INEQUALITIES AND MONOTONICITY RESULTS FOR SOME SPECIAL FUNCTIONS

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Abstract. The monotonicity, convexity, log-convexity and complete monotonicity property for some special functions are proved. Some Turán-type inequalities are established.

1. Introduction

Euler's Constant $\gamma = 0.57721566490153286\ldots$ was first introduced by Leonhard Euler (1707-1783) in 1734 as

$$\gamma = \lim_{n\to\infty} D_n, \quad \text{where} \quad D_n = \sum_{k=1}^{n} \frac{1}{k} - \log n. \tag{1}$$

It is also known as the Euler-Mascheroni constant. According to Glaisher [9], the use of the symbol γ is due to the geometer Lorenzo Mascheroni (1750-1800) who used it in 1790 while Euler used the letter C. Euler's constant plays an important role in Analysis (Gamma function, Bessel functions, exponential-integral, ...) and occurs frequently in Number Theory.

Direct use of formula (1) to compute the Euler's constant is of poor interest since the convergence is very slow. The Euler-Maclaurin summation can be used to have a complete asymptotic expansion of the harmonic numbers $H_n = \sum_{k=1}^{n} \frac{1}{k}$,

$$H_n - \log n = \gamma + \frac{1}{2n} - \sum_{k=1}^{\infty} \frac{B_{2k}}{2k} \frac{1}{n^{2k}},$$

where the B_{2k} are the Bernoulli numbers defined by

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}. \tag{2}$$

First four Bernoulli numbers with even indices are

$$B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad B_8 = -\frac{1}{30}, \tag{3}$$

and then

$$\gamma = H_n - \log n - \frac{1}{2n} + \frac{1}{12n^2} - \frac{1}{120n^4} + \frac{1}{252n^6} - \frac{1}{240n^8} + \cdots.$$

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In 1991, R. M. Young [13] presented an elegant geometrical proof for the double inequality

$$\frac{1}{2(n+1)} < D_n - \gamma < \frac{1}{2n}, \quad n = 1, 2, \ldots. \tag{4}$$

In [4, 5, 7, 12], other bounds for $D_n - \gamma$ were established. Since

$$\psi(n+1) = -\gamma + \sum_{k=1}^{n} \frac{1}{k} \tag{5}$$

(see [1, p. 258]), where $\psi = \Gamma'/\Gamma$ is the psi function, the inequality (4) can be written as

$$\frac{1}{2(n+1)} < \psi(n+1) - \log n < \frac{1}{2n}, \quad n = 1, 2, \ldots. \tag{6}$$

Let real sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be defined by

$$a_n = (n+1)\left[\psi(n+1) - \log n\right], \tag{7}$$

$$b_n = n\left[\psi(n+1) - \log n\right]. \tag{8}$$

From the asymptotic formula [1, p. 259]

$$\psi(x) = \log x - \frac{1}{2x} + O(x^{-2}) \quad \text{as} \quad x \to \infty, \tag{9}$$

and the functional equation [1, p. 258]

$$\psi(x+1) = \psi(x) + \frac{1}{x}, \tag{10}$$

we conclude that

$$\lim_{x\to\infty} a_n = \lim_{x\to\infty} b_n = \frac{1}{2}.$$

In order to prove (4) it suffices to show that the sequence $\{a_n\}_{n=1}^{\infty}$ strictly decreasing and $\{b_n\}_{n=1}^{\infty}$ is strictly increasing. The following Theorem 1 not only proves the monotonicity property of the sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$, but also considers the complete monotonicity property of $\{a_n\}_{n=1}^{\infty}$ and $\{\Delta b_n\}_{n=1}^{\infty}$. Recall that the kth order difference of a sequence $\{a_n\}_{n=1}^{\infty}$ is defined recuesively by

$$\Delta^0 a_n = a_n \quad \text{and} \quad \Delta^k a_n = \Delta^{k-1} a_{n+1} - \Delta^{k-1} a_n, \quad k = 1, 2, \ldots.$$

(Instead of $\Delta^1 = \Delta$.) A sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers is called strictly completely monotonic, if

$$(-1)^k \Delta^k a_n > 0 \quad \text{for all} \quad n \geq 1 \quad \text{and} \quad k \geq 0.$$

A sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers is called strictly convex (concave), if

$$\Delta^2 a_n = a_{n+2} - 2a_{n+1} + a_n > (<) 0 \quad \text{for all} \quad n \geq 1.$$

A sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers is called strictly log-convex (log-concave), if it is positive and

$$a_{n+1}^2 < (>) a_n a_{n+2} \quad \text{for all integers} \quad n \geq 1.$$

By the arithmetic-geometric mean inequality, the log-convexity implies the convexity, and the concavity implies the log-concavity.

Theorem 1. *Let real sequences* $\{a_n\}_{n=1}^{\infty}$ *and* $\{b_n\}_{n=1}^{\infty}$ *be defined recuesively by* (7) *and* (8). *Then the sequence* $\{a_n\}_{n=1}^{\infty}$ *is strictly completely monotonic, so is the sequence* $\{\Delta b_n\}_{n=1}^{\infty}$. *That is*

$$(-1)^k \Delta^k a_n > 0 \tag{11}$$

and

$$(-1)^k \Delta^{k+1} b_n > 0 \tag{12}$$

for all integers $n \geq 1$ *and* $k \geq 0$.

Clearly, Theorem 1 has shown the convexity of the sequences $\{a_n\}_{n=1}^{\infty}$. The following Theorem 2 further considers its log-convexity.

Theorem 2. *Let the sequence* $\{a_n\}_{n=1}^{\infty}$ *be defined by* (7). *Then the sequence* $\{a_n\}_{n=1}^{\infty}$ *is strictly log-convex.*

Remark 1. From the integral representations

$$\psi(x) = \int_0^{\infty} \left(\frac{e^{-t}}{t} - \frac{e^{-xt}}{1 - e^{-t}} \right) dt, \quad x > 0, \tag{13}$$

$$\ln x = \int_0^{+\infty} \frac{e^{-t} - e^{-xt}}{t} \,\mathrm{d}t, \quad x > 0, \tag{14}$$

we conclude that

$$D_n - \gamma = \int_0^{\infty} \left(\frac{1}{t} - \frac{1}{e^t - 1} \right) e^{-nt} \,\mathrm{d}t. \tag{15}$$

By using the following generalization of the Schwarz inequality

$$\int_0^{\infty} g(t)[f(t)]^m \,\mathrm{d}t \int_0^{\infty} g(t)[f(t)]^n \,\mathrm{d}t \geq \left[\int_0^{\infty} g(t)[f(t)]^{(m+n)/2} \,\mathrm{d}t \right]^2, \tag{16}$$

where f and g are two nonnegative functions of a real variable and m and n belong to a set S of real numbers, such that the integrals in (16) exist, A. Laforgia and P. Natalini [11] proved that the Turán-type inequality

$$(D_{n+1} - \gamma)^2 \leq (D_n - \gamma)(D_{n+2} - \gamma) \tag{17}$$

holds for all integers $n \geq 1$. The concavity of the sequence $\{b_n\}_{n=1}^{\infty}$ implies its log-concavity. From log-convexity of the sequence $\{a_n\}_{n=1}^{\infty}$ and log-concavity of the sequence $\{b_n\}_{n=1}^{\infty}$ we obtain the Turán-type inequality

$$\frac{(n+2)^2}{(n+1)(n+3)}(D_{n+1} - \gamma)^2 < (D_n - \gamma)(D_{n+2} - \gamma) < \frac{(n+1)^2}{n(n+2)}(D_{n+1} - \gamma)^2 \tag{18}$$

for all integers $n \geq 1$.

Obviously, the left inequality of (18) is an improvement of the inequality (17).

The convergence of the sequence D_n to γ is very slow. In 1993, D. W. DeTemple [7] studied a modified sequence which converges faster and proved

$$\frac{1}{24(n+1)^2} < R_n - \gamma < \frac{1}{24n^2}, \quad n \geq 1, \tag{19}$$

where

$$R_n = \sum_{k=1}^{n} \frac{1}{k} - \log\left(n + \frac{1}{2}\right).$$

Now let

$$H(n) = n^2(R_n - \gamma), \quad n \geq 1.$$

By (5), we see that

$$H(n) = (R_n - \gamma)n^2 = \left[\psi(n+1) - \log\left(n + \frac{1}{2}\right)\right] n^2. \tag{20}$$

Some computer experiments led M. Vuorinen to conjecture that $H(n)$ increases on the interval $[1, \infty)$ from $H(1) = -\gamma + 1 - \log(3/2) = 0.0173\ldots$ to $1/24 = 0.0416\ldots$. E. A. Karatsuba [10] proved that for all integers $n \geq 1$, $H(n) < H(n+1)$, by clever use of Stirling formula and Fourier series. Some computer experiments also seem to indicate that $[(n+1)/n]^2 H(n)$ is a decreasing convex function [6].

The following Theorem 3 shows the monotonicity and convexity property of $H(n)$, $[(n+1/2)/n]^2 H(n)$ and $[(n+1)/n]^2 H(n)$.

Theorem 3. *Let $H(n)$ $(n = 1, 2, \ldots)$ be defined by (20). Then for all integers $n \geq 1$, $H(n)$ and $[(n+1/2)/n]^2 H(n)$ are both strictly increasing concave sequence, while $[(n+1)/n]^2 H(n)$ is strictly decreasing convex sequence.*

Remark 2. By the asymptotic formula [2, p. 550]

$$\psi(x) = \log\left(x - \frac{1}{2}\right) + \frac{1}{24(x-1/2)^2} + O(x^{-4}) \quad \text{as} \quad x \to \infty,$$

we conclude that

$$\lim_{n\to\infty} H(n) = \frac{1}{24} \quad \text{and} \quad \lim_{n\to\infty} [(n+1)/n]^2 H(n) = \frac{1}{24}. \tag{21}$$

From the monotonicity of $H(n)$, $[(n+1)/n]^2 H(n)$ and (21), we obtain the inequality (19).

Remark 3. From the monotonicity of $[(n+1/2)/n]^2 H(n)$ and the limit relation $\lim_{n\to\infty} [(n+1/2)/n]^2 H(n) = \frac{1}{24}$, we obtain

$$R_n - \gamma < \frac{1}{24(n+\frac{1}{2})^2}, \quad n \geq 1. \tag{22}$$

Obviously, the upper in (22) is sharper than one in (19).

The inequality (22) can be written as

$$\frac{1}{\sqrt{24[\psi(n+1) - \log(n+1/2)]}} - n > \frac{1}{2}. \tag{23}$$

From the asymptotic formula [2, p. 550]

$$\psi(x) = \log\left(x - \frac{1}{2}\right) + \frac{1}{24(x-\frac{1}{2})^2} - \frac{7}{960(x-\frac{1}{2})^4} + O(x^{-6}) \quad \text{as} \quad x \to \infty,$$

we obtain

$$\begin{aligned}
&\frac{1}{\sqrt{24[\psi(x+1)-\log(x+1/2)]}}-x\\
&=\frac{1-x\sqrt{24[\psi(x+1)-\log(x+1/2)]}}{\sqrt{24[\psi(x+1)-\log(x+1/2)]}}\\
&=\frac{1-x\sqrt{\frac{1}{(x+1/2)^2}-\frac{7}{40(x+1/2)^4}+O(x^{-6})}}{\sqrt{\frac{1}{(x+1/2)^2}-\frac{7}{40(x+1/2)^4}+O(x^{-6})}}\\
&=\frac{x+\frac{1}{2}-x\sqrt{1-\frac{7}{40(x+1/2)^2}+O(x^{-4})}}{\sqrt{1-\frac{7}{40(x+1/2)^2}+O(x^{-4})}}\\
&=\frac{\frac{1}{2}+\frac{7x}{80(x+1/2)^2}+O(x^{-3})}{1-\frac{7}{80(x+1/2)^2}+O(x^{-4})}\to\frac{1}{2}\quad\text{as}\quad x\to\infty,
\end{aligned}$$

and then,

$$\lim_{n\to\infty}\left[\frac{1}{\sqrt{24[\psi(n+1)-\log(n+1/2)]}}-n\right]=\frac{1}{2}. \tag{24}$$

Hence, the constant $\frac{1}{2}$ in the upper of (22) is the best possible.

Remark 4. The Turán-type inequality

$$(R_{n+1}-\gamma)^2<(R_n-\gamma)(R_{n+2}-\gamma)<\frac{(n+\frac{3}{2})^4}{(n+\frac{1}{2})^2(n+\frac{5}{2})^2}(R_{n+1}-\gamma)^2 \tag{25}$$

holds for all integers $n\geq 1$. In fact, by using (13) and (14), we conclude that

$$R_n-\gamma=\int_0^\infty\left(\frac{1}{t}-\frac{1}{e^{t/2}-e^{-t/2}}\right)e^{-(n+1/2)t}\,\mathrm{d}t. \tag{26}$$

By (16), we obtain the left inequality of (25). The concavity of the sequence $[(n+1/2)/n]^2H(n)\,(n=1,2,\ldots)$ implies its log-concavity. From its log-concavity we obtain the right inequality of (25).

By Theorem 3, the sequences $[(n+1)/n]^2H(n)$ is convex. The following Theorem 4 further considers its log-convexity.

Theorem 4. *Let $H(n)\quad(n=1,2,\ldots)$ be defined by (20). Then for all integers $n\geq 1$, the sequences $[(n+1)/n]^2H(n)$ is strictly log-convex.*

Remark 5. The concavity of the sequence $[(n+1/2)/n]^2H(n)$ implies its log-concavity. From log-convexity of the sequence $[(n+1)/n]^2H(n)$ and log-concavity of the sequence $[(n+1/2)/n]^2H(n)$ we obtain the Turán-type inequality

$$\begin{aligned}
\frac{(n+2)^4}{(n+1)^2(n+3)^2}(R_{n+1}-\gamma)^2&<(R_n-\gamma)(R_{n+2}-\gamma)\\
&<\frac{(n+\frac{3}{2})^4}{(n+\frac{1}{2})^2(n+\frac{5}{2})^2}(R_{n+1}-\gamma)^2
\end{aligned} \tag{27}$$

for all integers $n\geq 1$.

2. **Proofs of theorems**

Proof of Theorem 1. It is clear that

$$\Delta^0 a_n = a_n > 0. \tag{28}$$

Using (10), we obtain

$$\Delta a_n = a_{n+1} - a_n = \psi(n+1) - (n+1)\log(n+1) + \log(n) + 1 \triangleq f(n). \tag{29}$$

This implies

$$\Delta^{k+1} a_n = \Delta^k f(n) = \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} g(n+i) = f^{(k)}(\xi), \tag{30}$$

where $\xi \in [n, n+k]$ (see [8, pp. 187-190]).

Using the representations (14) and

$$\psi'(x) = \int_0^\infty \frac{te^{-xt}}{1-e^{-t}} dt, \quad x > 0, \tag{31}$$

we conclude that

$$\begin{aligned} f'(x) &= \psi'(x+1) - \log(x+1) + \log(x) - \frac{1}{(x+1)^2} - \frac{1}{x+1} + \frac{1}{x} \\ &= \int_0^\infty \frac{\delta(t)}{te^t(e^t-1)} e^{-xt} dt, \end{aligned} \tag{32}$$

where

$$\begin{aligned} \delta(t) &= (t-1)e^{2t} - 2(t-1)e^t + t^2 + t - 1 \\ &= \sum_{n=4}^{\infty} [(n-2)2^{n-1} - 2(n-1)] \frac{t^n}{n!} > 0 \quad \text{for} \quad t > 0. \end{aligned} \tag{33}$$

Further, we get

$$(-1)^k f^{(k+1)}(x) = \int_0^\infty \frac{\delta(t)}{e^t(e^t-1)} t^{k-1} e^{-xt} dt > 0 \tag{34}$$

for $x > 0$ and $k = 0, 1, 2, \ldots$. By (30) and (34), we have

$$(-1)^{k+2} \Delta^{k+2} a_n > 0 \quad \text{for all integers } n \geq 1 \text{ and } k \geq 0. \tag{35}$$

Using (9) and (10), we conclude that $\lim_{x\to\infty} f(x) = 0$. By (34), the function f is strictly increasing on $(0, \infty)$, and then $f(x) < \lim_{x\to\infty} f(x) = 0$, which implies that

$$\Delta a_n = f(n) < 0. \tag{36}$$

From (28), (35) and (36), we see that

$$(-1)^k \Delta^k a_n > 0 \quad \text{for all integers } n \geq 1 \text{ and } k \geq 0. \tag{37}$$

Using (10), we obtain

$$\Delta b_n = b_{n+1} - b_n = \psi(n+1) - (n+1)\log(n+1) + n\log(n) + 1 \triangleq g(n). \tag{38}$$

This implies

$$\Delta^{k+1} b_n = \Delta^k g(n) = \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} g(n+i) = g^{(k)}(\zeta), \tag{39}$$

where $\xi \in [n, n+k]$. Hence, in order to prove (12) it suffices to show that the function g is strictly completely monotonic on $(0,\infty)$, that is

$$(-1)^k g^{(k)}(x) > 0 \quad \text{for} \quad x > 0 \quad \text{and} \quad k = 0, 1, 2, \ldots. \tag{40}$$

Using the representations (14) and (31), we conclude that

$$g'(x) = \psi'(x+1) - \log(x+1) + \log(x) = -\int_0^\infty \frac{\phi(t)}{te^t(e^t-1)} e^{-xt} dt, \tag{41}$$

where

$$\phi(t) = (e^t-1)^2 - t^2 e^t = \sum_{n=4}^{\infty} [2^n - n(n-1) - 2] \frac{t^n}{n!} > 0 \quad \text{for} \quad t > 0.$$

Further, we get

$$(-1)^k g^{(k)}(x) = \int_0^\infty \frac{\phi(t)}{e^t(e^t-1)} t^{k-2} e^{-xt} dt > 0 \text{ for } x > 0 \text{ and } k = 1, 2, \ldots. \tag{42}$$

Using (9) and (10), we conclude that $\lim_{x\to\infty} g(x) = 0$. By (42), the function g is strictly decreasing on $(0,\infty)$, and then

$$g(x) > \lim_{x\to\infty} g(x) = 0. \tag{43}$$

We obtain from (42) and (43) that the inequality (40) is true. The proof of Theorem 1 is complete. □

Proof of Theorem 2. Let $A(x)$ be defined by

$$A(x) = (x+1)\left[\psi(x+1) - \log x\right], \quad x > 0. \tag{44}$$

Direct calculation produces

$$\begin{aligned}
&[\psi(x+1) - \log x]^2 [\log A(x)]'' \\
&= \left[\psi''(x+1) + \frac{1}{x^2}\right] [\psi(x+1) - \log x] \\
&\qquad - \left[\psi'(x+1) - \frac{1}{x}\right]^2 - \frac{1}{(x+1)^2} [\psi(x+1) - \log x]^2 .
\end{aligned}$$

From the inequalities (see [3, Theorem 9])

$$\begin{aligned}
\log x - \frac{1}{2x} - \frac{1}{12x^2} &< \psi(x) < \log x - \frac{1}{2x} \\
\frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} &< \psi'(x) < \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3}, \\
-\frac{1}{x^2} - \frac{1}{x^3} - \frac{1}{2x^4} &< \psi''(x) < -\frac{1}{x^2} - \frac{1}{x^3} - \frac{1}{2x^4} + \frac{1}{6x^6},
\end{aligned} \tag{45}$$

and functional equation (10), we conclude that

$$\frac{1}{2x} - \frac{1}{12x^2} < \psi(x+1) - \log x < \frac{1}{2x} \tag{46}$$

$$\frac{1}{2x^2} - \frac{1}{6x^3} < \frac{1}{x} - \psi'(x+1) < \frac{1}{2x^2}, \tag{47}$$

$$\frac{1}{x^3} - \frac{1}{2x^4} < \psi''(x+1) + \frac{1}{x^2} < \frac{1}{x^3} - \frac{1}{2x^4} + \frac{1}{6x^6}, \tag{48}$$

and then, we obtain

$$[\psi(x+1)-\log x]^2\,[\log A(x)]''>\left(\frac{1}{x^3}-\frac{1}{2x^4}\right)\left(\frac{1}{2x}-\frac{1}{12x^2}\right)-\left(\frac{1}{2x^2}\right)^2 -\frac{1}{(x+1)^2}\left(\frac{1}{2x}\right)^2 =\frac{4x^3-9x^2-6x+1}{24x^6(x+1)^2}>0\quad\text{for}\quad x\ge 3.$$

A direct calculation produces

$$A(1)=2\,(1-\gamma)=0.8455686702\ldots,$$
$$A(2)=3\left(1+\frac{1}{2}-\gamma-\log 2\right)=0.6889114653\ldots,$$
$$A(3)=4\left(1+\frac{1}{2}+\frac{1}{3}-\gamma-\log 3\right)=0.6300215217\ldots,$$
$$A(4)=5\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}-\gamma-\log 4\right)=0.5991165365\ldots,$$
$$A(1)A(3)=0.53272646\ldots>0.474599007\ldots=A^2(2),$$
$$A(2)A(4)=0.412738251>0.3969271178\ldots=A^2(3).$$

Hence, the sequence $\{a_n\}_{n=1}^{\infty}$ is strictly log-convex for all integers $n\ge 1$. The proof of Theorem 2 is complete. □

In order prove our Theorem 3 and Theorem 4, we need to the following results [2]: For $x>\frac{1}{2}, N=0,1,2,\ldots,$

$$\log\left(x-\frac{1}{2}\right)-\sum_{k=1}^{2N}\frac{B_{2k}(1/2)}{2k(x-\frac{1}{2})^{2k}}<\psi(x) <\log\left(x-\frac{1}{2}\right)-\sum_{k=1}^{2N+1}\frac{B_{2k}(1/2)}{2k(x-\frac{1}{2})^{2k}} \tag{49}$$

and

$$\frac{(n-1)!}{(x-\frac{1}{2})^n}+\sum_{k=1}^{2N+1}\frac{B_{2k}(1/2)}{(2k)!}\frac{(n+2k-1)!}{(x-\frac{1}{2})^{n+2k}}<(-1)^{n+1}\psi^{(n)}(x) <\frac{(n-1)!}{(x-\frac{1}{2})^n}+\sum_{k=1}^{2N}\frac{B_{2k}(1/2)}{(2k)!}\frac{(n+2k-1)!}{(x-\frac{1}{2})^{n+2k}},\quad n=1,2,\ldots, \tag{50}$$

where

$$B_k(1/2)=-\left(1-\frac{1}{2^{k-1}}\right)B_k,\quad k=0,1,2,\ldots,$$

B_k are Bernoulli numbers defined by (2). By (3) we get

$$B_2(1/2)=-\frac{1}{12},\quad B_4(1/2)=\frac{7}{240},\quad B_6(1/2)=-\frac{31}{1344},\quad B_8(1/2)=\frac{127}{3840}.$$

From (49), we get

$$\log\left(x-\frac{1}{2}\right)+\frac{1}{24(x-\frac{1}{2})^2}-\frac{7}{960(x-\frac{1}{2})^4}<\psi(x)<\log\left(x-\frac{1}{2}\right)+\frac{1}{24(x-\frac{1}{2})^2}. \tag{51}$$

$$\log\left(x-\frac{1}{2}\right)+\frac{1}{24(x-\frac{1}{2})^2}-\frac{7}{960(x-\frac{1}{2})^4}<\psi(x)<\log\left(x-\frac{1}{2}\right)+\frac{1}{24(x-\frac{1}{2})^2}-\frac{7}{960(x-\frac{1}{2})^4}+\frac{31}{8064(x-\frac{1}{2})^6}. \tag{52}$$

From (50), we obtain

$$\frac{1}{x-\frac{1}{2}}-\frac{1}{12(x-\frac{1}{2})^3}<\psi'(x)<\frac{1}{x-\frac{1}{2}}-\frac{1}{12(x-\frac{1}{2})^3}+\frac{7}{240(x-\frac{1}{2})^5}, \tag{53}$$

$$\frac{1}{x-\frac{1}{2}}-\frac{1}{12(x-\frac{1}{2})^3}+\frac{7}{240(x-\frac{1}{2})^5}-\frac{31}{1344(x-\frac{1}{2})^7}<\psi'(x)<\frac{1}{x-\frac{1}{2}}-\frac{1}{12(x-\frac{1}{2})^3}+\frac{7}{240(x-\frac{1}{2})^5}-\frac{31}{1344(x-\frac{1}{2})^7}+\frac{127}{3840(x-\frac{1}{2})^9}. \tag{54}$$

$$-\frac{1}{(x-\frac{1}{2})^2}<\psi''(x)<-\frac{1}{(x-\frac{1}{2})^2}+\frac{1}{4(x-\frac{1}{2})^4}, \tag{55}$$

$$-\frac{1}{(x-\frac{1}{2})^2}+\frac{1}{4(x-\frac{1}{2})^4}-\frac{7}{48(x-\frac{1}{2})^6}<\psi''(x)<-\frac{1}{(x-\frac{1}{2})^2}++\frac{1}{4(x-\frac{1}{2})^4}-\frac{7}{48(x-\frac{1}{2})^6}+\frac{31}{920(x-\frac{1}{2})^8}. \tag{56}$$

Now we are in position to prove our Theorem 3.

Proof of Theorem 3. Let $a \geq 0$ be a real number and $f_a(x)$ be defined by

$$f_a(x)=(x+a)^2\left[\psi(x+1)-\log\left(x+\frac{1}{2}\right)\right],\quad x>-\frac{1}{2}. \tag{57}$$

Differentiation yields

$$f_a'(x)=2(x+a)\left[\psi(x+1)-\log\left(x+\frac{1}{2}\right)\right]+(x+a)^2\left[\psi'(x+1)-\frac{1}{x+\frac{1}{2}}\right].$$

$$f_a''(x)=2\left[\psi(x+1)-\log\left(x+\frac{1}{2}\right)\right]+4(x+a)\left[\psi'(x+1)-\frac{1}{x+\frac{1}{2}}\right]+(x+a)^2\left[\psi''(x+1)+\frac{1}{(x+\frac{1}{2})^2}\right].$$

From (49) and (50) we obtain the asymptotic formulas

$$\psi(x)=\log\left(x-\frac{1}{2}\right)+\frac{1}{24(x-\frac{1}{2})^2}-\frac{7}{960(x-\frac{1}{2})^4}+O(x^{-6}), \tag{58}$$

$$\psi'(x)=\frac{1}{x-\frac{1}{2}}-\frac{1}{12(x-\frac{1}{2})^3}+\frac{7}{240(x-\frac{1}{2})^5}+O(x^{-7}), \tag{59}$$

which concludes that

$$\lim_{x\to\infty} f_0'(x) = \lim_{x\to\infty} f_{1/2}'(x) = \lim_{x\to\infty} f_1'(x) = 0. \tag{60}$$

By (51), (53) and (55), we obtain

$$\begin{aligned} f_0''(x) &< 2\left[\frac{1}{24(x+\frac{1}{2})^2}\right] + 4x\left[-\frac{1}{12(x+\frac{1}{2})^3} + \frac{7}{240(x+\frac{1}{2})^5}\right] + x^2\left[\frac{1}{4(x+\frac{1}{2})^4}\right] \\ &= -\frac{5x^2 - \frac{23}{4}x - \frac{5}{8}}{60(x+\frac{1}{2})^5} < 0 \quad \text{for} \quad x \geq 2, \end{aligned}$$

and then, $f_0'(x) > \lim_{x\to\infty} f_0'(x) = 0$ for $x \geq 2$. Hence, the function f_0 is strictly increasing concave for $x \geq 2$.

A dierct calculation produces

$$\begin{aligned} f_0(1) &= \left(1 - \gamma - \log\frac{3}{2}\right) = 0.01731922699\ldots, \\ f_0(2) &= 4\left(1 + \frac{1}{2} - \gamma - \log\frac{5}{2}\right) = 0.025957441\ldots, \\ f_0(3) &= 9\left(1 + \frac{1}{2} + \frac{1}{3} - \gamma - \log\frac{7}{2}\right) = 0.03019229938\ldots, \\ f_0(1) &- 2f_0(2) + f_0(3) = -0.00440335563\ldots < 0. \end{aligned}$$

Thus, the sequence $H(n)$ is strictly increasing concave for all integers $n \geq 1$.

By (52), (53) and (56), we obtain

$$\begin{aligned} f_{1/2}''(x) &< 2\left[\frac{1}{24(x+\frac{1}{2})^2} - \frac{7}{960(x+\frac{1}{2})^4} + \frac{31}{8064(x+\frac{1}{2})^6}\right] \\ &\quad + 4\left(x+\frac{1}{2}\right)\left[-\frac{1}{12(x+\frac{1}{2})^3} + \frac{7}{240(x+\frac{1}{2})^5}\right] \\ &\quad + \left(x+\frac{1}{2}\right)^2\left[\frac{1}{4(x+\frac{1}{2})^4} - \frac{7}{48(x+\frac{1}{2})^6} + \frac{31}{192(x+\frac{1}{2})^6}\right] \\ &= -\frac{\frac{21}{480}(x+\frac{1}{2})^2 - \frac{341}{4032}}{(x+\frac{1}{2})^6} < 0 \quad \text{for} \quad x \geq 1, \end{aligned}$$

and then, $f_{1/2}'(x) > \lim_{x\to\infty} f_{1/2}'(x) = 0$ for $x \geq 1$. Hence, the function $f_{1/2}$ is strictly increasing concave for $x \geq 1$, and the sequence $[(n+1/2)/n]^2 H(n)$ is strictly increasing concave for all integers $n \geq 1$.

By (52), (54) and (56), we obtain

$$\begin{aligned} f_1''(x) > 2&\left[\frac{1}{24(x+\frac{1}{2})^2}-\frac{7}{960(x+\frac{1}{2})^4}\right]\\ &+4(x+1)\left[-\frac{1}{12(x+\frac{1}{2})^3}+\frac{7}{240(x+\frac{1}{2})^5}-\frac{31}{1344(x+\frac{1}{2})^7}\right]\\ &+(x+1)^2\left[\frac{1}{4(x+\frac{1}{2})^4}-\frac{7}{48(x+\frac{1}{2})^6}\right]\\ &=\frac{280(x+\frac{1}{2})^4+63(x+\frac{1}{2})^3-290(x+\frac{1}{2})^2-432.5(x+\frac{1}{2})-155}{3360(x+\frac{1}{2})^7}\\ &>0 \quad \text{for} \quad x\geq\frac{3}{2}. \end{aligned}$$

and then, $f_1'(x) < \lim_{x\to\infty} f_1'(x) = 0$ for $x \geq \frac{3}{2}$. Hence, the function f_1 is strictly decreasing convex for $x \geq \frac{3}{2}$.

A dierct calculation produces

$$\begin{aligned} f_1(1) &= 4\left(1-\gamma-\log\frac{3}{2}\right) = 0.06927690796\ldots,\\ f_1(2) &= 9\left(1+\frac{1}{2}-\gamma-\log\frac{5}{2}\right) = 0.0584042429\ldots,\\ f_1(3) &= 16\left(1+\frac{1}{2}+\frac{1}{3}-\gamma-\log\frac{7}{2}\right) = 0.0536751989\ldots,\\ f_1(1) &- 2f_1(2)+f_1(3) = 0.00614362106\ldots > 0. \end{aligned}$$

Hence, the sequence $[(n+1)/n]^2 H(n)$ is strictly decreasing convex for all integers $n \geq 1$. The proof of Theorem 3 is complete. □

Proof of Theorem 4. Let $f_1(x)$ be defined by

$$f_1(x) = (x+1)^2\left[\psi(x+1)-\log\left(x+\frac{1}{2}\right)\right], \quad x>0.$$

Direct calculation produces

$$\begin{aligned} &\left[\psi(x+1)-\log\left(x+\frac{1}{2}\right)\right][\log f_1(x)]''\\ =&\left[\psi''(x+1)+\frac{1}{(x+\frac{1}{2})^2}\right]\left[\psi(x+1)-\log\left(x+\frac{1}{2}\right)\right]\\ &-\left[\frac{1}{x+\frac{1}{2}}-\psi'(x+1)\right]^2-\frac{2}{(x+1)^2}\left[\psi(x+1)-\log\left(x+\frac{1}{2}\right)\right]^2. \end{aligned}$$

By (51), (53) and (56), we obtain

$$\begin{aligned}&\left[\psi(x+1)-\log\left(x+\frac{1}{2}\right)\right]^2[\log f_1(x)]''\\ &>\left[\frac{1}{4(x+\frac{1}{2})^4}-\frac{7}{48(x+\frac{1}{2})^6}\right]\left[\frac{1}{24(x+\frac{1}{2})^2}-\frac{7}{960(x+\frac{1}{2})^4}\right]\\ &\qquad-\left[\frac{1}{12(x+\frac{1}{2})^3}\right]^2-\frac{2}{(x+1)^2}\left[\frac{1}{24(x+\frac{1}{2})^2}\right]^2\\ &=\frac{40x^3-21x^2-142x-\frac{169}{2}}{11520(x+\frac{1}{2})^8(x+1)^2}+\frac{49}{46080(x+\frac{1}{2})^{10}}>0\quad\text{for}\quad x\geq 3.\end{aligned}$$

A dierct calculation produces

$$f_1(1)f_1(3)=0.003718451>0.003411055=f_1^2(2),$$
$$f_1(2)f_1(4)=0.002979014\ldots>0.002881026=f_1^2(3)>0.$$

Hence, the sequences $[(n+1)/n]^2H(n)$ $(n=1,2,\ldots)$ is strictly log-convex. The proof of Theorem 4 is complete. □

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