Inequalities Between The Sum of Power and The Exponential of Sum of Nonnegative Sequence

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Abstract. In this paper, we give a generalization of results of F. Qi [1]. We obtain by using a short proof, the following inequality between the sum of power and the exponential of sum of nonnegative sequence

$$\frac{e^p}{p^p}\sum_{i=1}^{n} x_i^p \leq \exp\left(\sum_{i=1}^{n} x_i\right),$$

where $p \geq 1$, $n \geq 2$, $x_i \geq 0$ for $1 \leq i \leq n$, and the constant $\frac{e^p}{p^p}$ is the best possible.

1 INTRODUCTION

In [1], F. QI proved the following:

Theorem A *For* $(x_1, x_{2,\ldots,}x_n) \in [0,\infty)^n$ *and* $n \geq 2$, *inequality*

$$\frac{e^2}{4}\sum_{i=1}^{n} x_i^2 \leq \exp\left(\sum_{i=1}^{n} x_i\right), \tag{1.1}$$

is valid. Equality in (1.1) *holds if* $x_i = 2$ *for some given* $1 \leq i \leq n$ *and* $x_j = 0$ *for all* $1 \leq j \leq n$ *with* $j \neq i$. *So, the constant* $\frac{e^2}{4}$ *in* (1.1) *is the best possible.*

Theorem B *Let* $\{x_i\}_{i=1}^{\infty}$ *be a nonnegative sequence such that* $\sum_{i=1}^{n} x_i < \infty$. *Then*

$$\frac{e^2}{4}\sum_{i=1}^{\infty} x_i^2 \leq \exp\left(\sum_{i=1}^{\infty} x_i\right). \tag{1.2}$$

Equality in (1.2) *holds if* $x_i = 2$ *for some given* $i \in \mathbb{N}$ *and* $x_j = 0$ *for all* $j \in \mathbb{N}$ *with* $j \neq i$. *So, the constant* $\frac{e^2}{4}$ *in* (1.2) *is the best possible.*

The aim of this paper is to give a generalization of inequality (1.1) and (1.2).

For our own convenience, we introduce the following notations:

$$[0,\infty)^n \triangleq \underbrace{[0,\infty) \times [0,\infty) \times ... \times [0,\infty)}_{n\ times} \tag{1.3}$$

and

$$(0,\infty)^n \triangleq \underbrace{(0,\infty) \times (0,\infty) \times ... \times (0,\infty)}_{n\ times} \tag{1.4}$$

for $n \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of all positive integers.

The main results of this paper are the following theorems.

Theorem 1.1 *Let* $p \geq 1$ *be a real number. For* $(x_1, x_{2,...,} x_n) \in [0,\infty)^n$ *and* $n \geq 2$, *inequality*

$$\frac{e^p}{p^p}\sum_{i=1}^{n} x_i^p \leq \exp\left(\sum_{i=1}^{n} x_i\right), \tag{1.5}$$

is valid. Equality in (1.5) *holds if* $x_i = p$ *for some given* $1 \leq i \leq n$ *and* $x_j = 0$ *for all* $1 \leq j \leq n$ *with* $j \neq i$. *So, the constant* $\frac{e^p}{p^p}$ *in* (1.5) *is the best possible.*

Theorem 1.2 *Let* $\{x_i\}_{i=1}^{\infty}$ *be a nonnegative sequence such that* $\sum\limits_{i=1}^{n} x_i < \infty$ *and* $p \geq 1$ *is a real number. Then*

$$\frac{e^p}{p^p}\sum_{i=1}^{\infty} x_i^p \leq \exp\left(\sum_{i=1}^{\infty} x_i\right). \tag{1.6}$$

Equality in (1.6) *holds if* $x_i = p$ *for some given* $i \in \mathbb{N}$ *and* $x_j = 0$ *for all* $j \in \mathbb{N}$ *with* $j \neq i$. *So, the constant* $\frac{e^p}{p^p}$ *in* (1.6) *is the best possible.*

Remark 1.1 In general we can't found $0 < \mu_n < \infty$, and $\lambda_n \in \mathbb{R}$ such that

$$\exp\left(\sum_{i=1}^{n} x_i\right) \leq \mu_n \sum_{i=1}^{n} x_i^{\lambda_n}.$$

Proof. We suppose that there exists $0 < \mu_n < \infty$, and $\lambda_n \in \mathbb{R}$ such that

$$\exp\left(\sum_{i=1}^{n} x_i\right) \leq \mu_n \sum_{i=1}^{n} x_i^{\lambda_n}.$$

Then for $(x_1, 1, ..., 1)$, we obtain as $x_1 \to +\infty$

$$1 \leq e^{1-n}\mu_n \left(n - 1 + x_1^{\lambda_n}\right) e^{-x_1} \to 0.$$

This is a contradiction.

Remark 1.2 Taking $p = 2$ in Theorems 1.1, 1.2 easily leads to Theorems A, B.

Remark 1.3 Inequality (1.5), can be rewritten as

$$\frac{e^p}{p^p} \sum_{i=1}^{n} x_i^p \leq \prod_{i=1}^{n} e^{x_i} \tag{1.7}$$

or

$$\frac{e^p}{p^p} \left\|x\right\|_p^p \leq \exp \left\|x\right\|_1, \tag{1.8}$$

where $x = (x_1, x_2, ..., x_n)$ and $\|.\|_p$ denotes the p-norm.

Remark 1.4 Inequality (1.6) can be rewritten as

$$\frac{e^p}{p^p} \sum_{i=1}^{\infty} x_i^p \leq \prod_{i=1}^{\infty} e^{x_i} \tag{1.9}$$

which is equivalent to inequality (1.8) for $x = (x_1, x_2, ...) \in [0, \infty)^{\infty}$.

Remark 1.5 Taking $x_i = \frac{1}{i}$ for $i \in \mathbb{N}$ in (1.5) and rearranging gives

$$p - p\ln p + \ln\left(\sum_{i=1}^{n} \frac{1}{i^p}\right) \leq \sum_{i=1}^{n} \frac{1}{i}. \tag{1.10}$$

Taking $x_i = \frac{1}{i^s}$ for $i \in \mathbb{N}$ and $s > 1$ in (1.6) and rearranging gives

$$p - p\ln p + \ln\left(\sum_{i=1}^{\infty} \frac{1}{i^{ps}}\right) = p - p\ln p + \ln\varsigma\left(ps\right) \leq \sum_{i=1}^{\infty} \frac{1}{i^s} = \varsigma\left(s\right), \tag{1.11}$$

where ς denotes the well-known Riemann Zêta function.

2 Lemmas

Lemma 2.1 *For* $x \in [0, \infty)$, $p \geq 1,$ *inequality*

$$\frac{e^p}{p^p} x^p \leq e^x, \tag{2.1}$$

is valid. Equality in (2.1) *holds if* $x = p$. *So, the constant* $\frac{e^p}{p^p}$ *in* (2.1) *is the best possible.*

Proof. Let $f(x) = p\ln x - x$ on the set $(0, \infty)$, it is easy to obtain that the function f has a maximal point $x = p$ and the maximal value equals $f(p) = p\ln p - p$. Then, we obtain (2.1). It is clear that inequality (2.1) holds also at $x = 0$.

Lemma 2.2 *Let* $p \geq 1$ *be a real number. For* $(x_1, x_{2,\ldots,}x_n) \in [0, \infty)^n$ *and* $n \geq 2,$ *inequality*

$$\sum_{i=1}^{n} x_i^p \leq \left(\sum_{i=1}^{n} x_i\right)^p, \qquad 2.2$$

is valid.

Proof. First we prove (2.2) for $n = 2$. We have for any $(x_1, x_2) \neq (0, 0)$

$$\begin{cases} \frac{x_1}{x_1+x_2} \leq 1 \\ \frac{x_2}{x_1+x_2} \leq 1 \end{cases} \tag{2.3}$$

Then, by $p \geq 1$ we get

$$\begin{cases} \left(\frac{x_1}{x_1+x_2}\right)^p \leq \frac{x_1}{x_1+x_2} \\ \left(\frac{x_2}{x_1+x_2}\right)^p \leq \frac{x_2}{x_1+x_2} \end{cases} \tag{2.4}$$

By addition from (2.4), we obtain

$$x_1^p + x_2^p \leq (x_1 + x_2)^p . \tag{2.5}$$

It is clear that inequality (2.5) holds also at the point $(0, 0)$.

Now we suppose that

$$\sum_{i=1}^{n} x_i^p \leq \left(\sum_{i=1}^{n} x_i\right)^p \tag{2.6}$$

and we prove that

$$\sum_{i=1}^{n+1} x_i^p \leq \left(\sum_{i=1}^{n+1} x_i\right)^p . \tag{2.7}$$

We have by (2.5)

$$\left(\sum_{i=1}^{n+1} x_i\right)^p = \left(\sum_{i=1}^{n} x_i + x_{n+1}\right)^p \geq \left(\sum_{i=1}^{n} x_i\right)^p + x_{n+1}^p \tag{2.8}$$

and by (2.6) and (2.8), we obtain

$$\sum_{i=1}^{n+1} x_i^p = \sum_{i=1}^{n} x_i^p + x_{n+1}^p \leq \left(\sum_{i=1}^{n} x_i\right)^p + x_{n+1}^p \leq \left(\sum_{i=1}^{n+1} x_i\right)^p . \tag{2.9}$$

Then for all $n \geq 2$ we have (2.2) holds.

3 Proofs of Theorems

Now we are in a position to prove our theorems.

Proof of Theorem 1.1 For $(x_1, x_{2,\ldots,}x_n) \in [0, \infty)^n$, we put $x = \sum_{i=1}^{n} x_i$. Then by (2.1) , we have

$$\frac{e^p}{p^p}\left(\sum_{i=1}^{n} x_i\right)^p \leq \exp\left(\sum_{i=1}^{n} x_i\right)$$

and by (2.2) we obtain (1.5) .

Proof of Theorem 1.2 This can be concluded by letting $n \to +\infty$ in the Theorem 1.1.

References

[1] **F. QI**, *Inequality between the sum of squares and the exponential of sum of nonnegative sequence*, J. Inequal. Pure and Appl. Math., 8 (3) Art.78, 2007, 1-5.