ON AN INTEGRAL INEQUALITY

SEVER S. DRAGOMIR AND QUỐC ANH NGÔ

Abstract. Let $f : [a,b] \to [0,\infty)$ be continuous function and $g : [a,b] \to [0,\infty)$ be non-decreasing, differentiable on (a,b). In this short paper, we prove that the following inequality

$$\int_x^b f^{\beta}(t)\,\mathrm{d}t \geq \int_x^b g^{\beta}(t)\,\mathrm{d}t$$

holds for $0 < \alpha < \beta$ and for all $x \in [a,b]$ provided

$$\int_x^b f^{\alpha}(t)\,\mathrm{d}t \geq \int_x^b g^{\alpha}(t)\,\mathrm{d}t$$

holds for all $x \in [a,b]$.

1. introduction

Let $f, g : [a,b] \to [0,\infty)$ be continuous functions and g is non-decreasing satisfying

$$\int_a^b f(t)\,\mathrm{d}t = \int_a^b g(t)\,\mathrm{d}t. \tag{1}$$

and

$$\int_x^b f(t)\,\mathrm{d}t \geq \int_x^b g(t)\,\mathrm{d}t \tag{2}$$

Recently, the authors obtained an interesting inequality for these functions f and g in [2]. Precisely, using the well-known second mean value theorem for integrals, they got

Theorem 1 (See [2]). *If* (1) *and* (2) *hold then*

$$\int_a^b h(f(t))\,\mathrm{d}t \geq \int_a^b h(g(t))\,\mathrm{d}t \tag{3}$$

validates for every convex function h on $[0,\infty)$.

2000 *Mathematics Subject Classification.* 26D15.

Key words and phrases. Integral inequality.

This paper was typeset using $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-LaTeX.

In particularly, if we choose $h(t) := t^\alpha$ for $\alpha > 1$ then function h is convex on $[0,\infty)$. Now (3) reads as following

$$\int_a^b f^\alpha(t)\,\mathrm{d}t \geq \int_a^b g^\alpha(t)\,\mathrm{d}t. \tag{4}$$

Besides, in the case $g(t) := t$ and without assumption (1), the authors in [1] obtained the following result

Theorem 2 (See [1])**.** *Let $f : [a,b] \to [0,\infty)$ be continuous function such that*

$$\int_x^b f(t)\ \mathrm{d}t \geq \int_x^b t\ \mathrm{d}t \tag{5}$$

for $x \in [a,b]$. Then

$$\int_a^b f^2(t)\,\mathrm{d}t \geq \int_a^b t^2\,\mathrm{d}t. \tag{6}$$

It is obvious to see that (6) can be rewritten to

$$\int_x^b f^2(t)\,\mathrm{d}t \geq \int_x^b t^2\,\mathrm{d}t \tag{7}$$

for all $x \in [a,b]$.

Summarizing all the above discussion, in this short note, without the assumption (1), we are interested in comparing the following two types of integral inequalities

$$\int_x^b f^\alpha(t)\,\mathrm{d}t \geq \int_x^b g^\alpha(t)\,\mathrm{d}t, \tag{8}$$

and

$$\int_x^b f^\beta(t)\,\mathrm{d}t \geq \int_x^b g^\beta(t)\,\mathrm{d}t, \tag{9}$$

where α, β are given positive numbers.

Our main results are included in a couple of theorems below. The first theorem concerns to the case when $\alpha = 1$.

Theorem 3. *Let $f : [a,b] \to [0,\infty)$ be continuous function and $g : [a,b] \to [0,\infty)$ be non-decreasing, differentiable on (a,b) satisfying (2). If*

$$\int_x^b f(t)\,\mathrm{d}t \geq \int_x^b g(t)\,\mathrm{d}t, \tag{10}$$

holds then (9) holds for $1 < \beta$.

And the second theorem concerns to the general case when $0 < \alpha$.

Theorem 4. *Let $f : [a,b] \to [0,\infty)$ be continuous function and $g : [a,b] \to [0,\infty)$ be non-decreasing, differentiable on (a,b) satisfying* (2)*. If* (8) *holds then* (9) *holds for $\alpha < \beta$.*

2. Proof of Theorem 3

First, we prove the inequality (9) for positive integers. That is, for positive integer n,

$$\int_x^b f^n(t)\,\mathrm{d}t \geq \int_x^b g^n(t)\,\mathrm{d}t, \tag{11}$$

The case that $n = 1$ is just the hypothesis. For $n = 2$, by (8) and integrating by parts,

$$\begin{aligned}
&\int_x^b g(t)\left(f(t) - g(t)\right)\mathrm{d}t = -\int_x^b g(t) d\left(\int_t^b \left(f(s) - g(s)\right)\mathrm{d}s\right) \\
&= -g(t)\int_t^b \left(f(s) - g(s)\right)\mathrm{d}s\,\Bigg|\begin{matrix} t=b \\ t=x \end{matrix} + \int_x^b \left(\int_t^b \left(f(s) - g(s)\right)\mathrm{d}s\right) g'(t)\,\mathrm{d}t \\
&= g(s)\int_x^b \left(f(t) - g(t)\right)\mathrm{d}t + \int_x^b \left(\int_t^b \left(f(s) - g(s)\right)\mathrm{d}s\right) g'(t)\,\mathrm{d}t \\
&\geq 0.
\end{aligned}$$

Therefore

$$\int_x^b f(t)\,g(t)\,\mathrm{d}t \geq \int_x^b g^2(t)\,\mathrm{d}t.$$

On the other hand, from the fact that $\left(f(t) - g(t)\right)^2 \geq 0$ we obtain

$$\int_x^b \left(f(t) - g(t)\right)^2 \mathrm{d}t \geq 0$$

which yields

$$\begin{aligned}
\int_x^b f^2(t)\,\mathrm{d}t &\geq 2\int_x^b f(t)\,g(t)\,\mathrm{d}t - \int_x^b g^2(t)\,\mathrm{d}t \\
&\geq \int_x^b g^2(t)\,\mathrm{d}t.
\end{aligned}$$

Suppose now that (9) is valid for $n \geq 2$. It is obvious that

$$\left(f^n(t) - g^n(t)\right)\left(f(t) - g(t)\right) \geq 0 \quad , \quad \forall t \in [a,b]$$

that is

$$f^{n+1}(t) \geq g(t) f^n(t) + g^n(t) f(t) - g^{n+1}(t) \quad , \quad \forall t \in [a,b]$$

Therefore, integrate both sides from x to b and with the hypothesis of the induction,

$$\int_x^b f^{n+1}(t)\,\mathrm{d}t \geq \int_x^b g(t)f^n(t)\,\mathrm{d}t + \int_x^b g^n(t)f(t)\,\mathrm{d}t - \int_x^b g^{n+1}(t)\,\mathrm{d}t \quad , \quad \forall x \in [a,b]$$

By simple calculation, we see that

$$\begin{aligned}
\int_x^b g(t)\, f^n(t)\,\mathrm{d}t &= -\int_x^b g(t)\, d\left(\int_t^b f^n(s)\,\mathrm{d}s\right) \\
&= -g(t)\int_t^b f^n(s)\,\mathrm{d}s \,\Bigg|\begin{matrix} t=b \\ t=x \end{matrix} + \int_x^b \left(\int_t^b f^n(s)\,\mathrm{d}s\right) g'(t)\,\mathrm{d}t \\
&= g(x)\int_x^b f^n(t)\,\mathrm{d}t + \int_x^b \left(\int_t^b f^n(s)\,\mathrm{d}s\right) g'(t)\,\mathrm{d}t \\
&\geq g(x)\int_x^b g^n(t)\,\mathrm{d}t + \int_x^b \left(\int_t^b g^n(s)\,\mathrm{d}s\right) g'(t)\,\mathrm{d}t \\
&= -g(t)\int_t^b g^n(s)\,\mathrm{d}s \,\Bigg|\begin{matrix} t=b \\ t=x \end{matrix} + \int_x^b \left(\int_t^b g^n(s)\,\mathrm{d}s\right) g'(t)\,\mathrm{d}t \\
&= -\int_x^b g(t)\, d\left(\int_t^b g^n(s)\,\mathrm{d}s\right) \\
&= \int_x^b g^{n+1}(t)\,\mathrm{d}t
\end{aligned}$$

and

$$\begin{aligned}
\int_x^b g^n(t)\, f(t)\,\mathrm{d}t &= -\int_x^b g^n(t)\, d\left(\int_t^b f(s)\,\mathrm{d}s\right) \\
&= -g^n(t)\int_t^b f(s)\,\mathrm{d}s \,\Bigg|\begin{matrix} t=b \\ t=x \end{matrix} + \int_x^b \left(\int_t^b f(s)\,\mathrm{d}s\right) d\left(g^n(t)\right) \\
&= g^n(x)\int_x^b f(t)\,\mathrm{d}t + \int_x^b \left(\int_t^b f(s)\,\mathrm{d}s\right) d\left(g^n(t)\right) \\
&\geq g^n(x)\int_x^b g(t)\,\mathrm{d}t + \int_x^b \left(\int_t^b g(s)\,\mathrm{d}s\right) d\left(g^n(t)\right) \\
&= -g^n(t)\int_t^b g(s)\,\mathrm{d}s \,\Bigg|\begin{matrix} t=b \\ t=x \end{matrix} + \int_x^b \left(\int_t^b g(s)\,\mathrm{d}s\right) d\left(g^n(t)\right) \\
&= -\int_x^b g^n(t)\, d\left(\int_t^b g(s)\,\mathrm{d}s\right)
\end{aligned}$$

$$= \int_x^b g^{n+1}(t)\,\mathrm{d}t.$$

Hence

$$\begin{aligned}\int_x^b f^{n+1}(t)\,\mathrm{d}t &\geq \int_x^b g^n(t)\,f(t)\,\mathrm{d}t + \int_x^b g(t)\,f^{n+1}(t)\,\mathrm{d}t - \int_x^b g^{n+1}(t)\,\mathrm{d}t \\ &\geq \int_x^b g^{n+1}(t)\,\mathrm{d}t.\end{aligned}$$

This means (9) is valid for all positive integers. Now for any $\beta > 1$, denote by $\lfloor\beta\rfloor$ the greatest integer not greater than α and set $\lceil\beta\rceil := \beta - \lfloor\beta\rfloor$. General Cauchy Inequality tells us that, for positive numbers A, B and γ, δ with $\gamma + \delta = 1$,

$$\gamma A + \delta B \geq A^\gamma B^\delta.$$

Therefore

$$\frac{\lfloor\beta\rfloor}{\beta} f^\beta(t) + \frac{\lceil\beta\rceil}{\beta} g^\beta(t) \geq g^{\lceil\beta\rceil}(t) f^{\lfloor a\rfloor}(t)\,, \forall t \in [a,b]\,.$$

This and (9) imply, for all $x \in [a,b]$, that

$$\begin{aligned}\frac{\lfloor\beta\rfloor}{\beta}\int_x^b f^\beta(t)\,\mathrm{d}t &\geq -\int_x^b \frac{\lceil\beta\rceil}{\beta} g^\beta(t)\,\mathrm{d}t + \int_x^b g^{\lceil\beta\rceil}(t)\,f^{\lfloor a\rfloor}(t)\,\mathrm{d}t \\
&= -\int_x^b \frac{\lceil\beta\rceil}{\beta} g^\beta(t)\,\mathrm{d}t - \int_x^b g^{\lceil\beta\rceil}(t)\,d\left(\int_t^b f^{\lfloor a\rfloor}(s)\,\mathrm{d}s\right) \\
&= -\int_x^b \frac{\lceil\beta\rceil}{\beta} g^\beta(t)\,\mathrm{d}t - g^{\lceil\beta\rceil}(t)\int_t^b f^{\lfloor a\rfloor}(s)\,\mathrm{d}s\,\Bigg|\begin{array}{l} t=b \\ t=x\end{array} \\
&\quad + \int_x^b \left(\int_t^b f^{\lfloor a\rfloor}(s)\,\mathrm{d}s\right) d\left(g^{\lceil\beta\rceil}(t)\right) \\
&\geq -\int_x^b \frac{\lceil\beta\rceil}{\beta} g^\beta(t)\,\mathrm{d}t + g^{\lceil\beta\rceil}(x)\int_x^b f^{\lfloor a\rfloor}(s)\,\mathrm{d}s \\
&\quad + \int_x^b \left(\int_t^b g^{\lfloor a\rfloor}(s)\,\mathrm{d}s\right) d\left(g^{\lceil\beta\rceil}(t)\right) \\
&= -\int_x^b \frac{\lceil\beta\rceil}{\beta} g^\beta(t)\,\mathrm{d}t - \int_x^b g^{\lceil\beta\rceil}(t)\,d\left(\int_t^b g^{\lfloor a\rfloor}(s)\,\mathrm{d}s\right) \\
&= -\int_x^b \frac{\lceil\beta\rceil}{\beta} g^\beta(t)\,\mathrm{d}t + \int_x^b g^\beta(t)\,\mathrm{d}t \\
&= \frac{\lfloor\beta\rfloor}{\beta}\int_x^b g^\beta(t)\,\mathrm{d}t.\end{aligned}$$

Thus

$$\frac{\lfloor\beta\rfloor}{\beta}\int_x^b f^\beta(t)\,\mathrm{d}t \geq \frac{\lfloor\beta\rfloor}{\beta}\int_x^b g^\beta(t)\,\mathrm{d}t.$$

The conclusion of the proposition follows since $\frac{\lfloor\beta\rfloor}{\beta} \neq 0$.

3. Proof of Theorem 4

Since $g(t)$ is non-decreasing then $g^\alpha(t)$ is also non-increasing for $\alpha > 0$. If replacing $f(x)$ by $f^\alpha(x)$ and $g(x)$ by $g^\alpha(x)$, then Proof of Theorem 4 is obtained by applying Theorem 3 with exponential $\frac{\beta}{\alpha}$. Therefore, we omit it.

References

[1] Quoc Anh Ngo, Du Duc Thang, Tran Tat Dat, and Dang Anh Tuan, Notes on an Integral Inequality, *JIPAM. J. Inequal. Pure Appl. Math.* **7** (2006), no. 4, Article 120, 5 pp, Available online at http://jipam.vu.edu.au/article.php?sid=737.

[2] Quoc Anh Ngo and Feng Qi, Generalizations of an integral inequality, submitted.

(Sever S. Dragomir) School of Computer Science and Mathematics,, Victoria University of Technology,, PO Box 14428, MCMC 8001, Victoria, Australia

E-mail address: sever.dragomir@vu.edu.au

(Q. A. Ngô) Department of Mathematics, Mechanics and Informatics, College of Science, Việt Nam National University, Hà Nội, Việt Nam

E-mail address: bookworm_vn@yahoo.com