THE GEOMETRICAL CONVEXITY AND CONCAVITY OF INTEGRAL FOR CONVEX AND CONCAVE FUNCTIONS

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Abstract: Suppose that $[a,b] \subseteq (0,\infty)$, $f:[a,b] \to (0,\infty)$ is a function. In this paper, the following two results are proved: (1)Both $g(x) = \int_x^b f(t)dt$ and $h(x) = \int_a^x f(t)dt$ are geometrically concave functions on $[a,b]$ if f is a geometrically concave function; (2) $p(x) = \int_a^x f(t)dt + af^2(a)/(f(a)+af'(a))$ is a geometrically convex function on $[a,b]$ if f is a twice differentiable geometrically convex function with $xf'(x)+f(x) > 0$ on $[a,b]$.

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1 Introduction

For the convenience of the readers, we recall the main definitions as following.

Definition 1. Let $I \subseteq R$ be an interval. A function $f: I \to R$ is called a convex function on I if

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}$$

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for all $x, y \in I$. And f is called a concave function if $-f$ is a convex function.

Definition 2. Let $I \subseteq (0, \infty)$ be an interval. A function $f : I \to (0, \infty)$ is called a geometrically convex function on I if

$$f(\sqrt{xy}) \leq \sqrt{f(x)f(y)}$$

for all $x, y \in I$. And f is called a geometrically concave function if $1/f$ is a geometrically convex function.

The property of convexity or concavity of a given function is one of the most powerful tools in establishing a wide range of analytic inequalities (see [1-12] and the bibliographies in those papers).

From definition 1 and definition 2, the following theorem A is obvious.

Theorem A. Suppose that I is a subinterval of $(0, \infty)$ and $f : I \to (0, \infty)$ is a geometrically convex function. Then

$$F = \log \circ f \circ \exp : \log(I) \to R$$

is a convex function. Conversely, if J is an interval(for which $\exp(J)$ is a subinterval of $(0, \infty)$) and $F : J \to R$ is a convex function, then

$$f = \exp \circ F \circ \log : \exp(J) \to (0, \infty)$$

is a geometrically convex function.

Equivalently, f is a geometrically convex function if and only if $\log f(x)$ is a convex function of $\log x$. Modulo this characterization, the class of all geometrically convex functions was first considered by P. Montel [13], in a beautiful paper discussing the analogues of the notion of convex function in n variables. However, the roots of the research in this area can be traced long before him.

In a long time, the subject of geometrical convexity seems to be even forgotten, which is a pity because of its richness. Recently, C.P. Niculescu [11] discussed the beautiful class of inequalities, which arise from the notion of geometrical convexity for functions. Niculescu's contribution is not only to call the attention to the beautiful zoo of inequalities falling in the realm of geometrical convexity, but also to prove that many classical inequalities can benefit of a better understanding via the geometrically approach of convexity.

One of the focus problems on geometrically convex and concave functions is how to distinguish the geometrical convexity or concavity of a function.

The following two results on geometrically convex functions were obtained by P. Montel and C.P. Niculescu, respectively.

Theorem B(see [13]). Let $f : [0, a) \to [0, \infty)$ be a continuous function, which is geometrically convex on $(0, a)$. Then

$$F(x) = \int_0^x f(t)dt$$

is also continuous on $[0, a)$ and geometrically convex on $(0, a)$.

Theorem C(see [11]). Let $f : I \to (0, \infty)$ be a differentiable function defined on a subinterval of $(0, \infty)$. then the following assertions are equivalent:
i) f is geometrically convex(or concave, resp.);
ii) The function $xf'(x)/f(x)$ is increasing(or decreasing, resp.);
iii) f verifies the inequality

$$\frac{f(x)}{f(y)} \geq (\text{or } \leq, \text{ resp.}) \left(\frac{x}{y}\right)^{yf'(y)/f(y)} \qquad \text{for every} \quad x, y \in I.$$

If moreover f is twice differentiable, then f is geometrically convex (or concave, resp.) if and only if

$$x(f(x)f''(x) - f'^2(x)) + f(x)f'(x) \geq (\text{or } \leq, \text{ resp.})0, \quad \text{for all} \quad x \in I.$$

The main purpose of this paper is to prove the following two theorems.

Theorem 1. Suppose that $f : [a, b] \subseteq (0, \infty) \to (0, \infty)$ is a geometrically concave function. If $g(x) = \int_x^b f(t)dt$ and $h(x) = \int_a^x f(t)dt$, then both g and h are geometrically concave on $[a, b]$.

Theorem 2. Suppose that $f : [a, b] \subseteq (0, \infty) \to (0, \infty)$ be a twice differentiable geometrically convex function, $p(x) = \int_a^x f(t)dt + af^2(a)/(f(a) + af'(a))$. If $xf'(x) + f(x) > 0$ for all $x \in [a, b]$, then $p(x)$ is geometrically convex on $[a, b]$.

2 Lemmas and the Proof of Theorems

For the sake of readability, we shall first give the following lemmas, they will be used to predigest the proof of theorem 1.

Lemma 1(see [14]). For each convex function $f : [a, b] \to R$, there exist infinitely differentiable convex function sequences $f_n : [a, b] \to R$, $n = 1, 2, 3, \cdots$, such that $\{f_n\}$ converge uniformly to f on $[a, b]$.

From definition 1, definition 2, theorem A and lemma 1 we can get the following lemma 2 immediately.

Lemma 2. For each geometrically convex(or concave, resp.) function $f : [a, b] \subseteq (0, \infty) \to (0, \infty)$, there exist infinitely differentiable geometrically convex (or concave, resp.) function sequences $f_n : [a, b] \to (0, \infty)$, $n = 1, 2, 3, \cdots$, such that $\{f_n\}$ converge uniformly to f on $[a, b]$.

The following lemma 3 can be derived from the definition of geometrically convex(or concave, resp.) function directly.

Lemma 3. Let $f_n : I \subseteq (0, \infty) \to (0, \infty)$, $n = 1, 2, 3, \cdots$, be geometrically convex(or concave, resp.) function sequences. If $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in I$, then $f : I \to (0, \infty)$ is also a geometrically convex(or concave, resp.) function.

Proof of theorem 1. By lemma 2 and lemma 3 we may assume that $f \in C^\infty[a, b]$. For $x \in [a, b]$, the expression of $g(x)$ leads to

$$\begin{aligned} & x(g(x)g''(x) - g'^2(x)) + g(x)g'(x) \\ = & -(xf'(x) + f(x)) \int_x^b f(t)dt - xf^2(x). \end{aligned} \tag{1}$$

To prove that $g(x)$ is geometrically concave on $[a, b]$ we need only to prove

$$-(xf'(x) + f(x)) \int_x^b f(t)dt - xf^2(x) \leq 0 \tag{2}$$

for all $x \in [a, b]$ by theorem C and (1).

Set

$$\begin{aligned} E &= \{x \in [a, b] : xf'(x) + f(x) \geq 0\} \\ &= \{x \in [a, b] : \frac{xf'(x)}{f(x)} \geq -1\}. \end{aligned}$$

Then since $xf'(x)/f(x)$ is decreasing, the following three cases will complete the proof.

Case 1. $b \in E$. Then $E = [a, b]$, and (2) is true.

Case 2. $a \notin E$. Then $E = \emptyset$, this implies that $xf'(x) + f(x) < 0$ for all $x \in [a, b]$. For $x \in [a, b]$, taking

$$g_1(x) = -\int_x^b f(t)dt - \frac{xf^2(x)}{xf'(x) + f(x)}. \tag{3}$$

Then the geometrical concavity of f and theorem C lead to

$$g_1(x) = xf(x)\frac{x(f(x)f''(x) - f'^2(x)) + f(x)f'(x)}{(xf'(x) + f(x))^2} \leq 0. \tag{4}$$

(4) yields

$$g_1(x) \geq g_1(b) = -\frac{bf^2(b)}{bf'(b) + f(b)} \geq 0 \tag{5}$$

for all $x \in [a, b]$, and (2) follows from (3) and (5).

Case 3. $a \in E$ and $b \notin E$. Then there exists $x_0 \in [a, b)$ such that $E = [a, x_0]$ and $xf'(x) + f(x) < 0$ for $x \in (x_0, b]$ since $xf'(x)/f(x)$ is decreasing on $[a, b]$. By the similar proof as in case 2 we know that (2) is true for $x \in (x_0, b]$. This and $E = [a, x_0]$ imply that (2) is true for all $x \in [a, b]$.

Making use of the parallel proof as above we can prove that $h(x)$ is a geometrically concave function on $[a, b]$.

Proof of theorem 2. For $x \in [a, b]$, the expression of $p(x)$ leads to

$$\begin{aligned} &x(p(x)p''(x) - p'^2(x)) + p(x)p'(x) \\ &= (xf'(x) + f(x))\int_a^x f(t)dt - xf^2(x) + \frac{af^2(a)}{f(a) + af'(a)}(xf'(x) + f(x)). \end{aligned} \tag{6}$$

Let

$$p_1(x) = \int_a^x f(t)dt - \frac{xf^2(x)}{xf'(x) + f(x)} + \frac{af^2(a)}{af'(a) + f(a)}, \tag{7}$$

then the geometrical convexity of f and theorem C lead to

$$p_1'(x) = xf(x)\frac{x(f(x)f''(x) - f'^2(x)) + f(x)f'(x)}{(xf'(x) + f(x))^2} \geq 0. \tag{8}$$

(8) yields

$$p_1(x) \geq p_1(a) = 0 \tag{9}$$

for all $x \in [a, b]$.

Theorem 2 follows from (6), (7), (9) and theorem C.

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