_1 a_i(\tau_i,\sigma_i)|^{p_i}\right\}^{\frac{1}{p_i}}\\ &\leq \frac{1}{\alpha}\left\{\sum_{i=1}^{n}\frac{(s_it_i)^{(p_i-1)\alpha}}{p_i}\right\}\prod_{i=1}^{n}\left\{\sum_{\tau_i=1}^{s_i}\sum_{\sigma_i=1}^{t_i}|\nabla_2\nabla_1 a_i(\tau_i,\sigma_i)|^{p_i}\right\}^{\frac{1}{p_i}}\\ &= \frac{1}{\sum_{i=1}^{n}\beta_i}\left\{\sum_{i=1}^{n}\alpha_i(s_it_i)^{(p_i-1)\alpha}\right\}\prod_{i=1}^{n}\left\{\sum_{\tau_i=1}^{s_i}\sum_{\sigma_i=1}^{t_i}|\nabla_2\nabla_1 a_i(\tau_i,\sigma_i)|^{p_i}\right\}^{\frac{1}{p_i}} \qquad (2.10)\end{aligned}$$

for $s_i \in N_{m_i}, t_i \in N_{n_i}$, $i = 1, 2, \cdots, n$. We observe that

$$\frac{\prod_{i=1}^{n}|a_i(s_i,t_i)|}{\sum_{i=1}^{n}\alpha_i(s_it_i)^{(p_i-1)\alpha}} \leq \frac{1}{\sum_{i=1}^{n}\beta_i}\prod_{i=1}^{n}\left\{\sum_{\tau_i=1}^{s_i}\sum_{\sigma_i=1}^{t_i}|\nabla_2\nabla_1 a_i(\tau_i,\sigma_i)|^{p_i}\right\}^{\frac{1}{p_i}} \qquad (2.11)$$

for $s_i \in N_{m_i}, t_i \in N_{n_i}$, $i = 1, 2, \cdots, n$. Taking the sum on both sides of (2.11) first over s_i, t_i $(i = 1, 2, \cdots, n)$ from 1 to m_i and then using Hölder's inequality and interchanging the order of summation, we observe that

$$\begin{aligned}&\sum_{s_1=1}^{m_1}\sum_{t_1=1}^{n_1}\cdots\sum_{s_n=1}^{m_n}\sum_{t_n=1}^{n_n}\frac{\prod_{i=1}^{n}|a_i(s_i,t_i)|}{\sum_{i=1}^{n}\beta_i(s_it_i)^{(p_i-1)\alpha}}\\ \leq\ &\sum_{s_1=1}^{m_1}\sum_{t_1=1}^{n_1}\cdots\sum_{s_n=1}^{m_n}\sum_{t_n=1}^{n_n}\frac{1}{\sum_{i=1}^{n}\beta_i}\prod_{i=1}^{n}\left\{\sum_{\tau_i=1}^{s_i}\sum_{\sigma_i=1}^{t_i}|\nabla_2\nabla_1 a_i(\tau_i,\sigma_i)|^{p_i}\right\}^{\frac{1}{p_i}}\\ =\ &\frac{1}{\sum_{i=1}^{n}\beta_i}\prod_{i=1}^{n}\left(\sum_{s_i=1}^{m_i}\sum_{t_i=1}^{n_i}\left\{\sum_{\tau_i=1}^{s_i}\sum_{\sigma_i=1}^{t_i}|\nabla_2\nabla_1 a_i(\tau_i,\sigma_i)|^{p_i}\right\}^{\frac{1}{p_i}}\right)\\ \leq\ &\frac{1}{\sum_{i=1}^{n}\beta_i}\prod_{i=1}^{n}(m_in_i)^{\frac{p_i-1}{p_i}}\prod_{i=1}^{n}\left\{\sum_{s_i=1}^{m_i}\sum_{t_i=1}^{n_i}\sum_{\tau_i=1}^{s_i}\sum_{\sigma_i=1}^{t_i}|\nabla_2\nabla_1 a_i(\tau_i,\sigma_i)|^{p_i}\right\}^{\frac{1}{p_i}}\\ =\ &\frac{1}{\sum_{i=1}^{n}\beta_i}\prod_{i=1}^{n}(m_in_i)^{\frac{p_i-1}{p_i}}\prod_{i=1}^{n}\left\{\sum_{s_i=1}^{m_i}\sum_{t_i=1}^{n_i}(m_i-s_i+1)(n_i-t_i+1)|\nabla_2\nabla_1 a_i(s_i,t_i)|^{p_i}\right\}^{\frac{1}{p_i}}.\end{aligned}$$

This completes the proof of Theorem 2.2.

Remark 2 By applying the inequality (2.4) on the right-hand sides of result inequalities in Theorem 2.1 and Theorem 2.2, we get the following inequalities

$$\sum_{s_1=1}^{m_1}\cdots\sum_{s_n=1}^{m_n}\frac{\prod_{i=1}^{n}|a_i(s_i)|}{\sum_{i=1}^{n}\beta_i s_i^{(p_i-1)\alpha}} \leq \frac{M(p_1,\cdots,p_n,m_1,\cdots,m_n)}{\alpha}\sum_{i=1}^{n}\frac{1}{p_i}\left\{\sum_{s_i=1}^{m_i}(m_i-s_i+1)|\nabla a_i(s_i)|^{p_i}\right\}^{\alpha} \qquad (2.12)$$

for $m_1, m_2, \cdots, m_n \in N$, where $M(p_1,\cdots,p_n,m_1,\cdots,m_n) = \frac{1}{\sum_{i=1}^{n}\beta_i}\prod_{i=1}^{n} m_i^{\frac{p_i-1}{p_i}}$, for $m_1, m_2, \cdots, m_n \in N$. And

$$\begin{aligned}\sum_{s_1=1}^{m_1}\sum_{t_1=1}^{n_1}\cdots\sum_{s_n=1}^{m_n}\sum_{t_n=1}^{n_n}\frac{\prod_{i=1}^{n}|a_i(s_i,t_i)|}{\sum_{i=1}^{n}\beta_i(s_it_i)^{(p_i-1)\alpha}} &\leq \frac{L(p_1,\cdots,p_n,m_1,\cdots,m_n,n_1,\cdots,n_n)}{\alpha}\\ &\times\sum_{i=1}^{n}\frac{1}{p_i}\left\{\sum_{s_i=1}^{m_i}\sum_{t_i=1}^{n_i}(m_i-s_i+1)(n_i-t_i+1)|\nabla_2\nabla_1 a_i(s_i,t_i)|^{p_i}\right\}^{\alpha} \qquad (2.13)\end{aligned}$$

for $m_1, m_2, \cdots, m_n, n_1, \cdots, n_n \in N$, where $L(p_1,\cdots,p_n,m_1,\cdots,m_n,n_1\cdots,n_n) = \frac{1}{\sum_{i=1}^{n}\beta_i}\prod_{i=1}^{n}(m_in_i)^{\frac{p_i-1}{p_i}}$, for $m_1, m_2, \cdots, m_n, n_1, \cdots, n_n \in N$.

3 Integral analogues

Now we give integral analogues of ours results as follows.

Theorem 3.1 Let $n \in N$, $n \geq 2$, $p_i > 1, (i = 1, 2, \cdots, n)$, $\alpha = \sum_{i=1}^n \frac{1}{p_i}$ and $\beta_i = \prod_{j=1, j\neq i}^n p_j, (i = 1, 2, \cdots, n)$. Let $f_i(x)$, $(i = 1, 2, \cdots, n)$, be real-valued continuous functions defined on I_{x_i} and $f_1(0) = f_2(0) = \cdots = f_n(0) = 0$. Then

$$\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |f_i(s_i)|}{\sum_{i=1}^n \beta_i s_i^{(p_i-1)\alpha}} ds_i \cdots ds_n \leq K(p_1, \cdots, p_n, x_1, \cdots, x_n) \prod_{i=1}^n \left\{ \int_0^{x_i} (x_i - s_i)|f_i'(s_i)|^{p_i} ds_i \right\}^{\frac{1}{p_i}} \tag{3.1}$$

for $x_1, x_2, \cdots, x_n \in I_0$, where $K(p_1, \cdots, p_n, x_1, \cdots, x_n) = \frac{1}{\sum_{i=1}^n \beta_i} \prod_{i=1}^n x_i^{\frac{p_i-1}{p_i}}$, for $x_1, x_2, \cdots, x_n \in I_0$.

Proof Along the lines of proof of the Theorem 2.1, from the hypotheses of Theorem 3.1, it is easy to observe that the following identities hold

$$f_i(s_i) = \int_0^{s_i} f_i'(\tau_i) d\tau_i, \quad i = 1, 2, \cdots, n \tag{3.2}$$

for $s_i \in I_{x_i}$, $i = 1, 2, \cdots, n$. From (2.2) and using Hölder's inequality with indices p_i, $p_i/(p_i - 1)$, we have

$$|f_i(s_i)| \leq s_i^{\frac{p_i-1}{p_i}} \left\{ \int_0^{s_i} |f_i'(\tau_i)|^{p_i} d\tau_i \right\}^{\frac{1}{p_i}}, \quad i = 1, 2, \cdots, n \tag{3.3}$$

for $s_i \in I_{x_i}$, $i = 1, 2, \cdots, n$. Let $x_i = s_i^{p_i-1}$, $\omega_i = \frac{1}{p_i}$, $i = 1, 2, \cdots, n$. and $r = \sum\limits_{i=1}^n \omega_i$, from (2.4) and (3.3), one obtains

$$\begin{aligned} \prod_{i=1}^n |f_i(s_i)| &\leq \prod_{i=1}^n s_i^{\frac{p_i-1}{p_i}} \left\{ \int_{\tau_i=1}^{s_i} |f_i'(\tau_i)|^{p_i} d\tau_i \right\}^{\frac{1}{p_i}} \\ &\leq \frac{1}{\alpha} \left\{ \sum_{i=1}^n \frac{s_i^{(p_i-1)\alpha}}{p_i} \right\} \prod_{i=1}^n \left\{ \int_0^{s_i} |f_i'(\tau_i)|^{p_i} d\tau_i \right\}^{\frac{1}{p_i}} \\ &= \frac{1}{\sum_{i=1}^n \beta_i} \left\{ \sum_{i=1}^n \alpha_i s_i^{(p_i-1)\alpha} \right\} \prod_{i=1}^n \left\{ \int_0^{s_i} |f_i'(\tau_i)|^{p_i} d\tau_i \right\}^{\frac{1}{p_i}} \end{aligned} \tag{3.4}$$

for $s_i \in I_{x_i}$, $i = 1, 2, \cdots, n$. We observe that

$$\frac{\prod_{i=1}^n |f_i(s_i)|}{\sum_{i=1}^n \alpha_i s_i^{(p_i-1)\alpha}} \leq \frac{1}{\sum_{i=1}^n \beta_i} \prod_{i=1}^n \left\{ \int_0^{s_i} |f_i'(\tau_i)|^{p_i} d\tau_i \right\}^{\frac{1}{p_i}} \tag{3.5}$$

for $s_i \in I_{x_i}$, $i = 1, 2, \cdots, n$. Taking the integral on both sides of (3.5) first over $s_i (i = 1, 2, \cdots, n)$ from 0 to x_i and then using Hölder's inequality and interchanging the order of integrals, we observe that

$$\int_0^{x_1}\cdots\int_0^{x_n}\frac{\prod_{i=1}^n|a_i(s_i)|}{\sum_{i=1}^n\alpha_i s_i^{(p_i-1)\alpha}}ds_1\cdots ds_n$$

$$\begin{aligned}
&\le \int_0^{x_1}\cdots\int_0^{x_n}\frac{1}{\sum_{i=1}^n\beta_i}\prod_{i=1}^n\left\{\int_0^{s_i}|f_i'(\tau_i)|^{p_i}d\tau_i\right\}^{\frac{1}{p_i}}ds_1\cdots ds_n\\
&= \frac{1}{\sum_{i=1}^n\beta_i}\prod_{i=1}^n\left(\int_0^{x_i}\left\{\int_0^{s_i}|f_i'(\tau_i)|^{p_i}d\tau_i\right\}^{\frac{1}{p_i}}ds_i\right)\\
&\le \frac{1}{\sum_{i=1}^n\beta_i}\prod_{i=1}^n x_i^{\frac{p_i-1}{p_i}}\prod_{i=1}^n\left\{\int_0^{x_i}\int_0^{s_i}|f_i'(\tau_i)|^{p_i}d\tau_i ds_i\right\}^{\frac{1}{p_i}}\\
&= \frac{1}{\sum_{i=1}^n\beta_i}\prod_{i=1}^n x_i^{\frac{p_i-1}{p_i}}\prod_{i=1}^n\left\{\int_0^{x_i}(x_i-s_i)|f_i'(s_i)|^{p_i}ds_i\right\}^{\frac{1}{p_i}}.
\end{aligned}$$

This completes the proof of Theorem 3.1.

In the following theorems we establish integral analogue of Theorem 2.2.

Theorem 3.2 Let $n\in N$, $n\ge 2$, $p_i>1,(i=1,2,\cdots,n)$, $\alpha=\sum_{i=1}^n\frac{1}{p_i}$ and $\beta_i=\prod_{j=1,j\ne i}^n p_j,(i=1,2,\cdots,n)$. Let $f_i(s_i,t_i):I_{x_i}\times I_{y_i}\to R$ and $f_1(0,t)=f_2(0,t)=\cdots=f_n(0,t)=f_1(s,0)=f_2(s,0)=\cdots=f_n(s,0)=0$. Then

$$\int_0^{x_1}\int_0^{y_1}\cdots\int_0^{x_n}\int_0^{y_n}\frac{\prod_{i=1}^n|f_i(s_i,t_i)|}{\sum_{i=1}^n\beta_i(s_it_i)^{(p_i-1)\alpha}}ds_1dt_1\cdots ds_ndt_n\le C(p_1,\cdots,p_n,x_1,\cdots,x_n,y_1,\cdots,y_n)$$

$$\times\prod_{i=1}^n\left\{\int_0^{x_i}\int_0^{y_i}(x_i-s_i)(y_i-t_i)|D_2D_1f_i(s_i,t_i)|^{p_i}ds_idt_i\right\}^{\frac{1}{p_i}}\tag{3.6}$$

for $x_1,\cdots,x_n,y_1,\cdots,y_n\in I_0$, where $C(p_1,\cdots,p_n,x_1,\cdots,x_n,y_1\cdots,y_n)=\frac{1}{\sum_{i=1}^n\beta_i}\prod_{i=1}^n(x_iy_i)^{\frac{p_i-1}{p_i}}$, for $x_1,\cdots,x_n,y_1,\cdots,y_n\in I_0$.

Proof Along the lines of proof of the Theorem 2.2, from the hypotheses of Theorem 3.2, it is easy to observe that the following identities hold

$$f_i(s_i,t_i)=\int_0^{s_i}\int_0^{t_i}D_2D_1f_i(\tau_i,\sigma_i)d\tau_id\sigma_i,\quad i=1,2,\cdots,n\tag{3.7}$$

for $s_i\in I_{x_i},t_i\in I_{y_i}$, $i=1,2,\cdots,n$. From (3.7) and using Hölder's inequality with indices p_i, $p_i/(p_i-1)$, we have

$$|f_i(s_i,t_i)|\le(s_it_i)^{\frac{p_i-1}{p_i}}\left\{\int_0^{s_i}\int_0^{t_i}|D_2D_1f_i(\tau_i,\sigma_i)|^{p_i}d\tau_id\sigma_i\right\}^{\frac{1}{p_i}},\quad i=1,2,\cdots,n\tag{3.8}$$

for $s_i\in I_{x_i},t_i\in I_{y_i}$, $i=1,2,\cdots,n$.Let $x_i=(s_it_i)^{p_i-1}$, $\omega_i=\frac{1}{p_i}$, $i=1,2,\cdots,n$. and $r=\sum_{i=1}^n\omega_i$, from (2.4) and (3.8), one obtains

$$\begin{aligned}\prod_{i=1}^{n}|f_i(s_i,t_i)| &\le \prod_{i=1}^{n}(s_it_i)^{\frac{p_i-1}{p_i}}\left\{\int_0^{s_i}\int_0^{t_i}|D_2D_1f_i(\tau_i,\sigma_i)|^{p_i}d\tau_id\sigma_i\right\}^{\frac{1}{p_i}}\\
&\le \frac{1}{\alpha}\left\{\sum_{i=1}^{n}\frac{(s_it_i)^{(p_i-1)\alpha}}{p_i}\right\}\prod_{i=1}^{n}\left\{\int_0^{s_i}\int_0^{t_i}|D_2D_1f_i(\tau_i,\sigma_i)|^{p_i}d\tau_id\sigma_i\right\}^{\frac{1}{p_i}}\\
&= \frac{1}{\sum_{i=1}^{n}\beta_i}\left\{\sum_{i=1}^{n}\alpha_i(s_it_i)^{(p_i-1)\alpha}\right\}\prod_{i=1}^{n}\left\{\int_0^{s_i}\int_0^{t_i}|D_2D_1f_i(\tau_i,\sigma_i)|^{p_i}d\tau_id\sigma_i\right\}^{\frac{1}{p_i}} \qquad (3.9)\end{aligned}$$

for $s_i\in I_{x_i}, t_i\in I_{y_i}$, $i=1,2,\cdots,n$. We observe that

$$\frac{\prod_{i=1}^{n}|f_i(s_i,t_i)|}{\sum_{i=1}^{n}\alpha_i(s_it_i)^{(p_i-1)\alpha}}\le\frac{1}{\sum_{i=1}^{n}\beta_i}\prod_{i=1}^{n}\left\{\int_0^{s_i}\int_0^{t_i}|D_2D_1f_i(\tau_i,\sigma_i)|^{p_i}d\tau_id\sigma_i\right\}^{\frac{1}{p_i}} \qquad (3.10)$$

for $s_i\in I_{x_i}, t_i\in I_{y_i}$, $i=1,2,\cdots,n$. Taking the integral on both sides of (2.11) first over s_i,t_i $(i=1,2,\cdots,n)$ from 0 to x_i and from 0 to y_i, respectively, and then using Hölder's inequality and interchanging the order of integrals, we observe that

$$\begin{aligned}&\int_0^{x_1}\int_0^{y_1}\cdots\int_0^{x_n}\int_0^{y_n}\frac{\prod_{i=1}^{n}|a_i(s_i,t_i)|}{\sum_{i=1}^{n}\beta_i(s_it_i)^{(p_i-1)\alpha}}ds_1dt_1\cdots ds_ndt_n\\
\le\ &\int_0^{x_1}\int_0^{y_1}\cdots\int_0^{x_n}\int_0^{y_n}\frac{1}{\sum_{i=1}^{n}\beta_i}\prod_{i=1}^{n}\left\{\int_0^{s_i}\int_0^{t_i}|D_2D_1f_i(\tau_i,\sigma_i)|^{p_i}d\tau_id\sigma_i\right\}^{\frac{1}{p_i}}ds_1dt_1\cdots ds_ndt_n\\
=\ &\frac{1}{\sum_{i=1}^{n}\beta_i}\prod_{i=1}^{n}\left(\int_0^{x_i}\int_0^{y_i}\left\{\int_0^{s_i}\int_0^{t_i}|D_2D_1f_i(\tau_i,\sigma_i)|^{p_i}d\tau_id\sigma_i\right\}^{\frac{1}{p_i}}ds_idt_i\right)\\
\le\ &\frac{1}{\sum_{i=1}^{n}\beta_i}\prod_{i=1}^{n}(x_iy_i)^{\frac{p_i-1}{p_i}}\prod_{i=1}^{n}\left\{\int_0^{x_i}\int_0^{y_i}\int_0^{s_i}\int_0^{t_i}|D_2D_1f_i(\tau_i,\sigma_i)|^{p_i}d\tau_id\sigma_ids_idt_i\right\}^{\frac{1}{p_i}}\\
=\ &\frac{1}{\sum_{i=1}^{n}\beta_i}\prod_{i=1}^{n}(x_iy_i)^{\frac{p_i-1}{p_i}}\prod_{i=1}^{n}\left\{\int_0^{x_i}\int_0^{y_i}(x_i-s_i)(y_i-t_i)|D_2D_1f_i(s_i,t_i)|^{p_i}ds_idt_i\right\}^{\frac{1}{p_i}}.\end{aligned}$$

This completes the proof of Theorem 3.2.

Remark 3 By applying the inequality (2.4) on the right-hand sides of result inequalities in Theorem 2.1 and Theorem 2.2, we get the following inequalities

$$\int_0^{x_1}\cdots\int_0^{x_n}\frac{\prod_{i=1}^{n}|f_i(s_i)|}{\sum_{i=1}^{n}\beta_is_i^{(p_i-1)\alpha}}ds_1\cdots ds_n\le\frac{K(p_1,\cdots,p_n,x_1,\cdots,x_n)}{\alpha}\sum_{i=1}^{n}\frac{1}{p_i}\left\{\int_0^{x_i}(x_i-s_i)|f_i'(s_i)|^{p_i}ds_i\right\}^{\alpha} \qquad (3.11)$$

for $x_1,x_2,\cdots,x_n\in I_0$, where $K(p_1,\cdots,p_n,x_1,\cdots,x_n)=\frac{1}{\sum_{i=1}^{n}\beta_i}\prod_{i=1}^{n}x_i^{\frac{p_i-1}{p_i}}$, for $x_1,x_2,\cdots,x_n\in I_0$. And

$$\int_0^{x_1}\int_0^{y_1}\cdots\int_0^{x_n}\int_0^{y_n}\frac{\prod_{i=1}^{n}|f_i(s_i,t_i)|}{\sum_{i=1}^{n}\beta_i(s_it_i)^{(p_i-1)\alpha}}ds_1dt_1\cdots ds_ndt_n\le\frac{C(p_1,\cdots,p_n,x_1,\cdots,x_n,y_1,\cdots,y_n)}{\alpha}$$

$$\times \sum_{i=1}^{n} \frac{1}{p_i} \left\{ \int_0^{x_i} \int_0^{y_i} (x_i - s_i)(y_i - t_i) |D_2 D_1 f_i(s_i, t_i)|^{p_i} ds_i dt_i \right\}^{\alpha} \tag{3.12}$$

for $x_1, x_2, \cdots, x_n, y_1, \cdots, y_n \in I_0$, where $C(p_1, \cdots, p_n, x_1, \cdots, x_n, y_1 \cdots, y_n) = \frac{1}{\sum_{i=1}^{n} \beta_i} \prod_{i=1}^{n} (x_i y_i)^{\frac{p_i - 1}{p_i}}$, for $x_1, x_2, \cdots, x_n, y_1, \cdots, y_n \in I_0$.

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