Some Inequalities of the Grüss Type for the Numerical Radius of Bounded Linear Operators in Hilbert Spaces

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Abstract. Some inequalities of the Grüss Type type for the numerical radius of bounded linear operators in Hilbert spaces are established.

1. Introduction

Let $(H; \langle \cdot, \cdot \rangle)$ be a complex Hilbert space. The *numerical range* of an operator T is the subset of the complex numbers $\mathbb{C}$ given by [**10**, p. 1]:

$$W(T) = \{\langle Tx, x \rangle, \ x \in H, \ \|x\| = 1\}.$$

The *numerical radius* $w(T)$ of an operator T on H is given by [**10**, p. 8]:

$$w(T) = \sup\{|\lambda|, \lambda \in W(T)\} = \sup\{|\langle Tx, x\rangle|, \|x\| = 1\}. \tag{1.1}$$

It is well known that $w(\cdot)$ is a norm on the Banach algebra $B(H)$ of all bounded linear operators $T : H \to H$. This norm is equivalent with the operator norm. In fact, the following more precise result holds [**10**, p. 9]:

Theorem 1 (Equivalent norm). *For any $T \in B(H)$ one has*

$$w(T) \le \|T\| \le 2w(T). \tag{1.2}$$

For other results on numerical radius, see [**11**], Chapter 11. For some recent and interesting results concerning inequalities for the numerical radius, see [**12**] and [**13**].

We recall some classical results involving the numerical radius of two linear operators A, B.

The following general result for the product of two operators holds [**10**, p. 37]:

Theorem 2. *If A, B are two bounded linear operators on the Hilbert space $(H, \langle \cdot, \cdot \rangle)$, then*

$$w(AB) \le 4w(A)w(B). \tag{1.3}$$

In the case that $AB = BA$, then

$$w(AB) \le 2w(A)w(B). \tag{1.4}$$

The following results are also well known [**10**, p. 38].

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Theorem 3. *If A is a unitary operator that commutes with another operator B, then*

$$w(AB) \le w(B). \tag{1.5}$$

If A is an isometry and $AB = BA$, then (1.5) also holds true.

We say that A and B *double commute* if $AB = BA$ and $AB^* = B^*A$.
The following result holds [**10**, p. 38].

Theorem 4 (Double commute). *If the operators A and B double commute, then*

$$w(AB) \le w(B)\|A\|. \tag{1.6}$$

As a consequence of the above, we have [**10**, p. 39]:

Corollary 1. *Let A be a normal operator commuting with B. Then*

$$w(AB) \le w(A)\,w(B). \tag{1.7}$$

For other results and historical comments on the above see [**10**, p. 39–41]. For more results on the numerical radius, see [**11**].
In the recent survey paper [**9**] we provided other inequalities for the numerical radius of the product of two operators. We list here some of the results:

Theorem 5. *Let $A, B : H \to H$ be two bounded linear operators on the Hilbert space $(H, \langle\cdot,\cdot\rangle)$, then*

$$\left\|\frac{A^*A + B^*B}{2}\right\| \le w(B^*A) + \frac{1}{2}\|A - B\|^2 \tag{1.8}$$

and

$$\left\|\frac{A + B}{2}\right\|^2 \le \frac{1}{2}\left[\left\|\frac{A^*A + B^*B}{2}\right\| + w(B^*A)\right], \tag{1.9}$$

respectively.

If more information regarding one operators is available, then the following results may be stated as well:

Theorem 6. *Let $A, B : H \to H$ be two bounded linear operators on H and B is invertible such that, for a given $r > 0$,*

$$\|A - B\| \le r. \tag{1.10}$$

Then

$$\|A\| \le \left\|B^{-1}\right\|\left[w(B^*A) + \frac{1}{2}r^2\right] \tag{1.11}$$

and

$$(0 \le)\,\|A\|\,\|B\| - w(B^*A) \le \frac{1}{2}r^2 + \frac{\|B\|^2\left\|B^{-1}\right\|^2 - 1}{\left\|B^{-1}\right\|^2}, \tag{1.12}$$

respectively.

Motivated by the natural question that arise in order to compare the quantity $w(AB)$ with other expressions comprising the norm or the numerical radius of the involved operators A and B (or certain expressions constructed with these operators), we establish in this paper some natural inequalities of the form

$$w(BA) \le w(A)\, w(B) + K_1 \text{ (additive Grüss' type inequality)}$$

or

$$\frac{w(BA)}{w(A)\, w(B)} \le K_2 \text{ (multiplicative Grüss' type inequality)}$$

where K_1 and K_2 are specified and desirably simple constants (depending on the given operators A and B).

Applications in providing upper bounds for the non negative quantities

$$\|A\|^2 - w^2(A) \text{ and } w^2(A) - w(A^2)$$

and the *super unitary* quantities

$$\frac{\|A\|^2}{w^2(A)} \text{ and } \frac{w^2(A)}{w(A^2)}$$

are also given.

2. Numerical Radius Inequalities of Grüss Type

For the complex numbers α, β and the bounded linear operator T we define the following transform

$$C_{\alpha,\beta}(T) := (T^* - \overline{\alpha} I)(\beta I - T), \tag{2.1}$$

where by T^* we denote the adjoint of T.

We list some properties of the transform $C_{\alpha,\beta}(\cdot)$ that are useful in the following:

(i) For any $\alpha, \beta \in \mathbb{C}$ and $T \in B(H)$ we have:

$$C_{\alpha,\beta}(I) = (1 - \overline{\alpha})(\beta - 1) I, \qquad C_{\alpha,\alpha}(T) = -(\alpha I - T)^*(\alpha I - T), \tag{2.2}$$

$$C_{\alpha,\beta}(\gamma T) = |\gamma|^2 C_{\frac{\alpha}{\gamma},\frac{\beta}{\gamma}}(T) \quad \text{for each } \gamma \in \mathbb{C}\setminus\{0\}, \tag{2.3}$$

$$[C_{\alpha,\beta}(T)]^* = C_{\beta,\alpha}(T) \tag{2.4}$$

and

$$C_{\overline{\beta},\overline{\alpha}}(T^*) - C_{\alpha,\beta}(T) = T^*T - TT^*. \tag{2.5}$$

(ii) The operator $T \in B(H)$ is normal if and only if $C_{\overline{\beta},\overline{\alpha}}(T^*) = C_{\alpha,\beta}(T)$ for each $\alpha, \beta \in \mathbb{C}$.

We recall that a bounded linear operator T on the complex Hilbert space $(H, \langle\cdot,\cdot\rangle)$ is called *accretive* if $\operatorname{Re}\langle Ty, y\rangle \ge 0$ for any $y \in H$.

Utilizing the following identity

$$\operatorname{Re}\langle C_{\alpha,\beta}(T)x, x\rangle = \operatorname{Re}\langle C_{\beta,\alpha}(T)x, x\rangle = \frac{1}{4}|\beta - \alpha|^2 - \left\| \left(T - \frac{\alpha+\beta}{2} I\right) x \right\|^2 \tag{2.6}$$

that holds for any scalars α, β and any vector $x \in H$ with $\|x\| = 1$ we can give a simple characterization result that is useful in the following:

Lemma 1. *For $\alpha, \beta \in \mathbb{C}$ and $T \in B(H)$ the following statements are equivalent:*
(i) The transform $C_{\alpha,\beta}(T)$ (or, equivalently $C_{\beta,\alpha}(T)$) is accretive;
(ii) The transform $C_{\overline{\alpha},\overline{\beta}}(T^*)$ $\left(\text{or, equivalently } C_{\overline{\beta},\bar{\alpha}}(T^*)\right)$ is accretive;
(iii) We have the norm inequality

$$\left\| T - \frac{\alpha + \beta}{2} \cdot I \right\| \leq \frac{1}{2} |\beta - \alpha| \tag{2.7}$$

or, equivalently,

$$\left\| T^* - \frac{\bar{\alpha} + \bar{\beta}}{2} \cdot I \right\| \leq \frac{1}{2} |\beta - \alpha| . \tag{2.8}$$

Remark 1. *In order to give examples of operators $T \in B(H)$ and numbers $\alpha, \beta \in \mathbb{C}$ such that the transform $C_{\alpha,\beta}(T)$ is accretive, it suffices to select a bounded linear operator S and the complex numbers z, w with the property that $\|S - zI\| \leq |w|$ and, by choosing $T = S$, $\alpha = \frac{1}{2}(z + w)$ and $\beta = \frac{1}{2}(z - w)$ we observe that T satisfies (2.7), i.e., $C_{\alpha,\beta}(T)$ is accretive.*

The following results compares the quantities $w(AB)$ and $w(A)w(B)$ provided that some information about the transforms $C_{\alpha,\beta}(A)$ and $C_{\gamma,\delta}(B)$ are available where $\alpha, \beta, \gamma, \delta \in \mathbb{K}$.

Theorem 7. *Let $A, B \in B(H)$ and $\alpha, \beta, \gamma, \delta \in \mathbb{K}$ be such that the transforms $C_{\alpha,\beta}(A)$ and $C_{\gamma,\delta}(B)$ are accretive, then*

$$w(BA) \leq w(A)w(B) + \frac{1}{4} |\beta - \alpha| |\gamma - \delta| . \tag{2.9}$$

Proof. Since $C_{\alpha,\beta}(A)$ and $C_{\gamma,\delta}(B)$ are accretive, then, on making use of Lemma 1 we have that

$$\left\| Ax - \frac{\alpha + \beta}{2} x \right\| \leq \frac{1}{2} |\beta - \alpha|$$

and

$$\left\| B^*x - \frac{\bar{\gamma} + \bar{\delta}}{2} x \right\| \leq \frac{1}{2} \left|\bar{\gamma} - \bar{\delta}\right|$$

for any $x \in H, \|x\| = 1$.

Now, we make use of the following Grüss type inequality for vectors in inner product spaces obtained by the author in [**1**] (see also [**2**] or [**8**, p. 43]):

Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$, $u, v, e \in H$, $\|e\| = 1$, and $\alpha, \beta, \gamma, \delta \in \mathbb{K}$ such that

$$\operatorname{Re} \langle \beta e - u, u - \alpha e \rangle \geq 0, \operatorname{Re} \langle \delta e - v, v - \gamma e \rangle \geq 0 \tag{2.10}$$

or equivalently,

$$\left\| u - \frac{\alpha + \beta}{2} e \right\| \leq \frac{1}{2} |\beta - \alpha| , \left\| v - \frac{\gamma + \delta}{2} e \right\| \leq \frac{1}{2} |\delta - \gamma| , \tag{2.11}$$

then

$$|\langle u, v \rangle - \langle u, e \rangle \langle e, v \rangle| \leq \frac{1}{4} |\beta - \alpha| |\delta - \gamma| . \tag{2.12}$$

Applying (2.12) for $u = Ax, v = B^*x$ and $e = x$ we deduce

$$|\langle BAx, x \rangle - \langle Ax, x \rangle \langle Bx, x \rangle| \leq \frac{1}{4} |\beta - \alpha| |\delta - \gamma| , \tag{2.13}$$

for any $x \in H, \|x\| = 1$, which is an inequality of interest in itself.

Observing that $|\langle BAx, x\rangle| - |\langle Ax, x\rangle \langle Bx, x\rangle| \leq |\langle BAx, x\rangle - \langle Ax, x\rangle \langle Bx, x\rangle|$, then by (2.12) we deduce the inequality

$$|\langle BAx, x\rangle| \leq |\langle Ax, x\rangle \langle Bx, x\rangle| + \frac{1}{4} |\beta - \alpha| |\delta - \gamma|, \tag{2.14}$$

for any $x \in H, \|x\| = 1$. On taking the supremum over $\|x\| = 1$ in (2.14) we deduce the desired result (2.9). ∎

The following particular case provides a upper bound for the nonnegative quantity $\|A\|^2 - w(A)^2$ when some information about the operator A is available:

Corollary 2. *Let $A \in B(H)$ and $\alpha, \beta \in \mathbb{K}$ be such that the transform $C_{\alpha,\beta}(A)$ is accretive, then*

$$(0 \leq) \|A\|^2 - w^2(A) \leq \frac{1}{4} |\beta - \alpha|^2. \tag{2.15}$$

Proof. Follows on applying the above Theorem 7 for the choice $B = A^*$, by taking into account that $C_{\alpha,\beta}(A)$ is accretive implies $C_{\overline{\alpha},\overline{\beta}}(A^*)$ is the same and $w(A^*A) = \|A\|^2$. ∎

Remark 2. *Let $A \in B(H)$ and $M > m > 0$ are such that the transform $C_{m,M}(A) = (A^* - mI)(MI - A)$ is accretive. Then*

$$(0 \leq) \|A\|^2 - w^2(A) \leq \frac{1}{4} (M - m)^2. \tag{2.16}$$

A sufficient simple condition for $C_{m,M}(A)$ to be accretive is that A is a selfadjoint operator on H and such that $MI \geq A \geq mI$ in the partial operator order of $B(H)$.

The following result may be stated as well:

Theorem 8. *Let $A, B \in B(H)$ and $\alpha, \beta, \gamma, \delta \in \mathbb{K}$ be such that $\operatorname{Re}(\beta\overline{\alpha}) > 0, \operatorname{Re}(\delta\overline{\gamma}) > 0$ and the transforms $C_{\alpha,\beta}(A), C_{\gamma,\delta}(B)$ are accretive, then*

$$\frac{w(BA)}{w(A)\,w(B)} \leq 1 + \frac{1}{4} \cdot \frac{|\beta - \alpha| |\delta - \gamma|}{[\operatorname{Re}(\beta\overline{\alpha}) \operatorname{Re}(\delta\overline{\gamma})]^{1/2}} \tag{2.17}$$

and

$$\begin{aligned} w(BA) &\leq w(A)\,w(B) \\ &+ \left[\left(|\alpha + \beta| - 2[\operatorname{Re}(\beta\overline{\alpha})]^{1/2}\right)\left(|\delta + \gamma| - 2[\operatorname{Re}(\delta\overline{\gamma})]^{1/2}\right)\right]^{1/2} \\ &\times [w(A)\,w(B)]^{1/2} \end{aligned} \tag{2.18}$$

respectively.

Proof. With the assumptions (2.10) (or, equivalently, (2.11) in the proof of Theorem 7) and if $\operatorname{Re}(\beta\overline{\alpha}) > 0, \operatorname{Re}(\delta\overline{\gamma}) > 0$ then

$$\begin{aligned} &|\langle u, v\rangle - \langle u, e\rangle \langle e, v\rangle| \\ &\leq \begin{cases} \frac{1}{4} \frac{|\beta-\alpha||\delta-\gamma|}{[\operatorname{Re}(\beta\overline{\alpha}) \operatorname{Re}(\delta\overline{\gamma})]^{1/2}} |\langle u, e\rangle \langle e, v\rangle|, \\ \\ \left[\left(|\alpha + \beta| - 2[\operatorname{Re}(\beta\overline{\alpha})]^{1/2}\right)\left(|\delta + \gamma| - 2[\operatorname{Re}(\delta\overline{\gamma})]^{1/2}\right)\right]^{1/2} \\ \times [|\langle u, e\rangle \langle e, v\rangle|]^{1/2}. \end{cases} \end{aligned} \tag{2.19}$$

The first inequality has been established in [**4**] (see [**8**, p. 62]) while the second one can be obtained in a canonical manner from the reverse of the Schwarz inequality given in [**6**]. The details are omitted.

Applying (2.12) for $u = Ax, v = B^*x$ and $e = x$ we deduce

$$|\langle BAx, x\rangle - \langle Ax, x\rangle \langle Bx, x\rangle| \leq \begin{cases} \frac{1}{4}\frac{|\beta-\alpha||\delta-\gamma|}{[\mathrm{Re}(\beta\overline{\alpha})\,\mathrm{Re}(\delta\overline{\gamma})]^{1/2}}\,|\langle A, x\rangle \langle Bx, x\rangle|\,, \\ \\ \left[\left(|\alpha+\beta| - 2\left[\mathrm{Re}\left(\beta\overline{\alpha}\right)\right]^{1/2}\right)\left(|\delta+\gamma| - 2\left[\mathrm{Re}\left(\delta\overline{\gamma}\right)\right]^{1/2}\right)\right]^{1/2} \\ \times \left[|\langle A, x\rangle \langle Bx, x\rangle|\right]^{1/2}. \end{cases} \tag{2.20}$$

for any $x \in H, \|x\| = 1$, which are of interest in themselves.

A similar argument to that in the proof of Theorem 7 yields the desired inequalities (2.17) and (2.18). The details are omitted. ∎

Corollary 3. *Let $A \in B(H)$ and $\alpha, \beta \in \mathbb{K}$ be such that $\mathrm{Re}\left(\beta\overline{\alpha}\right) > 0$ and the transform $C_{\alpha,\beta}(A)$ is accretive, then*

$$(1 \leq)\, \frac{\|A\|^2}{w^2(A)} \leq 1 + \frac{1}{4} \cdot \frac{|\beta-\alpha|^2}{\mathrm{Re}\left(\beta\overline{\alpha}\right)} \tag{2.21}$$

and

$$(0 \leq)\, \|A\|^2 - w^2(A) \leq \left(|\alpha+\beta| - 2\left[\mathrm{Re}\left(\beta\overline{\alpha}\right)\right]^{1/2}\right) w(A) \tag{2.22}$$

respectively.

The proof is obvious from Theorem 8 on choosing $B = A^*$ and the details are omitted.

Remark 3. *Let $A \in B(H)$ and $M > m > 0$ are such that the transform $C_{m,M}(A) = (A^* - mI)(MI - A)$ is accretive. Then, on making use of Corollary 3, we may state the following simpler results*

$$(1 \leq)\, \frac{\|A\|}{w(A)} \leq \frac{1}{2} \cdot \frac{M+m}{\sqrt{Mm}} \tag{2.23}$$

and

$$(0 \leq)\, \|A\|^2 - w^2(A) \leq \left(\sqrt{M} - \sqrt{m}\right)^2 w(A) \tag{2.24}$$

respectively. These two inequalities have been obtained earlier by the author using a different approach, see [**7**].

Problem 1. *Find general examples of bounded linear operators realizing the equality case in each of the inequalities (2.9), (2.17) and (2.18), respectively.*

3. Some Particular Cases of Interest

The following result is well known in the literature (see for instance [**14**]):

$$w(A^n) \leq w^n(A),$$

for each positive integer n and any operator $A \in B(H)$.

The following reverse inequalities for $n = 2$, can be stated:

Proposition 1. *Let $A \in B(H)$ and $\alpha, \beta \in \mathbb{K}$ be such that the transform $C_{\alpha,\beta}(A)$ is accretive, then*

$$(0 \leq)\, w^2(A) - w(A^2) \leq \frac{1}{4} |\beta - \alpha|^2 . \tag{3.1}$$

Proof. On applying the inequality (2.13) from Theorem 7 for the choice $B = A$, we get the following inequality of interest in itself:

$$\left| \langle Ax, x \rangle^2 - \langle A^2 x, x \rangle \right| \leq \frac{1}{4} |\beta - \alpha|^2 , \tag{3.2}$$

for any $x \in H, \|x\| = 1$. Since obviously,

$$|\langle Ax, x \rangle|^2 - \left| \langle A^2 x, x \rangle \right| \leq \left| \langle Ax, x \rangle^2 - \langle A^2 x, x \rangle \right| ,$$

then by (3.2) we get

$$|\langle Ax, x \rangle|^2 \leq \left| \langle A^2 x, x \rangle \right| + \frac{1}{4} |\beta - \alpha|^2 , \tag{3.3}$$

for any $x \in H, \|x\| = 1$. Taking the supremum over $\|x\| = 1$ in (3.3) we deduce the desired result (3.1). ∎

Remark 4. *Let $A \in B(H)$ and $M > m > 0$ are such that the transform $C_{m,M}(A) = (A^* - mI)(MI - A)$ is accretive. Then*

$$(0 \leq)\, w^2(A) - w(A^2) \leq \frac{1}{4} (M - m)^2 . \tag{3.4}$$

If $MI \geq A \geq mI$ in the partial operator order of $B(H)$, then (3.4) is valid.

Finally, we also have

Proposition 2. *Let $A \in B(H)$ and $\alpha, \beta \in \mathbb{K}$ be such that $\operatorname{Re}(\beta\overline{\alpha}) > 0$ and the transform $C_{\alpha,\beta}(A)$ is accretive, then*

$$(1 \leq)\, \frac{w^2(A)}{w(A^2)} \leq 1 + \frac{1}{4} \cdot \frac{|\beta - \alpha|^2}{\operatorname{Re}(\beta\overline{\alpha})} \tag{3.5}$$

and

$$(0 \leq)\, w^2(A) - w(A^2) \leq \left(|\alpha + \beta| - 2 \left[\operatorname{Re}(\beta\overline{\alpha}) \right]^{1/2} \right) w(A) \tag{3.6}$$

respectively.

Proof. On applying the inequality (2.20) from Theorem 8 for the choice $B = A$, we get the following inequality of interest in itself:

$$\left| \langle Ax, x \rangle^2 - \langle A^2 x, x \rangle \right| \leq \begin{cases} \frac{1}{4} \cdot \frac{|\beta-\alpha|^2}{\operatorname{Re}(\beta\overline{\alpha})} |\langle A, x \rangle|^2 , \\ \\ \left(|\alpha + \beta| - 2 \left[\operatorname{Re}(\beta\overline{\alpha}) \right]^{1/2} \right) |\langle A, x \rangle| . \end{cases} \tag{3.7}$$

for any $x \in H, \|x\| = 1$.

Now, on making use of a similar argument to the one in the proof of Proposition 1 we deduce the desired results (3.5) and (3.6). The details are omitted. ∎

Remark 5. *Let $A \in B(H)$ and $M > m > 0$ are such that the transform $C_{m,M}(A) = (A^* - mI)(MI - A)$ is accretive. Then, on making use of Proposition 2 we may state the following simpler results*

$$(1 \leq) \frac{w^2(A)}{w(A^2)} \leq \frac{1}{4} \cdot \frac{(M+m)^2}{Mm} \tag{3.8}$$

and

$$(0 \leq) w^2(A) - w\left(A^2\right) \leq \left(\sqrt{M} - \sqrt{m}\right)^2 w(A) \tag{3.9}$$

respectively.

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