.

INEQUALITIES FOR THE EULER-MASCHERONI CONSTANT

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ABSTRACT. Let $R_n = \sum_{k=1}^{n} \frac{1}{k} - \log\left(n + \frac{1}{2}\right), H(n) = n^2(R_n - \gamma), n = 1, 2, \ldots$, where γ is the Euler-Mascheroni constant. We prove that for all integers $n \geq 1$, $H(n)$ and $[(n+1/2)/n]^2 H(n)$ are strictly increasing, while $[(n+1)/n]^2 H(n)$ is strictly decreasing. For all integers $n \geq 1$,

$$\frac{1}{24(n+a)^2} \leq R_n - \gamma < \frac{1}{24(n+b)^2}$$

with the best possible constants

$$a = \frac{1}{\sqrt{24[-\gamma + 1 - \log(3/2)]}} - 1 = 0.55106\ldots \quad \text{and} \quad b = \frac{1}{2}.$$

This refines result of D. W. DeTemple, who proved that the double inequality holds with $a = 1$ and $b = 0$.

1. Introduction

The Euler-Mascheroni constant $\gamma = 0.57721\ldots$ is defined by

$$\gamma = \lim_{n\to\infty} D_n, \quad \text{where} \quad D_n = \sum_{k=1}^{n} \frac{1}{k} - \log n.$$

The speed of convergence of the sequence D_n has been studied by different authors. In 1971, S. R. Tims and J. A. Tyrrell [7] proved that

$$\frac{1}{2(n+1)} < D_n - \gamma < \frac{1}{2(n-1)}, \quad n = 2, 3, \ldots.$$

In 1991, R. M. Young [8] presented an elegant geometrical proof for the double inequality

$$\frac{1}{2(n+1)} < D_n - \gamma < \frac{1}{2n}, \quad n = 1, 2, \ldots.$$

In 1997, G. D. Anderson et al. [3] established the the double inequality

$$\frac{1-\gamma}{n} \leq D_n - \gamma < \frac{1}{2n}, \quad n = 1, 2, \ldots.$$

In 1998, H. Alzer [2] proved that for all integers $n \geq 1$, the inequality

$$\frac{1}{2(n+a)} \leq D_n - \gamma < \frac{1}{2(n+b)}$$

holds with the best possible constants

$$a = \frac{1}{2(1-\gamma)} - 1 = 0.1826\ldots \quad \text{and} \quad b = \frac{1}{6}.$$

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The convergence of the sequence D_n to γ is very slow. In 1993, D. W. DeTemple [5] studied a modified sequence which converges faster and proved

$$\frac{1}{24(n+1)^2} < R_n - \gamma < \frac{1}{24n^2}, \tag{1}$$

where

$$R_n = \sum_{k=1}^{n} \frac{1}{k} - \log\left(n + \frac{1}{2}\right).$$

Now let

$$H(n) = n^2(R_n - \gamma), \quad n \geq 1.$$

Since

$$\psi(n+1) = -\gamma + \sum_{k=1}^{n} \frac{1}{k},$$

we see that

$$H(n) = (R_n - \gamma)n^2 = \left[\psi(n+1) - \log\left(n + \frac{1}{2}\right)\right] n^2, \tag{2}$$

where $\psi = \Gamma'/\Gamma$ is the psi function. Some computer experiments led M. Vuorinen to conjecture that $H(n)$ increases on the interval $[1, \infty)$ from $H(1) = -\gamma + 1 - \log(3/2) = 0.0173\ldots$ to $1/24 = 0.0416\ldots$. E. A. Karatsuba [6] proved that for all integers $n \geq 1$, $H(n) < H(n+1)$, by clever use of Stirling formula and Fourier series. Some computer experiments also seem to indicate that $[(n+1)/n]^2 H(n)$ is a decreasing convex function [4].

The following Theorem 1 shows the monotonicity properties of $H(n)$, $[(n+1/2)/n]^2 H(n)$ and $[(n+1)/n]^2 H(n)$.

Theorem 1. *Let $H(n)$ $(n = 1, 2, \ldots)$ be defined by (2). Then for all integers $n \geq 1$, $H(n)$ and $[(n+1/2)/n]^2 H(n)$ are both strictly increasing, while $[(n+1)/n]^2 H(n)$ is strictly decreasing.*

Remark 1. By the asymptotic formula [1, p. 550]

$$\psi(x) = \log\left(x - \frac{1}{2}\right) + \frac{1}{24(x-1/2)^2} + O(x^{-4}) \quad \text{as} \quad x \to \infty,$$

we conclude that

$$\lim_{n\to\infty} H(n) = \frac{1}{24} \quad \text{and} \quad \lim_{n\to\infty} [(n+1)/n]^2 H(n) = \frac{1}{24}. \tag{3}$$

From the monotonicity of $H(n)$, $[(n+1)/n]^2 H(n)$ and (3), we obtain the inequality (1).

In view of the inequality (1) it is natural to ask: What is the smallest number a and What is the largest number b such that the inequality

$$\frac{1}{24(n+a)^2} \leq R_n - \gamma \leq \frac{1}{24(n+b)^2}$$

holds for all integers $n \geq 1$? The following Theorem 2 answers this question.

Theorem 2. *For all integers $n \geq 1$, then*

$$\frac{1}{24(n+a)^2} \leq R_n - \gamma < \frac{1}{24(n+b)^2} \tag{4}$$

with the best possible constants

$$a = \frac{1}{\sqrt{24[-\gamma + 1 - \log(3/2)]}} - 1 = 0.55106\ldots \quad \text{and} \quad b = \frac{1}{2}.$$

Remark 2. From the monotonicity of $[(n + 1/2)/n]^2 H(n)$ and the limit relation $\lim_{n\to\infty}[(n + 1/2)/n]^2 H(n) = \frac{1}{24}$, we obtain the right inequality of (4) with $b = \frac{1}{2}$.

2. Proofs of theorems

In order prove our Theorem 1 and Theorem 2 we need to the following results [1]: For $x > -\frac{1}{2}, N = 0, 1, 2, \ldots,$

$$\begin{aligned} \log\left(x - \frac{1}{2}\right) - \sum_{k=1}^{2N} \frac{B_{2k}(1/2)}{2k(x - \frac{1}{2})^{2k}} &< \psi(x) \\ &< \log\left(x - \frac{1}{2}\right) - \sum_{k=1}^{2N+1} \frac{B_{2k}(1/2)}{2k(x - \frac{1}{2})^{2k}} \end{aligned} \tag{5}$$

and

$$\begin{aligned} \frac{(n-1)!}{(x - \frac{1}{2})^n} + \sum_{k=1}^{2N+1} \frac{B_{2k}(1/2)}{(2k)!} \frac{(n+2k-1)!}{(x - \frac{1}{2})^{n+2k}} &< (-1)^{n+1}\psi^{(n)}(x) \\ &< \frac{(n-1)!}{(x - \frac{1}{2})^n} + \sum_{k=1}^{2N} \frac{B_{2k}(1/2)}{(2k)!} \frac{(n+2k-1)!}{(x - \frac{1}{2})^{n+2k}}, \quad n = 1, 2, \ldots, \end{aligned} \tag{6}$$

where

$$B_k(1/2) = -\left(1 - \frac{1}{2^{k-1}}\right) B_k, \quad k = 0, 1, 2, \ldots,$$

B_k are Bernoulli numbers defined by

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}.$$

First four Bernoulli numbers with even indices are

$$B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad B_8 = -\frac{1}{30},$$

this yields

$$B_2(1/2) = -\frac{1}{12}, \quad B_4(1/2) = \frac{7}{240}, \quad B_6(1/2) = -\frac{31}{1344}, \quad B_8(1/2) = \frac{127}{3840}.$$

From (5), we get

$$\begin{aligned} \log\left(x - \frac{1}{2}\right) + \frac{1}{24(x - \frac{1}{2})^2} - \frac{7}{960(x - \frac{1}{2})^4} &< \psi(x) \\ &< \log\left(x - \frac{1}{2}\right) + \frac{1}{24(x - \frac{1}{2})^2}. \end{aligned} \tag{7}$$

From (6), we obtain

$$\frac{1}{x - \frac{1}{2}} - \frac{1}{12(x - \frac{1}{2})^3} < \psi'(x) < \frac{1}{x - \frac{1}{2}} - \frac{1}{12(x - \frac{1}{2})^3} + \frac{7}{240(x - \frac{1}{2})^5}, \tag{8}$$

$$\frac{1}{x-\frac{1}{2}}-\frac{1}{12(x-\frac{1}{2})^3}+\frac{7}{240(x-\frac{1}{2})^5}-\frac{31}{1344(x-\frac{1}{2})^7}<\psi'(x)<\frac{1}{x-\frac{1}{2}}-\frac{1}{12(x-\frac{1}{2})^3}+\frac{7}{240(x-\frac{1}{2})^5}. \tag{9}$$

Now we are in position to prove our Theorem 1 and Theorem 2.

Proof of Theorem 1. Let $a \geq 0$ be a real number and $f_a(x)$ be defined by

$$f_a(x)=(x+a)^2\left[\psi(x+1)-\log\left(x+\frac{1}{2}\right)\right],\quad x>-\frac{1}{2}. \tag{10}$$

differentiation yields

$$\frac{1}{x+a}f_a'(x)=2\left[\psi(x+1)-\log\left(x+\frac{1}{2}\right)\right]+(x+a)\left[\psi'(x+1)-\frac{1}{x+\frac{1}{2}}\right].$$

By (7) and (8), we obtain

$$\begin{aligned}\frac{1}{x}f_0'(x)&=2\left[\psi(x+1)-\log\left(x+\frac{1}{2}\right)\right]+x\left[\psi'(x+1)-\frac{1}{x+\frac{1}{2}}\right]\\&>2\left[\frac{1}{24(x+\frac{1}{2})^2}-\frac{7}{960(x+\frac{1}{2})^4}\right]-\frac{x}{12(x+\frac{1}{2})^3}\\&=\frac{20x+3}{480(x+\frac{1}{2})^4}>0\quad\text{for}\quad x>-\frac{3}{20}.\end{aligned}$$

This means that the sequence $H(n)$ is strictly increasing for all integers $n \geq 1$.

By (7) and (9), we have

$$\begin{aligned}\frac{1}{x+\frac{1}{2}}f_{1/2}'(x)&=2\left[\psi(x+1)-\log\left(x+\frac{1}{2}\right)\right]+\left(x+\frac{1}{2}\right)\left[\psi'(x+1)-\frac{1}{x+\frac{1}{2}}\right]\\&>2\left[\frac{1}{24(x+\frac{1}{2})^2}-\frac{7}{960(x+\frac{1}{2})^4}\right]\\&\quad+\left(x+\frac{1}{2}\right)\left[-\frac{1}{12(x+\frac{1}{2})^3}+\frac{7}{240(x+\frac{1}{2})^5}-\frac{31}{1344(x+\frac{1}{2})^7}\right]\\&=\frac{98x^2+98x-131.5}{6720(x+\frac{1}{2})^6}>0\quad\text{for}\quad x>0.76168\ldots.\end{aligned}$$

This means that the sequence $[(n+1/2)/n]^2H(n)$ is strictly increasing for all integers $n \geq 1$.

By (7) and (8), we obtain

$$\begin{aligned}\frac{1}{x+1}f_1'(x)&=2\left[\psi(x+1)-\log\left(x+\frac{1}{2}\right)\right]+(x+1)\left[\psi'(x+1)-\frac{1}{x+\frac{1}{2}}\right]\\&<\frac{1}{12(x+\frac{1}{2})^2}+(x+1)\left[-\frac{1}{12(x+\frac{1}{2})^3}+\frac{7}{240(x+\frac{1}{2})^5}\right]\\&=-\frac{10x^2+3x-\frac{9}{2}}{240(x+\frac{1}{2})^5}<0\quad\text{for}\quad x>\frac{-3+\sqrt{189}}{20}=0.537\ldots.\end{aligned}$$

This means that the sequence $[(n+1)/n]^2H(n)$ is strictly decreasing for all integers $n \geq 1$. The proof of Theorem 1 is complete. □

Proof of Theorem 2. The inequality (4) can be written as

$$a < \frac{1}{\sqrt{24[\psi(n+1) - \log(n+1/2)]}} - n \leq b.$$

In order to prove (4) we define

$$f(x) = \frac{1}{\sqrt{24[\psi(x+1) - \log(x+1/2)]}} - x.$$

Differentiation yields

$$\begin{aligned} &\left[24\left(\psi(x+1) - \log\left(x+\frac{1}{2}\right)\right)\right]^{3/2} f'(x) \\ &= 12\left(\frac{1}{x+\frac{1}{2}} - \psi'(x+1)\right) - \left[24\left(\psi(x+1) - \log\left(x+\frac{1}{2}\right)\right)\right]^{3/2}. \end{aligned}$$

Now we show that there exists a positive real number x_0 such that the function f is strictly decreasing on (x_0, ∞). In order to find x_0, we consider

$$12\left(\frac{1}{x+\frac{1}{2}} - \psi'(x+1)\right) < \left[24\left(\psi(x+1) - \log\left(x+\frac{1}{2}\right)\right)\right]^{3/2}.$$

By (7) and (9), it is sufficient to consider

$$\begin{aligned} &12\left(\frac{1}{12(x+\frac{1}{2})^3} - \frac{7}{240(x+\frac{1}{2})^5} + \frac{31}{1344(x+\frac{1}{2})^7}\right) \\ &\qquad < \left[24\left(\frac{1}{24(x+\frac{1}{2})^2} - \frac{7}{960(x+\frac{1}{2})^4}\right)\right]^{3/2}. \end{aligned} \tag{11}$$

Set $u = x + \frac{1}{2}$, (11) become

$$1 - \frac{7}{20u^2} + \frac{31}{112u^4} < \left(1 - \frac{7}{40u^2}\right)^{3/2}. \tag{12}$$

By Bernoulli's inequality: Let $x \geq -1$, then for $\alpha < 0$ or $\alpha > 1$, $(1+x)^\alpha \geq 1 + \alpha x$, the equal sign holds if and only if $x = 0$, we have

$$1 - \frac{21}{80u^2} < \left(1 - \frac{7}{40u^2}\right)^{3/2}. \tag{13}$$

The inequality

$$1 - \frac{7}{20u^2} + \frac{31}{112u^4} < 1 - \frac{21}{80u^2} \tag{14}$$

holds for $u > 1.778557\ldots$, and then, $f'(x) < 0$ for $x > 1.278557\ldots$. Straightforward calculation produces $f(1) = 0.55106\ldots$, $f(2) = 0.53308\ldots$, thus, the sequence $f(n) = \frac{1}{\sqrt{24(R_n - \gamma)}} - n \quad (n = 1, 2, \ldots)$ is strictly decreasing. This leads to

$$\lim_{n\to\infty} f(n) < f(n) \leq f(1) = \frac{1}{\sqrt{24[-\gamma + 1 - \log(3/2)]}} - 1. \tag{15}$$

It remains to prove that

$$\lim_{n\to\infty} f(n) = \frac{1}{2}. \tag{16}$$

From the representation

$$\psi(x)=\log\left(x-\frac{1}{2}\right)+\frac{1}{24(x-\frac{1}{2})^2}-\frac{7}{960(x-\frac{1}{2})^4}+O(x^{-6})\quad\text{as}\quad x\to\infty,$$

we obtain

$$\begin{aligned}
f(x)&=\frac{1}{\sqrt{24[\psi(x+1)-\log(x+1/2)]}}-x\\
&=\frac{1-x\sqrt{24[\psi(x+1)-\log(x+1/2)]}}{\sqrt{24[\psi(x+1)-\log(x+1/2)]}}\\
&=\frac{1-x\sqrt{\frac{1}{(x+1/2)^2}-\frac{7}{40(x+1/2)^4}+O(x^{-6})}}{\sqrt{\frac{1}{(x+1/2)^2}-\frac{7}{40(x+1/2)^4}+O(x^{-6})}}\\
&=\frac{x+\frac{1}{2}-x\sqrt{1-\frac{7}{40(x+1/2)^2}+O(x^{-4})}}{\sqrt{1-\frac{7}{40(x+1/2)^2}+O(x^{-4})}}\\
&=\frac{\frac{1}{2}+\frac{7x}{80(x+1/2)^2}+O(x^{-3})}{1-\frac{7}{80(x+1/2)^2}+O(x^{-4})}\to\frac{1}{2}\quad\text{as}\quad x\to\infty,
\end{aligned}$$

and then, $\lim_{n\to\infty}f(n)=\frac{1}{2}$. The proof of Theorem 2 is complete. □

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